Ann. Funct. Anal. 10 (2019), no. 2, 180–195
https://doi.org/10.1215/20088752-2018-0017
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

G-FRAMES AND THEIR GENERALIZED MULTIPLIERS IN HILBERT SPACES

HESSAM HOSSEINNEZHAD,[1] GHOLAMREZA ABBASPOUR TABADKAN,[1*] and ASGHAR RAHIMI[2]

Communicated by K. F. Taylor

Abstract. In this article, we introduce the concept of generalized multipliers for g-frames. In fact, we show that every generalized multiplier for g-Bessel sequences is a generalized multiplier for the induced sequences, in a special sense. We provide some sufficient and/or necessary conditions for the invertibility of generalized multipliers. In particular, we characterize g-Riesz bases by invertible multipliers. We look at which perturbations of g-Bessel sequences preserve the invertibility of generalized multipliers. Finally, we investigate how to find a matrix representation of operators on a Hilbert space using g-frames, and then we characterize g-Riesz bases and g-orthonormal bases by applying such matrices.

1. Introduction

Frames in Hilbert spaces were originally introduced by Duffin and Schaeffer [12] to deal with some problems in nonharmonic Fourier analysis. These frames were later brought to life by Daubechies, Grossmann, and Meyer [10], after which they became popular subjects of research.

Let I be a subset of integers $\mathbb{Z}$. A *frame* for a separable Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i\in I}$ in $\mathcal{H}$ so that there are two positive constants A and B

Received Apr. 10, 2018; Accepted Jul. 17, 2018.
First published online Jan. 22, 2019.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 42C15; Secondary 47A05, 41A58.
Keywords. g-Bessel sequences, g-frames, g-Riesz bases, generalized multipliers.

satisfying

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in I} \left|\langle f, f_i\rangle\right|^2 \leq B\|f\|_{\mathcal{H}}^2, \quad (f \in \mathcal{H}). \tag{1.1}$$

The numbers A, B in (1.1) are called *frame bounds*.

Frames have been used as a powerful alternative to Hilbert space bases, and they allow for the development of a deep theory (for an overview, see [7]). They are also very important for applications in fields such as physics (see [9]), signal processing (see [5]), and acoustics (see [6]). Over the years, various extensions of frame theory have been investigated. Several of these are contained as special cases of the elegant theory for g-frames that was introduced by Sun [24]. Examples of these special cases include fusion frames, bounded quasiprojectors, outer frames, oblique frames, pseudoframes, and a class of time-frequency localization operators. G-frames and g-Riesz bases in complex Hilbert spaces have some properties similar to those of frames and Riesz bases, but not all the properties are similar. For example, exact g-frames are not equivalent g-Riesz bases (see [24], [25]).

The concept of Bessel multipliers in Hilbert spaces was introduced by Balazs [2] and later extended by the third author [22] to g-Bessel sequences. Bessel multipliers are operators that are defined by a fixed multiplication pattern which is inserted between the analysis and synthesis operators. This class of operators is not only of interest for applications in modern life, for example, in acoustics (see [26]), psychoacoustics (see [6]), and denoising (see [17]), but it is also important in different branches of functional analysis. In this respect, it is important to find the inverse of a multiplier if it exists.

The standard matrix description of operators on Hilbert spaces, using an orthonormal basis, was presented in [8]. This idea was developed for Bessel sequences, frames, and Riesz sequences by Balazs [3]. In this setting, multipliers are those operators that can be represented with diagonal matrices. Hence, it is a very natural idea to extend this representation to include more side-diagonals. Using this approach, a generalization of Bessel multipliers is obtained. For Gabor frames, this is a particular case of the "generalized Gabor multipliers" as found in [11]. Moreover, it is interesting to note that every bounded operator on a Hilbert space is a generalized frame multiplier (see [4]). The authors [1] recently used operator theory tools to obtain some detailed properties of generalized multipliers.

Motivated by the generalized multipliers for Bessel sequences, our aim here will be to extend generalized multipliers to more general cases, that is, for g-Bessel sequences. This article is organized as follows. In Sections 2 and 3, we briefly review some basic notation and preliminaries needed for the developments in the following sections. In Section 4, we define the concept of generalized multipliers for g-Bessel sequences, and then we investigate a number of properties with a special emphasis on the invertibility of generalized multipliers. Finally, in Section 5, we focus on studying the matrix representation of operators using g-frames.

2. Notation and preliminaries

Throughout this article, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\{\mathcal{H}_i : i \in I\}$ is a sequence of Hilbert spaces. Moreover, $\mathcal{B}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_i$. Note that for any sequence $\{\mathcal{H}_i : i \in I\}$, we can assume that there exists a Hilbert space $\mathcal{K}$ such that, for all $i \in I$, $\mathcal{H}_i \subseteq \mathcal{K}$ (e.g., $\mathcal{K} = \bigoplus_{i\in I} \mathcal{H}_i$). So throughout the article, we understand the inner product $\langle g_i, g_j\rangle$ for $g_i \in \mathcal{H}_i$ and $g_j \in \mathcal{H}_j$, as the inner product in the bigger Hilbert space $\mathcal{K}$.

Definition 2.1 ([24, Definition 1.1]). A sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a *generalized frame*, or simply a *g-frame*, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if there exist constants A and B, $0 < A \leq B < \infty$, such that

$$A\|f\|^2_{\mathcal{H}} \leq \sum_{i\in I} \|\Lambda_i f\|^2_{\mathcal{H}_i} \leq B\|f\|^2_{\mathcal{H}}, \quad (f \in \mathcal{H}). \tag{2.1}$$

The numbers A and B are called *g-frame bounds*.

The family $\{\Lambda_i : i \in I\}$ is called a *tight g-frame* if $A = B$ and a *normalized tight frame* if $A = B = 1$. If in (2.1) only the second inequality holds, then the sequence is called a *g-Bessel sequence*.

Definition 2.2 ([24, Definition 3.1]). A sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called *g-complete* if $\{f : \Lambda_i f = 0,\ i \in I\} = \{0\}$, and it is called a *g-orthonormal basis* for $\mathcal{H}$ if it satisfies

(1) $\langle \Lambda_i^* g_i, \Lambda_j^* g_j\rangle = \delta_{i,j}\langle g_i, g_j\rangle$, $(i, j \in I, g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j)$,
(2) $\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2$, $(f \in \mathcal{H})$.

A sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is said to be a *g-Riesz basis* if it is g-complete and there are positive constants A and B such that, for any finite subset $J \subset I$ and $f_i \in \mathcal{H}_i, i \in J$,

$$A\sum_{i\in J} \|f_i\|^2_{\mathcal{H}_i} \leq \Big\|\sum_{i\in J} \Lambda_i^* f_i\Big\|^2_{\mathcal{H}} \leq B\sum_{i\in J} \|f_i\|^2_{\mathcal{H}_i}.$$

For each sequence $\{\mathcal{H}_i : i \in I\}$, we define the space

$$\Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)_{\ell^2(I)} = \Big\{(f_i)_{i\in I} : f_i \in \mathcal{H}_i, i \in I, \text{ and} \sum_{i\in I} \|f_i\|^2_{\mathcal{H}_i} < \infty\Big\}$$

with the inner product defined by $\langle (f_i)_{i\in I}, (g_i)_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$. It is clear that $(\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)}$ is a Hilbert space.

Remark 2.3. Suppose that for each $i \in I$, $\{e^i_k\}_{k\in K_i}$ is an orthonormal basis for $\mathcal{H}_i$. For each $i \in I, k \in K_i$, we define $E^i_k = (\delta_{i,j} e^i_k)_{j\in I}$, where $\delta_{i,j}$ is the Kronecker delta. It is clear that $\{E^i_k\}_{i\in I,k\in K_i}$ is an orthonormal basis for $(\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)}$ and that, for each $(f_j)_{j\in I} \in (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)}$, we have

$$\big\langle (f_j)_{j\in I}, E^i_k\big\rangle = \langle f_i, e^i_k\rangle.$$

We define the *synthesis operator* for the g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ as follows:

$$T_\Lambda : \Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)_{\ell^2(I)} \to \mathcal{H}, \qquad T_\Lambda(f_i)_{i\in I} = \sum_{i\in I} \Lambda_i^* f_i.$$

The series converges unconditionally in the norm of $\mathcal{H}$. It is easy to show that the adjoint operator of T_Λ is

$$T_\Lambda^*: \mathcal{H} \to \Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)_{\ell^2(I)}, \qquad T_\Lambda^* f = (\Lambda_i f)_{i\in I}.$$

The operator T_Λ^* is called the *analysis operator* for $\{\Lambda_i\}_{i\in I}$. Composing T_Λ and T_Λ^*, the *g-frame operator* is obtained as follows:

$$S_\Lambda = T_\Lambda T_\Lambda^* : \mathcal{H} \to \mathcal{H}, \qquad S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f.$$

If $\Lambda = \{\Lambda_i : i \in I\}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ with bounds A and B, then the g-frame operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$ is a bounded, positive, and invertible operator.

