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PARTIAL HYPOELLIPTICITY FOR A CLASS OF ABSTRACT DIFFERENTIAL COMPLEXES ON BANACH SPACE SCALES

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Abstract. In this article we give sufficient conditions for the hypoellipticity in the first level of the abstract complex generated by the differential operators $L_j = \frac{\partial}{\partial t_j} + \frac{\partial \phi}{\partial t_j}(t, A)A$, $j = 1, 2, \ldots, n$, where $A : D(A) \subset X \longrightarrow X$ is a *sectorial* operator in a Banach space X, with $\Re\sigma(A) > 0$, and $\phi = \phi(t, A)$ is a series of nonnegative powers of A^{-1} with coefficients in $C^\infty(\Omega)$, Ω being an open set of $\mathbb{R}^n$ with $n \in \mathbb{N}$ arbitrary. Analogous complexes have been studied by several authors in this field, but only in the case $n = 1$ and with X a Hilbert space. Therefore, in this article, we provide an improvement of these results by treating the question in a more general setup. First, we provide sufficient conditions to get the partial hypoellipticity for that complex in the elliptic region. Second, we study the particular operator $A = 1 - \Delta : W^{2,p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N)$, for $1 \leq p \leq 2$, which will allow us to solve the problem of points which do not belong to the elliptic region.

1. Introduction and preliminaries

The present article extends to Banach space scales some ideas developed in [7] and [12] for Hilbert space scales. We provide sufficient conditions for the partial hypoellipticity, in the first degree, of the differential complex given by the operators

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial \phi}{\partial t_j}(t, A)A, \quad j = 1, 2, \ldots, n,$$

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where $A : D(A) \subset X \longrightarrow X$ is a sectorial linear operator with $\Re\sigma(A) > 0$, X is a Banach space (see [3], [6]), and $\phi = \phi(t, A)$ is a series of nonnegative powers of A^{-1} with coefficients in $C^\infty(\Omega)$, Ω being an open set of $\mathbb{R}^n$.

The map $\phi = \phi(t, A)$ has the form

$$\phi(t, A) = \sum_{k=0}^{\infty} \phi_k(t) A^{-k},$$

which converges in $\mathcal{L}(X)$—as well as in each of its t-derivatives—uniformly with respect to t on compacts of Ω, and $\phi_k \in C^\infty(\Omega) = C^\infty(\Omega; \mathbb{C})$ for every $k \in \mathbb{N} \cup \{0\}$. Moreover, we assume that its leading coefficient ϕ_0 is a real-valued function.

The key step in solving this problem is to establish an extension—to Banach spaces—of a method developed for Hilbert spaces in [12], [13], and [14]. This result says that the hypoellipticity (or solvability) of the differential complex generated by the operators above is equivalent to the hypoellipticity (resp., solvability) of a simpler complex, namely, the one generated by the differential operators

$$L_{j,0} := \frac{\partial}{\partial t_j} + \frac{\partial \phi_0}{\partial t_j}(t) A, \quad j = 1, 2, \ldots, n.$$

The local solvability of the transpose of this complex, in the case where X is a Hilbert space, was carried out by Yamaoka [14]. There he uses a result from [4] to obtain the local solvability for a class of undetermined systems by assuming that the leading coefficient in $\phi(t, A)$ is analytic and satisfies conditions (ψ_1) and (ψ_2). In this article, however, we avoid such conditions.

Here we first assume that the leading coefficient in $\phi(t, A)$ is just a real-valued function C^∞ and use some of its dynamical properties to obtain the partial hypoellipticity in the "elliptic region" (see Theorem 2.2). After that, we explore some of the techniques developed in [2], [5], [8], and [11] to study the problem of points which do not belong to the elliptic region. That will be the only case where we assume the analyticity of ϕ_0.

On the other hand, after suggestions from the referees, we realized that by following the proof of Proposition 9.2 in [14], if we also assume conditions (ψ_1) and (ψ_2) for the leading coefficient in $\phi(t, A)$, then it seems that these arguments could be adapted to our context (of the scale of fractional power spaces associated to a sectorial operator in a Banach space), leading us to the proof of the (partial) subellipticity of the system under study and thus giving us an improvement of our Theorem 2.2. (For that we are very thankful, and we look forward to exploring the possibility of proving these results in future work.)

Finally, we also must say that, in our opinion, there is a slight error in the statement of Proposition 9.2 in [14]. The conclusion cannot be "for any $\varepsilon > 0$" but rather "there is $\varepsilon > 0$." Yamaoka's arguments, however, can be fixed by making use of his last inequality on p. 241 in [14].

Now let $\mathcal{E} := \{t^* \in \Omega : \nabla\phi_0(t^*) = 0\}$. The set $\Omega \setminus \mathcal{E}$ will be called the *elliptic region*. We will prove that given a point $t_0 \in \Omega \setminus \mathcal{E}$, there exists an open set

$U \subset \Omega$, with $t_0 \in U$ and $U \cap \mathcal{E} = \varnothing$, such that for each $u \in C^\infty(U; X^{-\infty})$, if

$$\sum_{j=1}^{n} L_{j,0} u \, dt_j = f \quad \text{in } U$$

and $f \in \Lambda^1 C^\infty(U; X^\infty)$, then u belongs to $C^\infty(U; X^\infty)$.

To do so, we are going to use dynamical properties of the gradient system (see [1], [3]) generated by the real-valued function ϕ_0, that is, the system

$$\begin{cases} t'(s) = -\nabla\phi_0(t(s)), \quad s \geq 0, \\ t(0) = t_0 \in \Omega. \end{cases}$$

Below we will introduce the complex of differential operators that we want to study and then clarify every concept related to it, in order to understand its hypoellipticity and, finally, to solve this problem.

1.1. The complex under study. Let $A : D(A) \subset X \longrightarrow X$ be a sectorial operator, with $\Re\sigma(A) > 0$, in a Banach space X with norm $\|\cdot\|_X$. For each real α we can consider its fractional power space X^α. More precisely, for $\alpha > 0$, the fractional power space associated to A is defined by

$$X^\alpha := \{A^{-\alpha} f : f \in X\} \quad \text{equipped with the norm } \|u\|_\alpha := \|A^\alpha u\|_X,$$

where the operator $A^{-\alpha}$ is given by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty \theta^{\alpha-1} e^{-A\theta} \, d\theta,$$

and $\{e^{-A\theta} : \theta \geq 0\}$ is the analytic semigroup generated by $-A$. The operator $A^{-\alpha} : X \longrightarrow X$ is injective and its inverse is denoted by $A^\alpha : X^\alpha \longrightarrow X$. Also, for $\alpha < 0$, X^α is defined as the completion of the space $(X, \|\cdot\|_\alpha)$, where $\|u\|_\alpha := \|A^\alpha u\|_X$. Under these conditions, $X^0 = X$ and, for every real $\alpha \geq \beta$, we have $X^\alpha \subset X^\beta$ (see [3] and [6] for more properties).