The *canonical dual g-frame* of $\Lambda = \{\Lambda_i : i \in I\}$ is denoted by $\tilde{\Lambda} = \{\tilde{\Lambda}_i : i \in I\}$, where for each $i \in I$, $\tilde{\Lambda}_i := \Lambda_i S_\Lambda^{-1}$, which is also a g-frame for $\mathcal{H}$ with frame bounds B^{-1} and A^{-1}. Moreover, we have the *reconstruction formula* as

$$f = \sum_{i\in I} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i\in I} \tilde{\Lambda}_i^* \Lambda_i f, \quad (f \in \mathcal{H}).$$

Sometimes the reconstruction formula of g-frames is valid with other g-frames $\Theta = \{\Theta_i\}_{i\in I}$ instead of $\tilde{\Lambda} = \{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$. They are called *(alternative) dual g-frames* of $\Lambda = \{\Lambda_i\}_{i\in I}$. By using the analysis and synthesis operators of Λ and Θ, they are a dual pair if $T_\Lambda T_\Theta^* = I$ or $T_\Theta T_\Lambda^* = I$.

Let $\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)$. Suppose that for each $i \in I$, $\{e_k^i\}_{k\in K_i}$ is an orthonormal basis for $\mathcal{H}_i$, where K_i is a subset of $\mathbb{Z}$. Then

$$f \mapsto \langle \Lambda_i f, e_k^i\rangle$$

defines a bounded linear functional on $\mathcal{H}$. Consequently, for each $i \in I$ and $k \in K_i$, we can find $u_k^i \in \mathcal{H}$ such that for each $f \in \mathcal{H}$, $\langle f, u_k^i\rangle = \langle \Lambda_i f, e_k^i\rangle$. Hence

$$\Lambda_i f = \sum_{k\in K_i} \langle f, u_k^i\rangle e_k^i, \quad (f \in \mathcal{H}), \tag{2.2}$$

and

$$\Lambda_i^* g_i = \sum_{k\in K_i} \langle g_i, e_k^i\rangle u_k^i, \quad (i \in I, g_i \in \mathcal{H}_i). \tag{2.3}$$

In particular,

$$u_k^i = \Lambda_i^* e_k^i, \quad (i \in I, k \in K_i). \tag{2.4}$$

We call $\{u_k^i : i \in I, k \in K_i\}$ the *sequence induced by* $\{\Lambda_i : i \in I\}$ with respect to $\{e_k^i : i \in I, k \in K_i\}$.

In order to present the main results of this article, we need the following theorem that describes the relationship between the sequence $\{\Lambda_i : i \in I\}$ and its induced sequence $\{u_k^i : i \in I, k \in K_i\}$.

Theorem 2.4 ([24, Theorem 3.1])**.** *Let $\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)$, and let u_k^i be defined as in (2.4). Then we have the following:*

(1) $\{\Lambda_i : i \in I\}$ *is a g-frame (resp., g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for* $\mathcal{H}$ *if and only if* $\{u_k^i : i \in I, k \in K_i\}$ *is a frame (resp., Bessel sequence, tight frame, Riesz basis, orthonormal basis) for* $\mathcal{H}$*;*
(2) *the g-frame operator of* $\{\Lambda_i : i \in I\}$ *coincides with the frame operator for* $\{u_k^i : i \in I, k \in K_i\}$*;*
(3) $\{\Lambda_i : i \in I\}$ *and* $\{\tilde{\Lambda}_i : i \in I\}$ *are a pair of (canonical) dual g-frames if and only if the induced sequences are a pair of (canonical) dual frames.*

The following proposition gives a criterion for the invertibility of operators.

Proposition 2.5 ([14, Corollary 8.2]). *Let* $F : \mathcal{H} \to \mathcal{H}$ *be invertible on* $\mathcal{H}$*. Suppose that* $G : \mathcal{H} \to \mathcal{H}$ *is a bounded operator and that* $\|Gh - Fh\| \leq \upsilon \|h\|$*, for all* $h \in \mathcal{H}$*, where* $\upsilon \in [0, \frac{1}{\|F^{-1}\|})$*. Then* G *is invertible on* $\mathcal{H}$ *and* $G^{-1} = \sum_{k=0}^{\infty} [F^{-1}(F - G)]^k F^{-1}$*.*

3. Bessel multipliers and g-Bessel multipliers

Bessel multipliers in Hilbert spaces were introduced by Balazs in [2]. Given two Bessel sequences $G = \{g_i\}_{i\in I}$ and $F = \{f_i\}_{i\in I}$ in $\mathcal{H}$ and the weight sequence $m = \{m_i\}_{i\in I}$, the Bessel multiplier for these sequences is an operator on $\mathcal{H}$ defined by

$$\mathbf{M}_{m,G,F}(f) := \sum_{i\in I} m_i \langle f, f_i \rangle g_i, \quad (f \in \mathcal{H}). \tag{3.1}$$

Several basic properties of these operators were investigated in [2]. It should be mentioned that multipliers are not only interesting from a theoretical point of view, but they are also used in applications, particularly in the fields of audio and acoustics (see [26]). The concept of multipliers has been extended in several directions (for instance, multipliers for g-Bessel sequences were introduced by the third author in [22]). Moreover, X_d-Bessel sequences in Banach spaces and Bessel sequences in Hilbert modules have been studied in [13] and [16].

Let $\{\mathcal{H}_i : i \in I\}$ be a family of Hilbert spaces. Let $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ and $\Theta = \{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ be g-Bessel sequences for $\mathcal{H}$ with bounds B_Λ and B_Θ. For a weight $m = \{m_i\}_{i\in I} \in \ell^\infty(I)$, the operator

$$\mathbf{M}_{m,\Lambda,\Theta}(f) := \sum_{i\in I} m_i \Lambda_i^* \Theta_i f, \quad (f \in \mathcal{H}), \tag{3.2}$$

is called the *g-Bessel multiplier* of Λ, Θ with symbol m. In fact, a g-Bessel multiplier $\mathbf{M}_{m,\Lambda,\Theta}$ can be written as

$$\mathbf{M}_{m,\Lambda,\Theta} = T_\Lambda D_m T_\Theta^*, \tag{3.3}$$

where D_m is the diagonal operator

$$D_m : \Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)_{\ell^2(I)} \to \Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)_{\ell^2(I)}, \qquad D_m(\xi_i)_{i\in I} = (m_i \xi_i)_{i\in I}.$$

As was mentioned, multipliers are defined by a fixed multiplication pattern (the symbol) which is inserted between the analysis and synthesis operators. For a theoretical approach, it is very natural to extend this notion by replacing an arbitrary operator on the sequence space by the fixed multiplication operator and consider such operators in more general settings. (This idea was considered by Balazs in [3] and was investigated in more detail in [1].)

Definition 3.1. Let $G = \{g_i\}_{i\in I}$ and $F = \{f_i\}_{i\in I}$ be Bessel sequences in $\mathcal{H}$, and let U be an infinite matrix defining a bounded linear operator on $\ell^2(I)$, $(Uc)_i = \sum_k u_{i,k}c_k$. Then the operator $\mathbf{M}_{U,G,F} : \mathcal{H} \to \mathcal{H}$, defined by

$$\mathbf{M}_{U,G,F}(f) = T_G U T_F^*(f) := \sum_{k\in I}\sum_{j\in I} u_{k,j}\langle f, f_j\rangle g_k, \quad (f \in \mathcal{H}), \tag{3.4}$$

is considered the generalized multiplier for the Bessel sequences F and G with symbol U.