Now let $X^\infty := \bigcap_{\alpha\in\mathbb{R}} X^\alpha$ be equipped with the projective limit topology, that is, the locally convex topology generated by the family of norms $(\|\cdot\|_\alpha)_{\alpha>0}$. Here X^∞ is a Fréchet space. Also, let $X^{-\infty} := \bigcup_{\alpha\in\mathbb{R}} X^\alpha$ be equipped with the inductive limit topology, namely, the one such that: "a subset $U \subset X^{-\infty}$ is open if and only if $U \cap X^\alpha$ is an open set in X^α for every real α." Let $X^{-\infty}$ be a topological vector space.

Remark 1.1. Observe that the operator $A : D(A) \subset X \longrightarrow X$ "moves through" the scale of fractional power spaces $(X^\alpha)_{\alpha\in\mathbb{R}}$. That is, by restrictions and extensions (resp., $\alpha \geq 0$, $\alpha < 0$) for every $\alpha \in \mathbb{R}$, it holds that

$$A : X^{\alpha+1} \subset X^\alpha \longrightarrow X^\alpha.$$

Furthermore, this new $A : X^{\alpha+1} \subset X^\alpha \longrightarrow X^\alpha$ is a sectorial operator, and the analytic semigroup generated by $-A$ is such that, for all $\alpha \in \mathbb{R}$ and all $\theta > 0$, the following inclusion holds (see [3]):

$$e^{-A\theta}(X^\alpha) \subset X^\infty.$$

Example 1.2. As an example, let us consider the operator $A = 1-\Delta : W^{2,p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N)$, for $1 \leq p < \infty$, which satisfies all the abstract hypotheses above (see [3], [6]). As will be seen in Section 3, we have a very good understanding of the scale of its fractional power space.

On the other hand, consider the series

$$\phi(t, A) = \sum_{k=0}^{\infty} \phi_k(t) A^{-k}$$

which converges in $\mathcal{L}(X)$, uniformly for t on compacts of Ω, where Ω is an open set of $\mathbb{R}^n$, and $\phi_k \in C^\infty(\Omega)$ for every $k \in \mathbb{N} \cup \{0\}$, with ϕ_0 real-valued.

For $j = 1, 2, \ldots, n$, we define the differential operators $L_j : C^\infty(\Omega; X^{-\infty}) \longrightarrow C^\infty(\Omega; X^{-\infty})$ by

$$L_j u := \frac{\partial u}{\partial t_j} + \frac{\partial \phi}{\partial t_j}(t, A) A u. \tag{1.1}$$

Taking the leading coefficient in $\phi(t, A)$, that is, $\phi_0 \in C^\infty(\Omega)$, we also define, for each $j = 1, 2, \ldots, n$, the differential operator $L_{j,0} : C^\infty(\Omega; X^{-\infty}) \longrightarrow C^\infty(\Omega; X^{-\infty})$ by

$$L_{j,0} u := \frac{\partial u}{\partial t_j} + \frac{\partial \phi_0}{\partial t_j}(t) A u.$$

We note that, just by restriction, the space $C^\infty(\Omega; X^\infty)$ is invariant under L_j and $L_{j,0}$; that is, $L_j : C^\infty(\Omega; X^\infty) \longrightarrow C^\infty(\Omega; X^\infty)$ and $L_{j,0} : C^\infty(\Omega; X^\infty) \longrightarrow C^\infty(\Omega; X^\infty)$, because $X^\infty \hookrightarrow X^{-\infty}$.

The operators L_j and $L_{j,0}$ can be used to define complexes of differential operators

$$\mathbb{L} : \Lambda^p C^\infty(\Omega; X^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; X^{-\infty})$$

and by restriction

$$\mathbb{L} : \Lambda^p C^\infty(\Omega; X^{\infty}) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; X^{\infty}).$$

Also, for all $0 \leq p \leq n$,

$$\mathbb{L}_0 : \Lambda^p C^\infty(\Omega; X^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; X^{-\infty})$$

and also by restriction

$$\mathbb{L}_0 : \Lambda^p C^\infty(\Omega; X^{\infty}) \longrightarrow \Lambda^{p+1} C^\infty(\Omega; X^{\infty}).$$

For each $0 \leq p \leq n$, they are given by

$$\mathbb{L} u := \sum_{|J|=p} \sum_{j=1}^{n} L_j u_J \, dt_j \wedge dt_J \quad \text{for } u = \sum_{|J|=p} u_J \, dt_J$$

and

$$\mathbb{L}_0 u := \sum_{|J|=p} \sum_{j=1}^{n} L_{j,0} u_J \, dt_j \wedge dt_J \quad \text{for } u = \sum_{|J|=p} u_J \, dt_J,$$

where $dt_J = dt_{j_1} \wedge \cdots \wedge dt_{j_p}$, $J = (j_1, \ldots, j_p)$ is an ordered multi-index of integers with $1 \leq j_1 < \cdots < j_p \leq n$, and $|J|$ is its length. Thus, we get the global form of these complexes

$$\mathbb{L}u := d_t u + \omega(t, A) \wedge Au \tag{1.2}$$

and

$$\mathbb{L}_0 u := d_t u + \omega_0(t) \wedge Au, \tag{1.3}$$

with

$$\omega(t, A) := \sum_{k=0}^{\infty} \omega_k(t) A^{-k} \in \Lambda^1 C^\infty\big(\Omega; \mathcal{L}(X)\big),$$

where

$$\omega_k(t) := \sum_{j=1}^{n} \frac{\partial \phi_k}{\partial t_j}(t)\, dt_j$$

for every integer $k \geq 0$ and $t \in \Omega$.

In (1.2) and (1.3), d_t stands for the exterior derivative with respect to $t \in \Omega$, $u \in \Lambda^p C^\infty(\Omega; X^{-\infty})$ or $u \in \Lambda^p C^\infty(\Omega; X^\infty)$, and $Au := \sum_{|J|=p} Au_J\, dt_J$. Observe that $\mathbb{L} \circ \mathbb{L} = 0$ and $\mathbb{L}_0 \circ \mathbb{L}_0 = 0$; consequently, those operators actually define differential complexes. We can now introduce the kind of hypoellipticity that we are concerned with (see [12] and [13] for analogous definitions).