The interested reader can find the properties of this operator in [1] and [3]. In the next section, we introduce the concept of generalized multipliers for g-Bessel sequences and investigate some properties of such operators.

4. Generalized multipliers associated with G-Bessel sequences

It is well known (see [14]) that for given bounded linear operator U on a separable Hilbert space $\mathcal{H}$ and an orthonormal basis $\{e_i\}_{i\in I}$ for $\mathcal{H}$, the matrix that arises from U and the orthonormal basis is denoted by $U_{i,j} = [u_{ij}]$, where $u_{ij} = \langle Ue_j, e_i\rangle$. Moreover, the operator induced by the matrix $U_{i,j}$ on $\ell^2(I)$ is defined by

$$U' : \ell^2(I) \to \ell^2(I), \quad \big((U'c)_i\big)_{i\in I} := \Big(\sum_{k\in I}\langle Ue_k, e_i\rangle c_k\Big)_{i\in I}, \quad \big(c \in \ell^2(I)\big).$$

It is clear that $\|U\|_{\mathrm{op}} = \|U'\|_{\mathrm{op}}$. Indeed for each $c \in \ell^2(I)$,

$$\|U'c\|^2_{\ell^2(I)} = \sum_{i\in I}\Big|\sum_{k\in I}\langle Ue_k, e_i\rangle c_k\Big|^2 = \big\|T^*_{\{e_i\}}UT_{\{e_i\}}(c)\big\|^2_{\ell^2(I)} \le \|U\|^2_{\mathrm{op}}\|c\|^2_{\ell^2(I)}.$$

So $\|U'\|_{\mathrm{op}} \le \|U\|_{\mathrm{op}}$. On the other hand,

$$\|Uf\|_{\mathcal{H}} = \Big\|\sum_{i\in I}\Big\langle U\Big(\sum_{k\in I}\langle f, e_k\rangle e_k\Big), e_i\Big\rangle e_i\Big\|_{\mathcal{H}} = \|T_{\{e_i\}}U'T^*_{\{e_i\}}f\|_{\mathcal{H}} \le \|U'\|_{\mathrm{op}}\|f\|_{\mathcal{H}},$$

and hence $\|U\|_{\mathrm{op}} \le \|U'\|_{\mathrm{op}}$. Furthermore, U is invertible if and only if U' is invertible.

Now, suppose that $U : (\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)} \to (\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}$ is a bounded linear operator and $f = (f_i)_{i\in I} \in (\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}$. If for each $j \in I$, we define the operators V_j and W_j as

$$V_j : \Big(\bigoplus_{i\in I}\mathcal{H}_i\Big)_{\ell^2(I)} \to \mathcal{H}_j, \qquad V_j(f_i)_{i\in I} = f_j,$$

$$W_j : \mathcal{H}_j \to \Big(\bigoplus_{i\in I}\mathcal{H}_i\Big)_{\ell^2(I)}, \qquad W_jf = (0,\ldots,0,\underbrace{f}_{j\text{th}},0,\ldots),$$

then

$$(Uf)_i = V_iUf = V_iU\Big(\sum_{j\in I} W_jV_jf\Big) = \sum_{j\in I} V_iUW_jf_j = \sum_{j\in I} u_{ij}f_j, \tag{4.1}$$

where $u_{ij} = V_iUW_j \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$. So, we can consider the (infinite) matrix arising from U as

$$U_{i,j} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1j} & \dots \\ u_{21} & u_{22} & \dots & u_{2j} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.2}$$

On the other hand, since $u_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, $i, j \in I$, it can also induce a matrix as $[(u_{ij})_{k'k}] = \langle u_{ij}e^j_k, e^i_{k'}\rangle$, where for each $i \in I$, $\{e^i_k\}_{k\in K_i}$ is an orthonormal basis for $\mathcal{H}_i$. Therefore, $U_{i,j}$ looks like a block matrix of matrices. Hence, the operator $U' : (\bigoplus_j \ell^2(K_j))_{\ell^2(I)} \to (\bigoplus_j \ell^2(K_j))_{\ell^2(I)}$ induced by the matrix (4.2) is obtained as

$$\Big(\big(U'(c^j_k)_{j\in I,k\in K_j}\big)^i_{k'}\Big)_{i\in I,k'\in K_i} := \Big(\sum_{j\in I}\sum_{k\in K_j} \langle u_{ij}e^j_k, e^i_{k'}\rangle c^j_k\Big)_{i\in I,k'\in K_i}, \tag{4.3}$$

for every $(c^j_k)_{j\in I,k\in K_j} \in (\bigoplus_j \ell^2(K_j))_{\ell^2(I)}$. Furthermore, we have $\|U\|_{\text{op}} = \|U'\|_{\text{op}}$.

We now extend the notion of generalized multipliers to g-frames. In the remainder of this article, special attention is devoted to the study of invertible generalized multipliers.

Definition 4.1. Let $\{\mathcal{H}_i : i \in I\}$ be a family of Hilbert spaces. Let $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ and $\Theta = \{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ be g-Bessel sequences for $\mathcal{H}$ with bounds B_Λ and B_Θ. Moreover, we should let $U : (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)} \to (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)}$ be a bounded linear operator. Then the operator $\mathbf{M}_{U,\Lambda,\Theta} : \mathcal{H} \to \mathcal{H}$, defined as

$$\mathbf{M}_{U,\Lambda,\Theta} = T_\Lambda U T^*_\Theta, \tag{4.4}$$

is considered the generalized multiplier of g-Bessel sequences Λ and Θ with symbol U.

Clearly, $\mathbf{M}_{U,\Lambda,\Theta}$ is well defined, and $\|\mathbf{M}_{U,\Lambda,\Theta}\|_{\text{op}} \le \sqrt{B_\Lambda B_\Theta}\|U\|_{\text{op}}$.

As is mentioned above, every operator $U : (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)} \to (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)}$ has the matrix description defined by $U = [u_{ij}]$, where $u_{ij} = V_iUW_j$. Hence, by using (4.1), the generalized Bessel multiplier can be shown in the form

$$\mathbf{M}_{U,\Lambda,\Theta}(f) = T_\Lambda U T^*_\Theta(f) = \sum_{i\in I}\sum_{j\in I} \Lambda^*_i u_{ij}\Theta_j(f). \tag{4.5}$$

We will use both (4.4) and (4.5) in this article.

Example 4.2. Let Λ and Θ be two g-frames for $\mathcal{H}$. Then, as is proved in [18, Proposition 3.5], there exists a bounded linear operator $U : (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)} \to (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)}$ such that for any $f \in \mathcal{H}$, $T^*_\Theta(f) = UT^*_\Lambda(f)$. Hence, the frame operator

$$S_\Theta = T_\Theta T^*_\Theta = T_\Theta U T^*_\Lambda = \mathbf{M}_{U,\Theta,\Lambda}$$

is a generalized frame multiplier of Λ and Θ with symbol U.

Remark 4.3. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For a compact operator $U \in \mathcal{B}(\mathcal{H})$, let $\lambda_1(U), \lambda_2(U), \dots,$ denote the singular values of U, that is, the eigenvalues of the positive operator $|U| = (U^*U)^{1/2}$. For $1 \leq p < \infty$, the *Schatten p-class* $\mathcal{S}_p(\mathcal{H})$ is denoted by the set of all compact operators U for which $\sum_i \lambda_i^p(U) < \infty$. For $U \in \mathcal{S}_p(\mathcal{H})$, the Schatten p-norm of U is denoted by

$$\|U\|_p = \Big(\sum_i \lambda_i^p(U)\Big)^{1/p}.$$

It is worth mentioning that the concept of Schatten p-class can be defined for operators between different Hilbert spaces in a similar way (see [21]). As a consequence of [21, Theorem 15.5.7], it is easy to verify that, for Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ and operators $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), V \in \mathcal{B}(\mathcal{H}_4, \mathcal{H}_3)$, and $W \in \mathcal{S}_p(\mathcal{H}_3, \mathcal{H}_1)$, we have $UW \in \mathcal{S}_p(\mathcal{H}_3, \mathcal{H}_2)$ and $WV \in \mathcal{S}_p(\mathcal{H}_4, \mathcal{H}_1)$. Moreover, $\|UW\|_p \leq \|W\|_p \|U\|_{\text{op}}$ and $\|WV\|_p \leq \|W\|_p \|V\|_{\text{op}}$. (We refer the reader to [21] for more detailed information about these operators.) So, we conclude that if $U \in \mathcal{S}_p((\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)})$, for some $1 \leq p < \infty$, then by (4.4), the generalized multiplier $\mathbf{M}_{U,\Lambda,\Theta}$ also belongs to the same class of operators.

In [22, Theorem 4.7], the third author proved that for $m \in \ell^p(I)$, the multiplier $\mathbf{M}_m$ belongs to $\mathcal{S}_p(\mathcal{H})$ if $(\dim \mathcal{H}_i)_{i\in I} \in \ell^\infty(I)$. The following example shows that for some $m = (m_i)_{i\in I} \in (\bigoplus_{i\in I} \ell^p(K_i))_{\ell^p(I)}$, the generalized multiplier $\mathbf{M}_m \in \mathcal{S}_p(\mathcal{H})$ while $(\dim \mathcal{H}_i)_{i\in I} \notin \ell^\infty(I)$, where for each $1 \leq p \leq \infty$ the space $(\bigoplus_{j\in I} \ell^p(K_j))_{\ell^p(I)}$ is defined as follows:

$$\Big(\bigoplus_{j\in I} \ell^p(K_j)\Big)_{\ell^p(I)} = \big\{(m_j)_{j\in I} : m_j \in \ell^p(K_j) \text{ and } \big\{\|m_j\|_p\big\}_{j\in I} \in \ell^p(I)\big\}. \tag{4.6}$$

Example 4.4. Suppose that $\{\mathcal{H}_i : i \in I\}$ is a sequence of finite-dimensional Hilbert spaces and that, for each $i \in I$, $\{e_k^i\}_{k\in K_i}$ is an orthonormal basis for $\mathcal{H}_i$. Furthermore, let $\dim(\mathcal{H}_i) = |K_i|$ (cardinality of the set K_i). Consider the diagonal operator $U_m : (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)} \to (\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)}$ which is defined by

$$U_m E_k^i = c_k^i E_k^i, \quad (i \in I, k \in K_i),$$

where $E_k^i = (\delta_{i,j} e_k^i)_{j\in I}$ is an orthonormal basis for $(\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)}$ and $m = (m_i)_{i\in I} = (c_k^i) \in (\bigoplus_{i\in I} \ell^\infty(K_i))_{\ell^\infty(I)}$. Then, for two g-Bessel sequences $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Theta = \{\Theta_i\}_{i\in I}$, we can define the generalized multiplier $\mathbf{M}_{U_m,\Lambda,\Theta} = T_\Lambda U_m T_\Theta^*$, which is a more general form of (3.3).

Now, we are going to show that if $(m_i)_{i\in I} \in (\bigoplus_{i\in I} \ell^p(K_i))_{\ell^p(I)}$, for some $1 \leq p < \infty$, then $U_m \in \mathcal{S}_p((\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)})$. Indeed, let $\{\hat{E}_j\}_{j\in I}$ be the rearrangement of $\{E_k^i : i \in I, k \in K_i\}$ given by

$$\begin{cases} \hat{E}_j = E_j^1, & 1 \leq j \leq |K_1|, \\ \hat{E}_j = E_k^{n+1}, & j > |K_1|, \end{cases}$$

where $n = \max\{m \in I : j > |K_1| + \cdots + |K_m|\}$ and $k = j - (|K_1| + \cdots + |K_n|)$. Therefore, the sequence $\{\hat{E}_j\}_{j\in I}$ has the form

$$\{\hat{E}_j\}_{j\in I} = \{E_1^1, \ldots, E_{|K_1|}^1, E_1^2, \ldots, E_{|K_2|}^2, \ldots, E_1^i, \ldots, E_{|K_i|}^i, \ldots\}.$$

Thus, for every

$$f = (f_i)_{i\in I} \in \Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)_{\ell^2(I)} \quad \text{and} \quad m = (m_i)_{i\in I} \in \Big(\bigoplus_{i\in I} \ell^p(K_i)\Big)_{\ell^p(I)},$$

we have

$$U_m f = \sum_{i\in I}\sum_{k\in K_i} \langle f, E_k^i\rangle U(E_k^i) = \sum_{j\in I} \hat{m}_j \langle f, \hat{E}_j\rangle \hat{E}_j,$$

where

$$(\hat{m}_j)_{j\in I} = (c_1^1, \ldots, c_{|K_1|}^1, c_1^2, \ldots, c_{|K_2|}^2, \ldots, c_1^i, \ldots, c_{|K_i|}^i, \ldots).$$

Moreover,

$$\Big(\sum_{j\in I} |\hat{m}_j|^p\Big)^{1/p} = \Big(\sum_{i\in I} \|m_i\|_p^p\Big)^{1/p} < \infty,$$

which implies that $U_m \in \mathcal{S}_p((\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)})$ and so, by Remark 4.3, the multiplier $\mathbf{M}_{U_m,\Lambda,\Theta} \in \mathcal{S}_p(\mathcal{H})$. Now, let $\{\mathcal{H}_i : i \in I\}$ be a sequence of Hilbert spaces with $\dim(\mathcal{H}_i) = 2^{i-1}$, $i = 1, 2, \ldots$ and $m_1 = \{1\}$, $m_2 = \{\frac{1}{2}, \frac{1}{3}\}$, $m_3 = \{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\}, \ldots$. Then $\{\hat{m}_j\}_{j\in I} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \in \ell^p(I)$, $1 < p < \infty$, and so $\mathbf{M}_{U_m} \in \mathcal{S}_p(\mathcal{H})$, while $(\dim \mathcal{H}_i)_{i\in I} \notin \ell^\infty(I)$.

The following lemma gives the connection between generalized multipliers of g-Bessel sequences and generalized multipliers of Bessel sequences.

Lemma 4.5. *Every generalized multiplier of g-Bessel sequences is a generalized multiplier of the induced Bessel sequences.*

Proof. Let $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ and $\Theta = \{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ be g-Bessel sequences for $\mathcal{H}$ with induced sequences $\{\lambda_k^i : i \in I, k \in K_i\}$ and $\{\theta_k^i : i \in I, k \in K_i\}$, respectively. Then by using (4.5), (2.2), and (2.3), we get

$$\begin{aligned}
\mathbf{M}_{U,\Lambda,\Theta}(f) &= \sum_{i\in I}\sum_{j\in I} \Lambda_i^* u_{ij} \Theta_j(f) \\
&= \sum_{i\in I} \Lambda_i^* \Big(\sum_{j\in I}\sum_{k\in K_j} \langle f, \theta_k^j\rangle u_{ij} e_k^j\Big) \\
&= \sum_{i\in I}\sum_{k'\in K_i} \Big\langle \sum_{j\in I}\sum_{k\in K_j} \langle f, \theta_k^j\rangle u_{ij} e_k^j, e_{k'}^i \Big\rangle \lambda_{k'}^i \\
&= \sum_{i\in I}\sum_{j\in I}\sum_{k'\in K_i}\sum_{k\in K_j} \langle f, \theta_k^j\rangle \langle u_{ij} e_k^j, e_{k'}^i\rangle \lambda_{k'}^i \\
&= \mathbf{M}_{U',G,F}(f),
\end{aligned}$$

where $G = \{\lambda_k^i : i \in I, k \in K_i\}$, $F = \{\theta_k^i : i \in I, k \in K_i\}$, and $U' : (\bigoplus_j \ell^2(K_j))_{\ell^2(I)} \to (\bigoplus_j \ell^2(K_j))_{\ell^2(I)}$ is defined as (4.3). $\square$

The next proposition determines equivalent conditions under which the generalized multiplier is invertible.

Proposition 4.6. *Let* $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ *and* $\Theta = \{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ *be two g-frames for* $\mathcal{H}$*, and let* $U \in \mathcal{B}((\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)})$*. Then the following assertions are equivalent:*

(1) $\mathbf{M}_{U,\Lambda,\Theta}$ *is invertible if and only if* U *is invertible,*
(2) *both* Λ *and* Θ *are g-Riesz bases.*

Proof. Suppose that Λ and Θ are g-frames for $\mathcal{H}$ with induced sequences $\{\lambda_k^i : i \in I, k \in K_i\}$ and $\{\theta_k^i : i \in I, k \in K_i\}$, respectively.