Definition 1.3. Let Ω be an open set of $\mathbb{R}^n$. Given U an open set of Ω, we say that an operator $\mathbb{M} : C^\infty(\Omega; X^{-\infty}) \longrightarrow \Lambda^1 C^\infty(\Omega; X^{-\infty})$ is *hypoelliptic* in U if, for every distribution $u \in C^\infty(U; X^{-\infty})$,

$$\mathbb{M}u \in \Lambda^1 C^\infty(U; X^\infty) \qquad \text{implies that } u \in C^\infty(U; X^\infty).$$

When $\mathbb{M}$ is hypoelliptic in $U = \Omega$, we say that $\mathbb{M}$ is *globally hypoelliptic*, and when $\mathbb{M}$ is hypoelliptic in U, for every open set $U \subset \Omega$, we say that $\mathbb{M}$ is *locally hypoelliptic* in Ω.

Remark 1.4. We note that the kind of hypoellipticity introduced in Definition 1.3 is "partial" because we start with distributions which are regular in one of its variables, namely, the variable $t \in \Omega$, and we treat the regularity in the "variable x." In other words, the regularity relative to the scale of spaces X^α.

Under these circumstances, in the present article we will show that $\mathbb{L} : C^\infty(\Omega; X^{-\infty}) \longrightarrow \Lambda^1 C^\infty(\Omega; X^{-\infty})$ is hypoelliptic in $\Omega_0 := \Omega \setminus \mathcal{E}$, where $\mathcal{E} := \{t^* \in \Omega : \nabla\phi_0(t^*) = 0\}$. Here Ω_0 is called the *elliptic region* of $\mathbb{L}$ (and $\mathbb{L}_0$). After that, considering $A := 1 - \Delta$, by means of certain techniques we learned from [2], we will be able to study the hypoellipticity of $\mathbb{L}$ in points of $\mathcal{E}$. In other words, for the general case, we do not have all the information about $\mathbb{L}$ which would allow us to obtain its hypoellipticity in Ω. More precisely, we first use the dynamical properties of the solution of the Cauchy problem

$$\begin{cases} t' = -\nabla\phi_0(t), & s \geq 0, \\ t(0) = t_0 \in \Omega, \end{cases}$$

to study the hypoellipticity in Ω_0, and then we use the structure of the operator $1-\Delta$ to solve the problem in points of $\mathcal{E}$.

Our particular treatment of the case Ω_0 is inspired by Hounie in [7], who considered the same kind of problem as ours, but under the assumptions that Ω is an interval of the real line and X is a Hilbert space. Hounie also obtained a complete characterization of the global hypoellipticity by using the so-called *conditions* (ψ) and (τ). (We will not, however, assume such conditions here.)

We point out that Trèves [12] also studied the same problem as Hounie and that both authors obtained the same conclusions by means of different techniques. As was the case in [12], [13], and [14] in the framework of Hilbert spaces, the key step in studying the hypoellipticity of the operator $\mathbb{L}$ is that we can extend to the framework of Banach spaces a result which allows us to isolate the "principal part" of $\mathbb{L}$. This fact allows us to conclude that studying the $\mathbb{L}$'s hypoellipticity is equivalent to studying the hypoellipticity of the simpler operator $\mathbb{L}_0$. More precisely, we can prove the following.

Lemma 1.5. *If the leading coefficient ϕ_0 in $\phi(t,A)$ is a real-valued function, then for each $0 \le p \le n$ and each open set $U \subset \Omega$,*

$$\mathbb{L} : \Lambda^p C^\infty(U; X^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(U; X^{-\infty})$$

is hypoelliptic in U if and only if

$$\mathbb{L}_0 : \Lambda^p C^\infty(U; X^{-\infty}) \longrightarrow \Lambda^{p+1} C^\infty(U; X^{-\infty})$$

is hypoelliptic in U.

Proof. We begin by defining, for each $t \in \Omega$, the operator

$$\alpha(t,A) := \Re\phi_0(t) - \phi(t,A) = \phi_0(t) - \phi(t,A)$$

and then

$$\alpha(t,A)A = \big[\phi_0(t) - \phi(t,A)\big]A = -\sum_{k=1}^{\infty} \phi_k(t) A^{1-k}.$$

Observe that the composition $\alpha(t,A)A$ is, for all $t \in \Omega$, a bounded linear operator on X; hence, it is the infinitesimal generator of a *group* of linear operators (see [9]).

Thus, we can define the family of operators $U(t) := e^{\alpha(t,A)A}$, $t \in \Omega$. This family can then be used to generate an automorphism of $\Lambda^p C^\infty(U; X^\infty)$ and of $\Lambda^p C^\infty(U; X^{-\infty})$. For each $0 \le p \le n$, let

$$(\mathcal{U}u)(t) := U(t)u(t) = e^{\alpha(t,A)A}u(t) \quad \text{for } u \in C^\infty(U; X^\infty) \text{ and } t \in U.$$

It is not hard to see that $\mathcal{U} : C^\infty(U; X^\infty) \longrightarrow C^\infty(U; X^\infty)$ defines an automorphism, because $e^{\alpha(t,A)A}$ is invertible for every $t \in \Omega$, which has an extension $\mathcal{U} : C^\infty(U; X^{-\infty}) \longrightarrow C^\infty(U; X^{-\infty})$, because of Remark 1.1.

It is a straightforward consequence of the definition of $\mathcal{U}$ that

$$\big[L_j(\mathcal{U}u)\big](t) = \big[\mathcal{U}(L_{j,0}u)\big](t) \quad \text{for } u \in C^\infty(U; X^\infty), t \in U, \text{and } j = 1, 2, \ldots, n. \tag{1.4}$$

If we define, for $u = \sum_{|J|=p} u_J \, dt_J$,

$$\mathcal{U}u := \sum_{|J|=p} (\mathcal{U}u_J) \, dt_J,$$

then the equality (1.4) tells us that

$$\mathbb{L}(\mathcal{U}u) = (\mathcal{U}\mathbb{L}_0)u \quad \text{for } u \in C^\infty(U; X^\infty).$$

As the equality above is also true for every $u \in C^\infty(U; X^{-\infty})$, our claim follows. □

Remark 1.6. The proof we gave above does not work if ϕ_0 is not real-valued, for if that were the case, then we would have

$$\alpha(t, A)A := \Re\phi_0(t)A - \phi(t, A)A = \big[\phi_0(t) - \phi(t, A)\big]A - i\Im\phi_0(t)A.$$

Since A is an arbitrary sectorial operator, iA might not be a generator of a group in X.

On the other hand, if $\operatorname{Im}\phi_0(t) \neq 0$, then the number $-i\operatorname{Im}\phi_0(t)$ does not belong to the sector $\{z \in \mathbb{C} : |\arg z| < \varepsilon\}$ on which the analytic semigroup $\{e^{-A\theta} : \theta \geq 0\}$ has an analytic extension (see page 21 in [6] for more details). So the operator $U(t) = e^{\alpha(t,A)A}$ might not be well defined and the proof breaks down. However, when X is a Hilbert space H, $A : D(A) \subset H \longrightarrow H$ is a positive self-adjoint linear operator with $0 \in \rho(A)$, iA generates a group in H, and now $e^{\alpha(t,A)A}$ is well defined (see [12], [13] for more details).

2. Main theorems

Let us start with a simple result from ODE theory which will be extremely useful for proving Theorem 2.2 and whose proof we leave to the reader. In it for $B \subset \Omega$, the symbol $\mathcal{O}_\delta(B)$ will stand for the union of all the open balls with radius $\delta > 0$ and centered at some point of B.

Lemma 2.1. *Let $\phi_0 \in C^\infty(\Omega; \mathbb{R})$, $\mathcal{E} := \{t^* \in \Omega : \nabla\phi_0(t^*) = 0\}$, and consider the Cauchy problem*

$$\begin{cases} t'(s) = -\nabla\phi_0(t(s)), & s \geq 0, \\ t(0) = t_0 \in \Omega. \end{cases} \tag{2.1}$$

For each $t_0 \in \Omega$, let $\omega(t_0) > 0$ be the maximal time of existence of the solution $s \mapsto T(s)t_0$ of (2.1). Then for each $t_0 \in \Omega_0 := \Omega \setminus \mathcal{E}$ and each $\delta > 0$ with $d(t_0, \mathcal{E}) > 2\delta$, there exist an open set $U \subset \Omega$, with $t_0 \in U$, and $\tau > 0$ such that

(i) *$\omega(t) \geq \tau$ for every $t \in U$,*
(ii) *$T(s)U \subset \mathcal{O}_\delta(\mathcal{E} \cup \partial\Omega)$ whenever $s \geq \tau$,*
(iii) *$T(s)U \subset \Omega_0$ when $0 \leq s \leq \tau$, and*
(iv) *$U \cap \mathcal{O}_\delta(\mathcal{E} \cup \partial\Omega) = \varnothing$.*

We can now prove our main result, which will be based on ideas from [7]. According to Lemma 1.5, it is sufficient to study the complex generated by $\mathbb{L}_0$. This fact will be implicit in the following results.

Theorem 2.2. *If the leading coefficient ϕ_0 in $\phi(t, A)$ is a real-valued function, then, given any $t_0 \in \Omega \setminus \mathcal{E}$, there exists an open set $U \subset \Omega \setminus \mathcal{E}$, with $t_0 \in U$, such that $\mathbb{L}$ is hypoelliptic in U. Consequently, $\mathbb{L}$ is locally hypoelliptic in $\Omega \setminus \mathcal{E}$.*

Proof. Indeed, given $t_0 \in \Omega_0 = \Omega \setminus \mathcal{E}$ and $\delta > 0$ with $d(t_0, \mathcal{E}) > 2\delta$, let U and $\tau > 0$ be the ones obtained in the last lemma. Also, let $\{e^{-A\theta} : \theta \geq 0\}$ be the analytic semigroup in X generated by $-A$. As we saw in Remark 1.1, $e^{-A\theta}u \in X^{\infty}$ for every $u \in X^{-\infty}$ whenever $\theta > 0$.

Now, for $\omega \in \Lambda^1 C^{\infty}(U; X^{\infty})$ (or $\omega \in \Lambda^1 C^{\infty}(U; X^{-\infty})$) and $t \in U$, we define the linear operator

$$(K\omega)(t) := -\int_{\gamma_t} e^{(\phi_0(z) - \phi_0(t))A}\omega(z)\, dz, \tag{2.2}$$

where the integration path is $\gamma_t(s) := T(s)t$, $s \in [0, \tau]$, the solution of (2.1) through t.

In the same way, we can define K on each open subset W of U. Observe that the value $(K\omega)(t)$ is well defined because the function ϕ_0 is a Lyapunov function for the Cauchy problem (2.1) (see [1], [3]). So $\phi_0(T(s)t) \leq \phi_0(t)$ for every $s \in [0, \tau]$ and $t \in U$, and hence we can evaluate the semigroup $\{e^{-A\theta} : \theta \geq 0\}$ at time

$$\theta = -\big(\phi_0\big(T(s)t\big) - \phi_0(t)\big) \geq 0.$$

Furthermore, it is not hard to see that K maps $\Lambda^1 C^{\infty}(U'; X^{\infty})$ into $C^{\infty}(U'; X^{\infty})$ and $\Lambda^1 C^{\infty}(U'; X^{-\infty})$ into $C^{\infty}(U'; X^{-\infty})$ for every open subset $U' \subset U$.

Note that for $t \in U$ fixed, the equality $\phi_0(T(s)t) = \phi_0(t)$ is only possible for a finite number of $s \in [0, \tau]$. Otherwise, there exists a sequence $(s_j)_{j \in \mathbb{N}}$ in $[0, \tau]$ with $s_j \longrightarrow s_0 \in [0, \tau]$ and $\phi_0(T(s_j)t) = \phi_0(t)$, so $\nabla\phi_0(T(s_0)t) = 0$; that is, $T(s_0)t \in \mathcal{E}$. Thus, we are led to a contradiction, since $T(s)U \subset \Omega_0$ when $0 \leq s \leq \tau$, according to Lemma 2.1.

This fact says that the coefficients of the 1-form $e^{(\phi_0(T(s)t) - \phi_0(t))A}\omega(T(s)t)$ are in X^{∞}, except for a finite number of $s \in [0, \tau]$. Then for every $t \in U$ we have $(K\omega)(t) \in X^{\infty}$. By derivation under integral sign, we obtain $K\omega \in C^{\infty}(U; X^{\infty})$ for every $\omega \in C^{\infty}(U; X^{-\infty})$.