To derive the first statement from the second one, we assume that Λ and Θ are two g-Riesz bases for $\mathcal{H}$. Then by Theorem 2.4, $\{\lambda_k^i : i \in I, k \in K_i\}$ and $\{\theta_k^i : i \in I, k \in K_i\}$ are Riesz bases for $\mathcal{H}$. If the generalized multiplier $\mathbf{M}_{U,\Lambda,\Theta}$ is invertible, then by Lemma 4.5, the multiplier $\mathbf{M}_{U',G,F}$ is also invertible. Since the sequences $G = \{\lambda_k^i\}_{i\in I, k\in K_i}$ and $F = \{\theta_k^i\}_{i\in I,k\in K_i}$ are Riesz bases, then by [1, Proposition 4.1], U' is invertible and so is U. Conversely, if U is invertible, then U' is invertible. Due to the fact that F and G are Riesz bases for $\mathcal{H}$, it follows from [1, Proposition 4.1] that $\mathbf{M}_{U',G,F}$ is invertible and hence $\mathbf{M}_{U,\Lambda,\Theta}$ is also invertible.

To obtain the second statement from the first one, set $U = T^*_{\tilde{\Lambda}} T_{\tilde{\Theta}}$. Thus,

$$\mathbf{M}_{U,\Lambda,\Theta} = T_\Lambda U T^*_\Theta = T_\Lambda (T^*_{\tilde{\Lambda}} T_{\tilde{\Theta}}) T^*_\Theta = I.$$

So by the assumption, $U = T^*_{\tilde{\Lambda}} T_{\tilde{\Theta}}$ is invertible. In particular, $T_{\tilde{\Theta}}$ is injective and has closed range. Hence by [20, Lemma 7], $\tilde{\Theta}$ is a g-Riesz basis, and by [19, Theorem 4.2] we conclude that Θ is also a g-Riesz basis. Furthermore, since $T_{\tilde{\Lambda}}$ is also injective, the same technique can be used for showing that Λ is also a g-Riesz basis. □

In the following, we show that the invertibility of the generalized multiplier implies that the underlying g-Bessel sequences are g-frames.

Proposition 4.7. *Let* $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ *and* $\Theta = \{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ *be two g-Bessel sequences for* $\mathcal{H}$*. If the generalized multiplier* $\mathbf{M}_{U,\Lambda,\Theta}$ *is invertible, then both* Λ *and* Θ *are g-frames for* $\mathcal{H}$*.*

Proof. Suppose that Λ and Θ are g-Bessel sequences for $\mathcal{H}$ with induced sequences $\{\lambda_k^i : i \in I, k \in K_i\}$ and $\{\theta_k^i : i \in I, k \in K_i\}$, respectively. Since $\mathbf{M}_{U,\Lambda,\Theta}$ is invertible, by Lemma 4.5 $\mathbf{M}_{U',G,F}$ is invertible. Now since, by Theorem 2.4, the sequences $G = \{\lambda_k^i\}_{i\in I,k\in K_i}$ and $F = \{\theta_k^i\}_{i\in I,k\in K_i}$ are Bessel, it follows that, by [1, Proposition 4.2], both F and G are frames for $\mathcal{H}$. Hence, Theorem 2.4 implies that the sequences Λ and Θ are g-frames for $\mathcal{H}$. □

We immediately obtain the following useful corollary.

Corollary 4.8. *Let* $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ *be a g-Riesz basis, and let* $\Theta = \{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}$ *be a g-Bessel sequence for* $\mathcal{H}$*. Moreover, let* $U \in \mathcal{B}((\bigoplus_{i\in I} \mathcal{H}_i)_{\ell^2(I)})$ *be invertible. Then* $\mathbf{M}_{U,\Lambda,\Theta}$ *is invertible if and only if* Θ *is a g-Riesz basis for* $\mathcal{H}$*.*

In the following two propositions, we show that under certain conditions the invertibility of generalized multipliers can be stable.

Proposition 4.9. *Let $\Lambda = \{\Lambda_i : i \in I\}$ be a g-frame for $\mathcal{H}$ with bounds A, B, and let $\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)$. Moreover, suppose that there exists a constant $0 < \mu < 1/\sqrt{B}$ such that*

$$\Big(\sum_{i\in I}\big\|(\Theta_i - \tilde{\Lambda}_i)f\big\|^2_{\mathcal{H}_i}\Big)^{1/2} \leq \mu\|f\|_{\mathcal{H}}. \tag{4.7}$$

Then

(1) *$\{\Theta_i : i \in I\}$ is a g-frame for $\mathcal{H}$,*
(2) *$\mathbf{M}_{I,\Lambda,\Theta}$ is invertible.*

Furthermore, suppose that U is a bounded linear operator on $(\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}$ with $\|U\| < \frac{1}{2}$ and $\|U - I\| < \frac{\sqrt{A}}{2\sqrt{B}}$. If (4.7) holds, then $\mathbf{M}_{U,\Lambda,\Theta}$ is invertible.

Proof. (1) By [25, Theorem 3.1], the proof is evident.

(2) For every $f \in \mathcal{H}$, we have

$$\begin{aligned}\|\mathbf{M}_{I,\Lambda,\Theta}f - f\|_{\mathcal{H}} &= \|T_\Lambda T^*_\Theta f - T_\Lambda T^*_{\tilde{\Lambda}} f\|_{\mathcal{H}}\\ &= \big\|T_\Lambda(T^*_\Theta - T^*_{\tilde{\Lambda}})f\big\|_{\mathcal{H}}\\ &\leq \sqrt{B}\Big(\sum_{i\in I}\big\|(\Theta_i - \tilde{\Lambda}_i)f\big\|^2_{\mathcal{H}_i}\Big)^{1/2}\\ &\leq \sqrt{B}\mu\|f\|_{\mathcal{H}} < \|f\|_{\mathcal{H}},\end{aligned}$$

which shows that $\|\mathbf{M}_{I,\Lambda,\Theta} - I\|_{\text{op}} < 1$. Therefore, $\mathbf{M}_{I,\Lambda,\Theta}$ is invertible.

Let us deal with the second claim. For each $f \in \mathcal{H}$,

$$\begin{aligned}\|\mathbf{M}_{U,\Lambda,\Theta}f - f\|_{\mathcal{H}} &\leq \|\mathbf{M}_{U,\Lambda,\Theta}f - \mathbf{M}_{U,\Lambda,\tilde{\Lambda}}f\|_{\mathcal{H}} + \|\mathbf{M}_{U,\Lambda,\tilde{\Lambda}}f - \mathbf{M}_{I,\Lambda,\tilde{\Lambda}}f\|_{\mathcal{H}}\\ &= \big\|T_\Lambda U(T^*_\Theta - T^*_{\tilde{\Lambda}})f\big\|_{\mathcal{H}} + \big\|T_\Lambda(U - I)T^*_{\tilde{\Lambda}}f\big\|_{\mathcal{H}}\\ &\leq \sqrt{B}\mu\|U\|_{\text{op}}\|f\|_{\mathcal{H}} + \frac{\sqrt{B}}{\sqrt{A}}\|U - I\|_{\text{op}}\|f\|_{\mathcal{H}} < \|f\|_{\mathcal{H}}.\end{aligned}$$

Since $\|\mathbf{M}_{U,\Lambda,\Theta} - I\|_{\text{op}} < 1$, it follows that $\mathbf{M}_{U,\Lambda,\Theta}$ is invertible. ☐

Proposition 4.10. *Let $\Lambda = \{\Lambda_i : i \in I\}$, $\Theta = \{\Theta_i : i \in I\}$, and $\Gamma = \{\Gamma_i : i \in I\}$ be g-Bessel sequences for $\mathcal{H}$, and assume that there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ such that*

$$\Big\|\sum_{i\in I_1}(\Lambda^*_i - \Gamma^*_i)f_i\Big\|_{\mathcal{H}} \leq \lambda_1\Big\|\sum_{i\in I_1}\Lambda^*_i f_i\Big\|_{\mathcal{H}} + \lambda_2\Big\|\sum_{i\in I_1}\Gamma^*_i f_i\Big\|_{\mathcal{H}} + \mu\Big(\sum_{i\in I_1}\|f_i\|^2_{\mathcal{H}_i}\Big)^{1/2}, \tag{4.8}$$

for any finite subsets $I_1 \subset I$ and $f_i \in \mathcal{H}_i$. Moreover, suppose that there are $U_1, U_2 \in \mathcal{B}((\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2})$ with $\|U_1 - U_2\|_{\text{op}} < \epsilon$. If $\mathbf{M}_{U_1,\Lambda,\Theta}$ is invertible and

$$\sqrt{B_\Lambda B_\Theta}\epsilon + \sqrt{B}_\Theta(\lambda_1\sqrt{B}_\Lambda + \lambda_2\sqrt{B}_\Gamma + \mu)\|U_2\|_{\text{op}} < \frac{1}{\|\mathbf{M}^{-1}_{U_1,\Lambda,\Theta}\|_{\text{op}}}, \tag{4.9}$$

then $\mathbf{M}_{U_2,\Gamma,\Theta}$ is also invertible.