On the other hand, let $g \in C_c^{\infty}(U; X^{-\infty})$, and consider $\mathbb{L}_0 g \in \Lambda^1 C^{\infty}(U; X^{-\infty})$. For every $t \in U$ we have, by Lemma 2.1, that $T(\tau)t \notin U$ and hence $T(\tau)t \notin \operatorname{supp}(g)$, so integrating by parts and using the fact that $T(s)t$ is the solution of (2.1), we see that

$$\begin{aligned}
\big[K(\mathbb{L}_0 g)\big](t) &= -\int_{\gamma_t} e^{(\phi_0(z) - \phi_0(t))A}(\mathbb{L}_0 g)(z)\, dz \\
&= -\int_{\gamma_t} e^{(\phi_0(z) - \phi_0(t))A}(d_t g)(z)\, dz - \int_{\gamma_t} e^{(\phi_0(z) - \phi_0(t))A}\omega_0(z) \wedge Ag(z)\, dz \\
&= -\big[e^{(\phi_0(T(\tau)t) - \phi_0(t))A} g\big(T(\tau)t\big) - e^{(\phi_0(t) - \phi_0(t))A} g(t)\big] = g(t);
\end{aligned}$$

that is,

$$\big[K(\mathbb{L}_0 g)\big](t) = g(t) \quad \text{for every } t \in U. \tag{2.3}$$

Thus, if $u \in C^{\infty}(U; X^{-\infty})$ is such that $\mathbb{L}_0 u = f \in \Lambda^1 C^{\infty}(U; X^{\infty})$, for each $t' \in U$ we can choose $\varphi \in C_c^{\infty}(U; \mathbb{R})$, with $\varphi = 1$ in some neighborhood U' of t'. Then

$g := \varphi u \in C_c^\infty(U; X^{-\infty})$ and we have

$$\mathbb{L}_0(\varphi u) = \varphi \mathbb{L}_0 u + \sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}(t)u(t)\,dt_j = \varphi f + \sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}(t)u(t)\,dt_j.$$

So, by (2.3) we obtain

$$\big[K(\varphi f)\big](t) + \Big[K\Big(\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j} u\,dt_j\Big)\Big](t) = \big[K\mathbb{L}_0(\varphi u)\big](t) = (\varphi u)(t) \quad \text{for all } t \in U.$$

Since $\varphi f \in \Lambda^1 C^\infty(U; X^\infty)$, we have $K(\varphi f) \in C^\infty(U; X^\infty)$. Also, since U' was arbitrary, the theorem will be proved if we show that

$$K\Big(\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j} u\,dt_j\Big) \in C^\infty(U'; X^\infty).$$

Indeed, since φ is constant on U', we have $\sum_{j=1}^n \frac{\partial\varphi}{\partial t_j}(r)u(r)\,dt_j = 0$ as long as $r \in U'$. On the other hand, each $t' \in U'$ has a neighborhood $V' \subset U'$ and there exists $\tau_1 > 0$ such that $T(s)t \in U'$ whenever $s \in [0, \tau_1]$ and $t \in V'$. So, if $t \in V'$, then

$$\begin{aligned} &K\Big(\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j} u\,dt_j\Big)(t) \\ &\qquad = -\int_{\tau_1}^{\tau} e^{(\phi_0(T(s)t) - \phi_0(t) + \eta)A}\Big[e^{-\eta A}\Big(\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}\big(T(s)t\big)u\big(T(s)t\big)\frac{d(T(s)t)_j}{ds}\Big)\Big]\,ds, \end{aligned}$$

where $\tau_1 > 0$ may be chosen such that $\eta := (\phi_0(t) - \phi_0(T(\tau_1)t) > 0$, and $\frac{d(T(s)t)_j}{ds}$ stands for the components $j = 1, 2, \ldots, n$ of the vector $\frac{dT(s)t}{ds} \in \mathbb{R}^n$.

Finally, it is not hard to see that if $\alpha \in \mathbb{R}$ is fixed, then for every $h \in C^\infty([\tau_1, \tau]; X^{-\infty})$ we have $e^{-\eta A}h \in C^\infty([\tau_1, \tau]; X^\alpha)$ and so

$$e^{(\phi_0(T(s)t) - \phi_0(t) + \eta)A}[e^{-\eta A}h] \in C^\infty\big([\tau_1, \tau]; X^\infty\big).$$

All these facts together give us that, for every $t \in U'$,

$$\begin{aligned} (\varphi u)(t) = K(\varphi f)(t) - \int_{\tau_1}^{\tau} & e^{(\phi_0(T(s)t) - \phi_0(t) + \eta)A} \\ & \times e^{-\eta A}\Big[\sum_{j=1}^n \frac{\partial \varphi}{\partial t_j}\big(T(s)t\big)u\big(T(s)t\big)\frac{d(T(s)t)_j}{ds}\Big]\,ds \end{aligned}$$

holds, so the second term in the last sum also defines an element of $C^\infty(U'; X^\infty)$; therefore, $\varphi u \in C^\infty(U'; X^\infty)$. But $\varphi u = u$ in U', and the proof is complete. □

The last theorem does not provide an answer for the hypoellipticity in points of $\mathcal{E}$. The next result shows that there may be points in $\mathcal{E}$ where we cannot have hypoellipticity, which means that we will need something else if we want to obtain hypoellipticity in points of $\mathcal{E}$. In Section 3 we provide the conditions needed to overcome this difficulty.

Theorem 2.3. *If $t^* \in \mathcal{E}$ is a local minimal point for ϕ_0, then t^* has a neighborhood V in Ω where $\mathbb{L}$ is not hypoelliptic.*

Proof. Indeed, let V be an open set of Ω with $t^* \in V$ such that $\phi_0(t^*) \leq \phi_0(t)$ for all $t \in V$. Fix $u_0 \in X \setminus X^\infty$, and define $u : V \longrightarrow X^{-\infty}$ by $u(t) := e^{-(\phi_0(t)-\phi_0(t^*))A}u_0$, $t \in V$. It follows that u is well defined and $u \in C^\infty(V; X^{-\infty})$. It is easy to see that $\mathbb{L}_0 u = 0$ in V, so $\mathbb{L}_0 u \in \Lambda^1 C^\infty(V; X^\infty)$. However, since $u(t^*) = u_0 \notin X^\infty$, we do not have $u \in C^\infty(V; X^\infty)$, as we wish. □

Remark 2.4. It is easy to see that when $t^* \in \mathcal{E}$ is an isolated saddle point, ϕ_0 is an open map on some neighborhood of t^*; therefore, t^* is not a local minimal point for ϕ_0.