Proof. By (4.8) and (4.9), we have

$$\begin{aligned}
&\|\mathbf{M}_{U_1,\Lambda,\Theta} - \mathbf{M}_{U_2,\Gamma,\Theta}\|_{\mathrm{op}} \\
&\quad \le \|\mathbf{M}_{U_1,\Lambda,\Theta} - \mathbf{M}_{U_2,\Lambda,\Theta}\|_{\mathrm{op}} + \|\mathbf{M}_{U_2,\Lambda,\Theta} - \mathbf{M}_{U_2,\Gamma,\Theta}\|_{\mathrm{op}} \\
&\quad = \left\|T_\Lambda(U_1 - U_2)T_\Theta^*\right\|_{\mathrm{op}} + \left\|(T_\Lambda - T_\Gamma)U_2T_\Theta^*\right\|_{\mathrm{op}} \\
&\quad \le \sqrt{B_\Lambda B_\Theta}\|U_1 - U_2\|_{\mathrm{op}} + \sqrt{B}_\Theta(\lambda_1\sqrt{B}_\Lambda + \lambda_2\sqrt{B}_\Gamma + \mu)\|U_2\|_{\mathrm{op}} \\
&\quad \le \sqrt{B_\Lambda B_\Theta}\epsilon + \sqrt{B}_\Theta(\lambda_1\sqrt{B}_\Lambda + \lambda_2\sqrt{B}_\Gamma + \mu)\|U_2\|_{\mathrm{op}} \\
&\quad < \frac{1}{\|\mathbf{M}_{U_1,\Lambda,\Theta}^{-1}\|_{\mathrm{op}}},
\end{aligned}$$

which implies, by Proposition 2.5, that $\mathbf{M}_{U_2,\Gamma,\Theta}$ is invertible. ☐

As a consequence of Proposition 4.10, we have the following corollary.

Corollary 4.11. *Let $\Gamma = \{\Gamma_i : i \in I\}$ be a g-frame with bounds A_Γ, B_Γ, and let $\Lambda = \{\Lambda_i : i \in I\}$ be a g-Bessel sequence with bound B_Λ for $\mathcal{H}$ such that (4.8) holds. Moreover, let $U \in \mathcal{B}((\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)})$ such that $\|U - I\|_{\mathrm{op}} < \epsilon$, for some positive constant ϵ. If*

$$\sqrt{B_\Lambda B_\Gamma}\epsilon + \sqrt{B_\Gamma}(\lambda_1\sqrt{B}_\Lambda + \lambda_2\sqrt{B}_\Gamma + \mu) < A_\Gamma,$$

then $\mathbf{M}_{U,\Lambda,\Gamma}$ is invertible.

Proof. For every $f \in \mathcal{H}$,

$$\begin{aligned}
\|\mathbf{M}_{U,\Lambda,\Gamma}f - S_\Gamma f\|_{\mathcal{H}} &\le \|\mathbf{M}_{U,\Lambda,\Gamma}f - T_\Lambda T_\Gamma^* f\|_{\mathcal{H}} + \|T_\Lambda T_\Gamma^* f - S_\Gamma f\|_{\mathcal{H}} \\
&\le \left\|T_\Lambda(U - I)T_\Gamma^* f\right\|_{\mathcal{H}} + \left\|(T_\Lambda - T_\Gamma)T_\Gamma^* f\right\|_{\mathcal{H}} \\
&\le \sqrt{B_\Lambda B_\Gamma}\epsilon\|f\|_{\mathcal{H}} + \sqrt{B}_\Gamma(\lambda_1\sqrt{B}_\Lambda + \lambda_2\sqrt{B}_\Gamma + \mu)\|f\|_{\mathcal{H}} \\
&< A_\Gamma\|f\|_{\mathcal{H}} \le \frac{1}{\|S_\Gamma^{-1}\|_{\mathrm{op}}}\|f\|_{\mathcal{H}}.
\end{aligned}$$

Now, Proposition 2.5 implies that $\mathbf{M}_{U,\Lambda,\Gamma}$ is invertible. ☐

5. Generalized cross-Gram matrix for g-frames

The matrix representation of operators in Hilbert spaces using an orthonormal basis (see [8]), Gabor frames (see [15]), and linear independent Gabor systems (see [23]) led Balazs to develop this idea in full generality for Bessel sequences, frames, and Riesz sequences (see [3]). Those matrices are constructed by composing the given operator V with the synthesis and analysis operators. Therefore, they can be considered as a generalization of Gram matrices. In this section, we are going to develop this idea for g-Bessel sequences from a different viewpoint.

Definition 5.1. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Theta = \{\Theta_i\}_{i\in I}$ be g-Bessel sequences in $\mathcal{H}$. For given $V \in \mathcal{B}(\mathcal{H})$, the matrix $\mathbf{G}_{V,\Lambda,\Theta}$, defined as

$$(\mathbf{G}_{V,\Lambda,\Theta})_{ij} = \Lambda_i V \Theta_j^*, \tag{5.1}$$

is called the *generalized cross-Gram matrix* for Λ and Θ with symbol V. Therefore, $\mathbf{G}_{V,\Lambda,\Theta}$ looks like a block matrix of operators in $\mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$.

It is obvious that the generalized cross-Gram matrix defines a bounded linear operator on $(\bigoplus_{j\in I}\mathcal{H}_j)_{\ell^2(I)}$. In fact, for every $f=(f_j)_{j\in I}\in(\bigoplus_{j\in I}\mathcal{H}_j)_{\ell^2(I)}$,

$$(\mathbf{G}_{V,\Lambda,\Theta}f)_i=\sum_{j\in I}(\mathbf{G}_{V,\Lambda,\Theta})_{ij}f_j=\sum_{j\in I}\Lambda_i V\Theta_j^*f_j=\Lambda_i V\Big(\sum_{j\in I}\Theta_j^*f_j\Big)=\Lambda_i(VT_\Theta f),$$

which shows that as operator we have $\mathbf{G}_{V,\Lambda,\Theta}=T_\Lambda^*VT_\Theta$. Moreover, $\mathbf{G}_{V,\Lambda,\Theta}^*=\mathbf{G}_{V^*,\Theta,\Lambda}$ and

$$\|\mathbf{G}_{V,\Lambda,\Theta}\|_{\text{op}}\leq\sqrt{B_\Lambda B_\Theta}\|V\|_{\text{op}}.$$

Since matrices are helpful in solving operator equations, the algebraic properties of matrices may then be used to solve the equations. For instance, the invertibility of matrices plays a key role in this topic. Therefore, it would be very interesting to provide some conditions for the invertibility of the matrix description of operators.

The following proposition gives us a sufficient and necessary condition for the invertibility of the generalized cross-Gram matrix. Before proceeding, we need the following lemma.