We finish our contribution by studying the particular operator $A : D(A) \subset X \longrightarrow X$, where $A = 1 - \Delta$, $D(A) = W^{2,p}(\mathbb{R}^N)$, and $X = L^p(\mathbb{R}^N)$.

3. Application to the operator $1 - \Delta$

In this section we present very important information about the scale of fractional power spaces associated to the operator

$$1 - \Delta : W^{2,p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N),$$

where $1 \leq p \leq 2$, which will be useful in the proof of Theorem 3.6. This operator satisfies all the hypotheses we have made so far in this work, namely, that it is a sectorial operator with $\Re\sigma(A) > 0$, as we can see in [3], [6], and [10]. Furthermore, we have a very good understanding of the embeddings of its scale of fractional power spaces into known spaces, as we explain below.

Definition 3.1. For every $1 \leq p \leq 2$ and $\alpha \in \mathbb{R}$, we define the space

$$\mathscr{L}^p_\alpha(\mathbb{R}^N) := \left\{u \in \mathscr{S}'(\mathbb{R}^N) : \left(1 + 4\pi^2|\xi|^2\right)^{\alpha/2}\widehat{u} \in L^{p'}(\mathbb{R}^N)\right\}, \tag{3.1}$$

where $1/p + 1/p' = 1$, $\mathscr{S}'(\mathbb{R}^N)$ stands for the space of tempered distributions on $\mathbb{R}^N$ and the "hat" stands for the Fourier transform with respect to the variable x. Also, $\mathscr{L}^p_\alpha(\mathbb{R}^N)$ is equipped with the norm $\|u\|_{\mathscr{L}^p_\alpha} := \|(1 + 4\pi^2|\xi|^2)^{\alpha/2}\widehat{u}\|_{L^{p'}}$.

On the other hand, in [6] and [10] we find the following.

Definition 3.2. For every $1 \leq p < \infty$ and $\alpha > 0$, we define

$$L^p_\alpha(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : u = G_\alpha * f, \text{for some } f \in L^p(\mathbb{R}^N)\right\}, \tag{3.2}$$

where $G_\alpha : \mathbb{R}^N \longrightarrow \mathbb{C}$ is the function such that $\widehat{G_\alpha}(\xi) = (1+4\pi^2|\xi|^2)^{-\alpha/2}$, and the $*$ stands for the convolution product. Also, $L^p_\alpha(\mathbb{R}^N)$ is equipped with the norm $\|u\|_{L^p_\alpha} := \|f\|_{L^p}$, where $u = G_\alpha * f$.

Now, in [6] it is shown that, for every $\alpha > 0$,

$$\left[L^p(\mathbb{R}^N)\right]^{\alpha/2} = L^p_\alpha(\mathbb{R}^N),$$

holds; that is, the fractional power space $[L^p(\mathbb{R}^N)]^{\alpha/2}$ associated to the operator $1 - \Delta$ coincides with the space $L^p_\alpha(\mathbb{R}^N)$ for each $\alpha > 0$, because $(1 - \Delta)^{-\alpha/2}\varphi = G_\alpha * \varphi$, for each $\varphi \in \mathscr{S}(\mathbb{R}^N)$.

For every $\alpha \geq 0$ and $1 \leq p \leq 2$, it easily follows that $L^p_\alpha(\mathbb{R}^N) \subset \mathscr{L}^p_\alpha(\mathbb{R}^N)$, because of the Hausdorff–Young inequality and the formula $\widehat{G_\alpha * f} = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}\widehat{f}$, $f \in L^p(\mathbb{R}^N)$. On the other hand, when $\alpha < 0$ we have the embedding

$$\big[L^p(\mathbb{R}^N)\big]^{\alpha/2} \subset \mathscr{L}^p_\alpha(\mathbb{R}^N). \tag{3.3}$$

Remark 3.3. Since $\mathscr{S}(\mathbb{R}^N)$ is dense in $\mathscr{L}^p_\alpha(\mathbb{R}^N)$, for every $\alpha \in \mathbb{R}$ and $1 \leq p \leq 2$, we obtain that $\alpha > m + \frac{N}{p'} \Longrightarrow \mathscr{L}^p_\alpha(\mathbb{R}^N) \subset C^m(\mathbb{R}^N)$. This embedding with (3.3) give us the inclusion

$$\big[L^p(\mathbb{R}^N)\big]^{\infty} \subset \bigcap_{\alpha>0} \mathscr{L}^p_\alpha(\mathbb{R}^N) \subset C^\infty(\mathbb{R}^N). \tag{3.4}$$

The framework L^p allows us to use the Fourier transform to obtain the regularity of the solutions $u \in C^\infty(\Omega; [L^p(\mathbb{R}^N)]^{-\infty})$ of the equation $\mathbb{L}u = f$ by means of adapted techniques from [2].

Next we recall Lemmas 4.4 and 4.5 from [2], which we will need to prove Theorem 3.6. Each result will be introduced with a slight modification of its statement, but its proof remains unchanged. These results are consequences of more general results due to Maire [8], Hardt [5], and Teissier [11] involving bonds for the length of curves joining points in analytic spaces.

Lemma 3.4 ([2, Lemma 4.4])**.** *Suppose that ϕ_0 is an analytic function. Let $t^* \in \mathcal{E}$, let B be an open ball contained in Ω such that $B \cap \mathcal{E}$ is* connected by piecewise smooth paths *(i.e., given two points in $B \cap \mathcal{E}$, there exists a piecewise smooth path in $B \cap \mathcal{E}$ which connects these two points), and take $t_0 \in B \cap \mathcal{E}$. Then there exist*

- (a) *an open neighborhood $B^* \subset B$ of t^*,*
- (b) *a constant $K > 0$ and $\varepsilon > 0$, and*
- (c) *a family $(\gamma_t)_{t\in B^*}$ of piecewise smooth paths $\gamma_t : [0,1] \longrightarrow B$ such that*
  - (I) *$\gamma_t(0) = t$, for every $t \in B^*$,*
  - (II) *$\phi_0(\gamma_t(s)) \leq \phi_0(t)$ for all $s \in [0,1]$ and all $t \in B^*$,*
  - (III) *the length $l(\gamma_t)$ of γ_t is such that $l(\gamma_t) \leq K$ for all $t \in B^*$,*
  - (IV) *if $t \in B^*$, then one of the following properties holds:*
    - $(\mathrm{IV})_1$ *$\gamma_t(1) = t_0$,*
    - $(\mathrm{IV})_2$ *$\phi_0(\gamma_t(1)) \leq \phi_0(t) - \varepsilon$.*