Lemma 5.2. *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ be a g-Riesz basis for $\mathcal{H}$ with the unique (canonical) dual g-Riesz basis $\tilde{\Lambda}=\{\tilde{\Lambda}_i\}_{i\in I}$. Then $T_{\tilde{\Lambda}}^*T_\Lambda=I$.*

Proof. For every $f=(f_i)_{i\in I},g=(g_i)_{i\in I}\in(\bigoplus_i\mathcal{H}_i)_{\ell^2(I)}$,

$$\begin{aligned}\langle T_{\tilde{\Lambda}}^*T_\Lambda f,g\rangle&=\langle T_\Lambda f,T_{\tilde{\Lambda}}g\rangle=\Big\langle\sum_{i\in I}\Lambda_i^*f_i,\sum_{j\in I}\tilde{\Lambda}_j^*g_j\Big\rangle=\sum_{i\in I}\sum_{j\in I}\langle\Lambda_i^*f_i,\tilde{\Lambda}_j^*g_j\rangle\\&=\sum_{i\in I}\sum_{j\in I}\delta_{i,j}\langle f_i,g_j\rangle=\langle f,g\rangle,\end{aligned}$$

which implies that $T_{\tilde{\Lambda}}^*T_\Lambda=I$. □

Proposition 5.3. *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ and $\Theta=\{\Theta_i\}_{i\in I}$ be g-Riesz bases for $\mathcal{H}$. Then $\mathbf{G}_{V,\Lambda,\Theta}$ is invertible if and only if V is invertible.*

Proof. Suppose that $\tilde{\Lambda}=\{\tilde{\Lambda}_i\}_{i\in I}$ and $\tilde{\Theta}=\{\tilde{\Theta}_i\}_{i\in I}$ are the canonical dual g-Riesz bases of Λ and Θ, respectively. If V is invertible, then

$$(T_\Lambda^*VT_\Theta)(T_{\tilde{\Theta}}^*V^{-1}T_{\tilde{\Lambda}})=T_\Lambda^*VV^{-1}T_{\tilde{\Lambda}}=T_\Lambda^*T_{\tilde{\Lambda}}=I.$$

Furthermore, one can check that $(T_{\tilde{\Theta}}^*V^{-1}T_{\tilde{\Lambda}})(T_\Lambda^*VT_\Theta)=I$, and so $\mathbf{G}_{V,\Lambda,\Theta}$ is invertible. Conversely, let $\mathbf{G}_{V,\Lambda,\Theta}$ have the inverse $\mathbf{G}_{V,\Lambda,\Theta}^{-1}$. Then

$$V(T_\Theta\mathbf{G}_{V,\Lambda,\Theta}^{-1}T_\Lambda^*)=T_{\tilde{\Lambda}}T_\Lambda^*(VT_\Theta\mathbf{G}_{V,\Lambda,\Theta}^{-1})T_\Lambda^*=T_{\tilde{\Lambda}}(T_\Lambda^*VT_\Theta\mathbf{G}_{V,\Lambda,\Theta}^{-1})T_\Lambda^*=T_{\tilde{\Lambda}}T_\Lambda^*=I,$$

and by the similar argument $(T_\Theta\mathbf{G}_{V,\Lambda,\Theta}^{-1}T_\Lambda^*)V=I$. Hence, V is invertible. □

In the next result, we characterize generalized cross-Gram matrices of g-orthonormal bases.

Proposition 5.4. *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$. Then Λ is a g-orthonormal basis for $\mathcal{H}$ if and only if $\mathbf{G}_{I,\Lambda,\Lambda}=I$.*

Proof. First, assume that $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$. Then for every $f=(f_i)_{i\in I}, g=(g_i)_{i\in I}\in(\bigoplus_i\mathcal{H}_i)_{\ell^2(I)}$,

$$\begin{aligned}\langle\mathbf{G}_{I,\Lambda,\Lambda}f,g\rangle&=\Big\langle\Big\{\sum_{i\in I}\Lambda_j\Lambda_i^*f_i\Big\}_{j\in I},\{g_j\}_{j\in I}\Big\rangle=\sum_{j\in I}\Big\langle\sum_{i\in I}\Lambda_j\Lambda_i^*f_i,g_j\Big\rangle\\&=\sum_{j\in I}\sum_{i\in I}\langle\Lambda_i^*f_i,\Lambda_j^*g_j\rangle\\&=\sum_{j\in I}\langle f_j,g_j\rangle=\langle f,g\rangle,\end{aligned}$$

which shows that $\mathbf{G}_{I,\Lambda,\Lambda}=I$.

Conversely, assume that $\mathbf{G}_{I,\Lambda,\Lambda}=T_\Lambda^*T_\Lambda=I$. Then

$$S_\Lambda^2=T_\Lambda T_\Lambda^*T_\Lambda T_\Lambda^*=T_\Lambda T_\Lambda^*=S_\Lambda.$$

Since S_Λ is invertible, we conclude that $S_\Lambda=I$. So, for every $f\in\mathcal{H}$,

$$\|f\|_{\mathcal{H}}^2=\big|\langle S_\Lambda f,f\rangle\big|=\Big|\Big\langle\sum_{i\in I}\Lambda_i^*\Lambda_if,f\Big\rangle\Big|=\Big|\sum_{i\in I}\langle\Lambda_i^*\Lambda_if,f\rangle\Big|=\sum_{i\in I}\|\Lambda_if\|_{\mathcal{H}_i}^2.\quad(5.2)$$

Now we prove that, for every $f_i\in\mathcal{H}_i$ and $f_j\in\mathcal{H}_j$, $i,j\in I$,

$$\langle\Lambda_i^*f_i,\Lambda_j^*f_j\rangle=\delta_{i,j}\langle f_i,f_j\rangle.$$

Assume that $f=\{0,\ldots,0,f_j,0,\ldots\}$, for an arbitrary $j\in I$. Then

$$f=T_\Lambda^*T_\Lambda(f)=\{\Lambda_i\Lambda_j^*f_j\}_{i\in I},$$

which implies that $\Lambda_i\Lambda_j^*=\delta_{i,j}, i,j\in I$. Therefore, for every $f_i\in\mathcal{H}_i$ and $f_j\in\mathcal{H}_j$, $i,j\in I$, we have

$$\langle\Lambda_i^*f_i,\Lambda_j^*f_j\rangle=\langle f_i,\Lambda_i^*\Lambda_j^*f_j\rangle=\langle f_i,\delta_{i,j}f_j\rangle=\delta_{i,j}\langle f_i,f_j\rangle.$$ □

Our next result shows that the invertibility of the generalized cross-Gram matrix of two g-Bessel g-complete sequences implies that both of them are g-Riesz bases.

Proposition 5.5. *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ and $\Theta=\{\Theta_i\}_{i\in I}$ be two g-complete g-Bessel sequences for $\mathcal{H}$, and let $V\in\mathcal{B}(\mathcal{H})$. If $\mathbf{G}_{V,\Lambda,\Theta}$ is invertible, then both Λ and Θ are g-Riesz bases for $\mathcal{H}$.*

Proof. Suppose that Λ is a g-complete g-Bessel sequence for $\mathcal{H}$. We show that Θ is a g-Riesz basis for $\mathcal{H}$. For every $f=(f_i)_{i\in I}\in(\bigoplus_i\mathcal{H}_i)_{\ell^2(I)}$,

$$\begin{aligned}\|f\|^2_{\bigoplus_i\mathcal{H}_i}&=\big|\langle f,f\rangle\big|\\&=\big|\langle\mathbf{G}_{V,\Lambda,\Theta}^{-1}\mathbf{G}_{V,\Lambda,\Theta}f,f\rangle\big|\\&\le\|\mathbf{G}_{V,\Lambda,\Theta}^{-1}\|_{\mathrm{op}}\|\mathbf{G}_{V,\Lambda,\Theta}f\|_{(\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}}\|f\|_{(\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}}\\&\le\sqrt{B_\Lambda}\|\mathbf{G}_{V,\Lambda,\Theta}^{-1}\|_{\mathrm{op}}\|V\|_{\mathrm{op}}\|T_\Theta f\|_{\mathcal{H}}\|f\|_{(\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}}\\&=\sqrt{B_\Lambda}\|\mathbf{G}_{V,\Lambda,\Theta}^{-1}\|_{\mathrm{op}}\|V\|_{\mathrm{op}}\Big\|\sum_{i\in I}\Theta_i^*f_i\Big\|_{\mathcal{H}}\|f\|_{(\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}},\end{aligned}$$

which implies that

$$\frac{1}{B_\Lambda \|\mathbf{G}_{V,\Lambda,\Theta}^{-1}\|_{\mathrm{op}}^2 \|V\|_{\mathrm{op}}^2} \|f\|^2_{(\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}} \leq \Big\|\sum_{i\in I}\Theta_i^* f_i\Big\|_{\mathcal{H}}^2. \tag{5.3}$$

Moreover, clearly

$$\Big\|\sum_{i\in I}\Theta_i^* f_i\Big\|_{\mathcal{H}}^2 = \|T_\Theta f\|_{\mathcal{H}}^2 \leq B_\Theta \|f\|^2_{(\bigoplus_{i\in I}\mathcal{H}_i)_{\ell^2(I)}}. \tag{5.4}$$

By considering the inequalities (5.3) and (5.4) and due to the fact that Θ is g-complete, we conclude that Θ is a g-Riesz basis for $\mathcal{H}$. For the second part, it is enough to imply the same process for $\mathbf{G}_{V,\Lambda,\Theta}^* = \mathbf{G}_{V^*,\Theta,\Lambda}$. □

Now, we immediately obtain the following useful corollary.