Lemma 3.5 ([2, Lemma 4.5])**.** *Let ϕ_0 be a real-valued, real-analytic function. Suppose that, for each $t^* \in \mathcal{E}$, ϕ_0 is an open function at t^* and that there exists an open ball B centered at t^*, with $B \subset \Omega$, such that $B \cap \mathcal{E}$ is* piecewise smooth connected. *Then there exist*

- (a) *an open neighborhood $B^* \subset B$ of t^*,*
- (b) *a constant $K > 0$ and $\varepsilon > 0$,*
- (c) *a family $(\gamma_t)_{t\in B^*}$ of piecewise smooth paths $\gamma_t : [0,1] \longrightarrow B$ such that*
  - (I) *$\gamma_t(0) = t$, for every $t \in B^*$,*
  - (II) *$\phi_0(\gamma_t(s)) \leq \phi_0(t)$ for all $s \in [0,1]$ and all $t \in B^*$,*
  - (III) *the length $l(\gamma_t)$ of γ_t is such that $l(\gamma_t) \leq K$ for all $t \in B^*$,*
  - (IV) *for each $t \in B^*$ we have $\phi_0(\gamma_t(1)) \leq \phi_0(t) - \varepsilon$.*

The modification we made in the statement of Lemma 4.4 in [2] was to assume that "$B\cap\mathcal{E}$ is *piecewise smooth connected*" instead of "$B\cap\mathcal{E}$ is *connected*," as the authors consider in [2]. We do that because our ϕ_0 does not need to be a constant equal to zero on $\mathcal{E}$, as they have there. The hypothesis "$B\cap\mathcal{E}$ is piecewise smooth connected" allows us to obtain that ϕ_0 is constant on $B\cap\mathcal{E}$. This modification does not change the proof introduced in [2] for this result.

We note that the hypothesis "$B\cap\mathcal{E}$ is piecewise smooth connected" is always satisfied whenever $\mathcal{E}$ is discrete because one can take B small enough so that $B\cap\mathcal{E}$ is a singleton. Now we can prove our main application.

Theorem 3.6. *Let $\phi_0 : \Omega \longrightarrow \mathbb{R}$ be a real-valued analytic function such that for every $t^*\in\mathcal{E}$ there exists an open ball B centered at t^*, with $B\subset\Omega$ and $B\cap\mathcal{E}$ piecewise smooth connected. Let $A = 1-\Delta : W^{2,p}(\mathbb{R}^N)\subset L^p(\mathbb{R}^N)\longrightarrow L^p(\mathbb{R}^N)$, $u\in C^\infty(\Omega;[L^p(\mathbb{R}^N)]^{-\infty})$ with $\mathbb{L}_0 u = f\in\Lambda^1 C^\infty(\Omega;[L^p(\mathbb{R}^N)]^{\infty})$, fix $t^*\in\mathcal{E}$, and suppose that one of the following properties holds:*

(i) *ϕ_0 is an open map at t^*.*
(ii) *there exists $t_0\in B\cap\mathcal{E}$ such that $u(t_0,\cdot)\in[L^p(\mathbb{R}^N)]^\infty$.*

Then, $u\in C^\infty(B'\times\mathbb{R}^N)$ for some neighborhood $B'\subset B$ of t^*.

Proof. In the first place, we can apply the Fourier transform on $\mathbb{R}^N$ to the equality $\mathbb{L}_0 u = f$ and obtain

$$d_t\hat{u}(t,\xi) + \omega_0(t)\wedge a(\xi)\hat{u}(t,\xi) = \hat{f}(t,\xi) \quad \text{for } t\in B, \xi\in\mathbb{R}^N, \tag{3.5}$$

where the "hat" stands for the Fourier transform in the variable x, and $a(\xi) = 1+4\pi^2|\xi|^2$ is the symbol of the operator $1-\Delta$.

One must observe that (3.5) makes sense for $\xi\in\mathbb{R}^N$, since $u(t,\cdot)\in\mathscr{L}^p_\alpha(\mathbb{R}^N)$, for some real α; therefore, $\hat{u}(t,\cdot)$ is a function. So by multiplying the equality (3.5) by $e^{a(\xi)\phi_0(t)}$, for $t\in\Omega$, we can write

$$d_t\big(e^{a(\xi)\phi_0(t)}\hat{u}(t,\xi)\big) = e^{a(\xi)\phi_0(t)}\hat{f}(t,\xi) \quad \text{for all } t\in B \text{ and all } \xi\in\mathbb{R}^N.$$

Also by Lemma 3.4 or 3.5, considering the family of paths $(\gamma_t)_{t\in B^*}$ and by integrating the last equality along γ_t, we get

$$\begin{aligned} &e^{a(\xi)\phi_0(\gamma_t(1))}\hat{u}\big(\gamma_t(1),\xi\big) - e^{a(\xi)\phi_0(t)}\hat{u}(t,\xi)\\ &\qquad = \int_{\gamma_t} d_t\big(e^{a(\xi)\phi_0(z)}\hat{u}(z,\xi)\big)\,dz = \int_{\gamma_t} e^{a(\xi)\phi_0(z)}\hat{f}(z,\xi)\,dz. \end{aligned}$$

So, for all $t\in B^*$ and $\xi\in\mathbb{R}^N$, it holds that

$$\hat{u}(t,\xi) = e^{a(\xi)[\phi_0(\gamma_t(1))-\phi_0(t)]}\hat{u}\big(\gamma_t(1),\xi\big) - \int_{\gamma_t} e^{a(\xi)[\phi_0(z)-\phi_0(t)]}\hat{f}(z,\xi)\,dz,$$

and hence

$$\big|\hat{u}(t,\xi)\big| \le e^{a(\xi)[\phi_0(\gamma_t(1))-\phi_0(t)]}\big|\hat{u}\big(\gamma_t(1),\xi\big)\big| + \Big|\int_{\gamma_t} e^{a(\xi)[\phi_0(z)-\phi_0(t)]}\hat{f}(z,\xi)\,dz\Big|. \tag{3.6}$$

At this point, we divide the proof into two cases.