Corollary 5.6. *Let Θ be a g-Riesz basis, let Λ be a g-complete g-Bessel sequence for $\mathcal{H}$, and let $V \in \mathcal{B}(\mathcal{H})$. Then $\mathbf{G}_{V,\Lambda,\Theta}$ is invertible if and only if Λ is a g-Riesz basis for $\mathcal{H}$ and V is invertible.*

Acknowledgments. The authors would like to thank the referees for their valuable comments and suggestions, which improved the final manuscript.

References

1. G. Abbaspour Tabadkan, H. Hosseinnezhad, and A. Rahimi, *Generalized Bessel multipliers in Hilbert spaces*, Results Math. **73** (2018), no. 2, art. ID 85. Zbl 06909704. MR3806074. DOI 10.1007/s00025-018-0841-6. 181, 185, 189
2. P. Balazs, *Basic definition and properties of Bessel multipliers*, J. Math. Anal. Appl. **325** (2007), no. 1, 571–585. Zbl 1105.42023. MR2273547. DOI 10.1016/j.jmaa.2006.02.012. 181, 184
3. P. Balazs, *Matrix representation of operators using frames*, Sampl. Theory Signal Image Process. **7** (2008), no. 1, 39–54. Zbl 1182.41029. MR2455829. 181, 185, 191
4. P. Balazs, *Matrix representation of bounded linear operators by Bessel sequences, frames and Riesz sequence*, preprint, http://hal.archives-ouvertes.fr/hal-00453173/document (accessed 23 November 2018). 181
5. P. Balazs, M. Dörfler, F. Jaillet, N. Holighaus, and G. Velasco, *Theory, implementation and applications of nonstationary Gabor frames*, J. Comput. Appl. Math. **236** (2011), no. 6, 1481–1496. Zbl 1236.94026. MR2854065. DOI 10.1016/j.cam.2011.09.011. 181
6. P. Balazs, B. Laback, G. Eckel, and W. A. Deutsch, *Time-frequency sparsity by removing perceptually irrelevant components using a simple model of simultaneous masking*, IEEE Trans. Audio Speech. Lang. Process. **18** (2010), no. 1, 34–49. DOI 10.1109/TASL.2009.2023164. 181
7. O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd ed., Appl. Numer. Harmon. Anal., Birkhäuser, Cham, 2016. Zbl 1348.42033. MR3495345. DOI 10.1007/978-3-319-25613-9. 181
8. J. B. Conway, *A Course in Functional Analysis*, Grad. Texts in Math. **96**, Springer, New York, 1985. Zbl 0558.46001. MR0768926. DOI 10.1007/978-1-4757-3828-5. 181, 191
9. N. Cotfas and J. P. Gazeau, *Finite tight frames and some applications*, J. Phys. A **43** (2010), no. 19, art. ID 193001. Zbl 1189.81045. MR2639123. DOI 10.1088/1751-8113/43/19/193001. 181
10. I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), no. 5, 1271–1283. Zbl 0608.46014. MR0836025. DOI 10.1063/1.527388. 180

11. M. Dörfler and B. Torrésani, *Representation of operators in the time-frequency domain and generalized Gabor multipliers*, J. Fourier Anal. Appl. **16** (2010), no. 2, 261–293. Zbl 1205.47017. MR2600960. DOI 10.1007/s00041-009-9085-x. 181
12. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72**, no. 2 (1952), 341–366. Zbl 0049.32401. MR0047179. DOI 10.2307/1990760. 180
13. M. H. Faroughi, E. Osgooei, and A. Rahimi, (X_d, X_d^*)*-Bessel multipliers in Banach spaces*, Banach J. Math. Anal. **7** (2013), no. 2, 146–161. Zbl 1266.42083. MR3039944. DOI 10.15352/bjma/1363784228. 184
14. I. Gohberg, S. Goldberg, and M. Kaashoek, *Basic Classes of Linear Operators*, Birkhäuser, Basel, 2003. Zbl 1065.47001. MR2015498. DOI 10.1007/978-3-0348-7980-4. 184, 185
15. K. Gröchenig, *Time-frequency analysis of Sjöstrand's class*, Rev. Mat. Iberoam. **22** (2006), no. 2, 703–724. Zbl 1127.35089. MR2294795. DOI 10.4171/RMI/471. 191
16. A. Khosravi and M. Mirzaee Azandaryani, *Bessel multipliers in Hilbert C^*-modules*, Banach J. Math. Anal. **9** (2015), no. 3, 153–163. Zbl 1311.42083. MR3296131. DOI 10.15352/bjma/09-3-11. 184
17. P. Majdak, P. Balazs, W. Kreuzer, and M. Dörfler, *A time-frequency method for increasing the signal-to-noise ratio in system identification with exponential sweeps*, preprint, http://ieeexplore.ieee.org/document/5947182/ (accessed 24 November 2018). 181
18. A. Najati and A. Rahimi, *Generalized frames in Hilbert spaces*, Bull. Iranian Math. Soc. **35** (2009), no. 1, 97–109. Zbl 1180.41027. MR2547619. 186
19. E. Osgooei, *G-Riesz bases and clear structure for duals*, preprint, http://www.ispacs.com/journals/cacsa/2015/cacsa-00034/ (accessed 24 November 2018). 189
20. E. Osgooei and M. H. Faroughi, *Hilbert-Schmidt sequences and dual of G-frames*, Acta Univ. Apulensis Math. Inform. **36** (2013), 165–179. Zbl 1340.41046. MR3266687. 189
21. A. Pietsch, *Operator Ideals*, North-Holland Math. Libr. **20**, North-Holland, Amsterdam, 1980. Zbl 0434.47030. MR0582655. 187
22. A. Rahimi, *Multipliers of generalized frames in Hilbert spaces*, Bull. Iranian Math. Soc. **37** (2011), no. 1, 63–80. Zbl 1231.42031. MR2850104. 181, 184, 187
23. T. Strohmer, *Pseudodifferential operators and Banach algebras in mobile communications*, Appl. Comput. Harmon. Anal. **20** (2006), no. 2, 237–249. Zbl 1099.94029. MR2207837. DOI 10.1016/j.acha.2005.06.003. 191
24. W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), no. 1, 437–452. Zbl 1129.42017. MR2239250. DOI 10.1016/j.jmaa.2005.09.039. 181, 182, 183
25. W. Sun, *Stability of g-frames*, J. Math. Anal. Appl. **326** (2007), no. 2, 858–868. Zbl 1130.42307. MR2280948. DOI 10.1016/j.jmaa.2006.03.043. 181, 190
26. D. Wang and G. J. Brown, *Computational Auditory Scene Analysis: Principles, Algorithms, and Applications*, Wiley-IEEE, Hoboken, 2006. DOI 10.1109/9780470043387. 181, 184

[1]Department of Pure Mathematics, School of Mathematics and Computer Science, Damghan University, Damghan, Iran.

E-mail address: hosseinnezhad_h@yahoo.com; abbaspour@du.ac.ir

[2]Department of Mathematics, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.

E-mail address: rahimi@maragheh.ac.ir