Case 1. The hypothesis (ii) and conclusion (IV)$_1$ from Lemma 3.4 hold. Therefore, for every $\alpha \in \mathbb{R}$ we have

$$\left(1+|\xi|^2\right)^{\alpha/2}\hat{u}(t_0,\cdot) \in L^{p'}(\mathbb{R}^N). \tag{3.7}$$

Since $f \in \Lambda^1 C^\infty(\Omega;[L^p(\mathbb{R}^N)]^\infty)$, for every $\alpha \in \mathbb{R}$ and every $t \in \Omega$ we have

$$\left(1+|\xi|^2\right)^{\alpha/2}\hat{f}_j(t,\cdot) \in L^{p'}(\mathbb{R}^N).$$

Also, the map $\Omega \ni t \mapsto f_j(t,\cdot) \in [L^p(\mathbb{R}^N)]^\infty$ is C^∞ for all j, where $f = \sum_{j=1}^n f_j\, dt_j$.

Thus, using these facts and conclusion (III) of Lemma 3.5 in (3.6), we obtain that, for each real α, $\xi \in \mathbb{R}^N$ and $t \in B^*$,

$$\left(1+|\xi|^2\right)^{\alpha/2}\left|\hat{u}(t,\xi)\right| \le \left(1+|\xi|^2\right)^{\alpha/2}\left|\hat{u}(t_0,\xi)\right| + \left|\int_{\gamma_t}\left(1+|\xi|^2\right)^{\alpha/2}\hat{f}(z,\xi)\,dz\right|. \tag{3.8}$$

For the last integral, by Minkowski's inequality for integrals we have

$$\begin{aligned}&\left(\int_{\mathbb{R}^N}\left|\int_{\gamma_t}\left(1+|\xi|^2\right)^{\alpha/2}\hat{f}(z,\xi)\,dz\right|^{p'}d\xi\right)^{1/p'}\\ &\qquad\le \int_{\gamma_t}\left(\int_{\mathbb{R}^N}\left(1+|\xi|^2\right)^{\alpha p'/2}\left|\hat{f}(z,\xi)\right|^{p'}d\xi\right)^{1/p'}|\,dz| \le K\sup_{z\in B}\left\|f(z,\cdot)\right\|_{\mathscr{L}^p_\alpha} < \infty.\end{aligned}$$

This and (3.7) give us that $(1+|\xi|^2)^{\alpha/2}|\hat{u}(t,\cdot)| \in L^{p'}(\mathbb{R}^N)$ for all real α.

Case 2. The conclusion (IV)$_2$ of Lemma 3.4 or hypothesis (i) holds. In this case the estimate (3.6) gives us that, for each real α,

$$\begin{aligned}&\left(1+|\xi|^2\right)^{\alpha/2}\left|\hat{u}(t,\xi)\right|\\ &\qquad\le \left(1+|\xi|^2\right)^{\alpha/2}e^{-\varepsilon a(\xi)}\left|\hat{u}\big(\gamma_t(1),\xi\big)\right| + \left|\int_{\gamma_t}\left(1+|\xi|^2\right)^{\alpha/2}\hat{f}(z,\xi)\,dz\right|,\end{aligned} \tag{3.9}$$

from where we see that to take care of $|\int_{\gamma_t}(1+|\xi|^2)^{\alpha/2}\hat{f}(z,\xi)\,dz|$, we can use the same method we used in case 1.

Now, since for some real γ, $(1+|\xi|^2)^\gamma\hat{u}(\gamma_t(1),\cdot) \in L^{p'}(\mathbb{R}^N)$, we have that for every real α, $(1+|\xi|^2)^{\alpha/2}e^{-\varepsilon a(\xi)}\hat{u}(\gamma_t(1),\cdot) \in L^{p'}(\mathbb{R}^N)$. Hence, for all $t \in B^*$, $\xi \in \mathbb{R}^N$, and $\alpha \in \mathbb{R}$,

$$\left(1+|\xi|^2\right)^{\alpha/2}\hat{u}(t,\cdot) \in L^{p'}(\mathbb{R}^N),$$

completing the proof of this case. From these two cases we conclude that for every $\alpha \in \mathbb{R}$, there exists a constant $K_\alpha > 0$ such that

$$\sup_{t\in B^*}\left\|u(t,\cdot)\right\|_{\mathscr{L}^p_\alpha} \le K_\alpha.$$

Finally take $\psi \in C_c^\infty(B^*;\mathbb{R})$ with $\psi \equiv 1$ on some neighborhood $B' \subset B$ of t^*. Differentiating with respect to t_k the equation

$$\mathbb{L}_{j,0}(\psi u) = \frac{\partial\psi}{\partial t_j}u + \psi f_j,$$

we get

$$\begin{aligned}\frac{\partial}{\partial t_k}\Big(\frac{\partial(\psi u)}{\partial t_j}\Big)(t) &+ \frac{\partial \phi_0}{\partial t_k}(t)A\Big(\frac{\partial(\psi u)}{\partial t_j}(t)\Big) \\ &= \frac{\partial}{\partial t_k}\Big(\frac{\partial \psi}{\partial t_j}u + \psi f_j\Big)(t) - \psi(t)\frac{\partial^2 \phi_0}{\partial t_k \partial t_j}(t)Au(t).\end{aligned}$$

So replacing u by $\frac{\partial(\psi u)}{\partial t_j}$ and f_j by $\frac{\partial}{\partial t_k}(\frac{\partial\psi}{\partial t_j}u + \psi f_j)(t) - \psi(t)\frac{\partial^2\phi_0}{\partial t_k\partial t_j}(t)Au(t)$, we can repeat the same procedure as above to conclude that for every $\alpha \in \mathbb{R}$ and $j = 1, 2, \ldots, n$, there exists a constant $K_{\alpha,j} > 0$ such that

$$\sup_{t\in B'}\Big\|\frac{\partial u}{\partial t_j}(t,\cdot)\Big\|_{\mathscr{L}^p_\alpha} \le K_{\alpha,j}.$$

Observe that $\frac{\partial u}{\partial t_k} = f_k - \frac{\partial \phi_0}{\partial t_k}(t)Au$, which helps us to obtain the estimates we need to replace $\sup_{z\in B}\|f(z,\cdot)\|_{\mathscr{L}^p_\alpha}$, as we did in case 1. Thus proceeding by induction, we will obtain that, for every $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}^n_+$, there exists a constant $K_{\alpha,\beta} > 0$ such that

$$\sup_{t\in B'}\big\|\partial_t^\beta u(t,\cdot)\big\|_{\mathscr{L}^p_\alpha} \le K_{\alpha,\beta},$$

from where we can easily get that $u \in C^\infty(B'; [L^p(\mathbb{R}^N)]^\infty) \subset C^\infty(B' \times \mathbb{R}^N)$. □

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