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CYCLIC WEIGHTED SHIFT MATRIX WITH REVERSIBLE WEIGHTS

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ABSTRACT. We characterize a class of matrices that is unitarily similar to a complex symmetric matrix via the discrete Fourier transform.

1. Introduction

The numerical range $W(A)$ of an $n \times n$ matrix A is defined as

$$W(A) = \{\xi^* A\xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.$$

Toeplitz introduced the compact set $W(A)$, and Hausdorff proved its convexity. Kippenhahn developed a birational algebraic-geometric method to study the set $W(A)$. He introduced a real ternary homogeneous form

$$F_A(x, y, z) = \det\bigl(x\Re(A) + y\Im(A) + zI_n\bigr),$$

where $\Re(A) = (A + A^*)/2$, $\Im(A) = (A - A^*)/(2i)$ for the conjugate transpose A^* of A. He showed that the form F_A completely determines the range $W(A)$. In particular, he showed that the convex hull of the points $z = x_0 + iy_0$ (with $(x_0, y_0) \in \mathbb{R}^2$), for which the line $x_0x + y_0y + 1 = 0$ is a tangent of the real affine curve $F_A(x, y, 1) = 0$ at some point, coincides with the range $W(A)$. The real form $F_A(x, y, z)$ satisfies $F_A(0, 0, 1) = 1$, and every solution of the equation $F_A(x_1, y_1, z) = 0$ in z is real for every $(x_1, y_1) \in \mathbb{R}^2$. Recently, Plaumann and

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Vinzant [13] proved that a ternary form $F(x, y, z)$ possessing the above property is expressed as

$$F(x, y, z) = \det(xH_1 + yH_2 + zI_n)$$

by using some real Hermitian matrices H_1, H_2. Their proof is rather elementary. Lentzos and Pasley [11] proved that the matrices $H_1 + iH_2$ can be taken as a cyclic weighted shift matrix if the hyperbolic form F is weakly circular invariant. A strict assertion for an arbitrary hyperbolic form

$$F(x, y, z) = \det(xS_1 + yS_2 + zI_n)$$

has been proved by Helton and Vinnikov in [10]. Using the result in [10], Helton and Spitkovsky [9] proved that the numerical range $W(A)$ of an arbitrary $n \times n$ matrix A has some $n \times n$ complex symmetric matrix S satisfying $W(A) = W(S)$. These results provide new motivation for considering the following question: What matrix A is unitarily similar to a complex symmetric matrix? In particular, what cyclic weighted shift matrix is unitarily similar to a symmetric matrix? In addition, complex symmetric matrices or operators have been widely studied over the past decade (see [1], [6], [7]). Chien, Liu, Nakazato, and Tam [4] recently provided some unitary matrices which uniformly turn Toeplitz matrices into symmetric matrices. We wish to provide another class of matrices satisfying a similar property.

An $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ with the entries $a_{12} = w_1, a_{23} = w_2, \dots, a_{n-1,n} = w_{n-1}$, $a_{n,1} = w_n$, $a_{ij} = 0$ for (i, j) other than $(i, j) = (1, 2), \dots, (n-1, n), (n, 1)$ is called a *weighted shift matrix*. It is given by

$$\left[\begin{array}{c|ccccc} & w_1 & & & & \\ & & w_2 & & & \\ 0 & & & w_3 & & \\ & & & & \ddots & \\ & & & & & w_{n-1} \\ \hline w_n & & & 0 & & \end{array}\right], \tag{1.1}$$

where the w_j's are called *weights*. Various interesting properties are known for weighted shift matrices (see [8], [14]). As it was shown in [5], the weighted shift matrix

$$\begin{bmatrix} 0 & 8 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

is not unitarily similar to a complex symmetric matrix.

The characteristic polynomial of a weighted shift matrix is given by

$$\lambda^n - w_1 w_2 \cdots w_n.$$

Hence if none of the w_j's vanish, then the weighted shift matrix is similar to a diagonal matrix

$$(w_1 w_2 \cdots w_n)^{1/n} \operatorname{diag}(1, \omega, \omega^2, \dots, \omega^{n-1}),$$

by an invertible matrix $g \in \mathrm{GL}(n : \mathbb{C})$, where $(w_1 w_2 \cdots w_n)^{1/n}$ is one of the nth root of $w_1 w_2 \cdots w_n$ in the field $\mathbb{C}$ and $\omega = \exp(2\pi\sqrt{-1}/n)$. In the case where one of the w_j's vanishes, the weighted shift matrix S is nilpotent. So various studies of weighted shift matrices are usually based on the different methods according to whether $w_1 w_2 \cdots w_n \neq 0$ or $w_1 w_2 \cdots w_n = 0$. However, the method used in this article does not need the assumption $w_1 w_2 \cdots w_n \neq 0$. A weighted shift matrix satisfying this condition is called *cyclic*. A weight sequence $W = (w_1, w_2, \ldots, w_n)$ is called *reversible* if $w_{n-k+1} = w_k$ for $k = 1, 2, \ldots, n$. We mainly treat the matrix (1.1) with reversible weights.

2. Main result

The Fourier transform $\tilde{A}$ of an $n \times n$ matrix A is defined as U^*AU, where U is the $n \times n$ unitary matrix defined by

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdot & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdot & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdot & \omega^{3(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdot & \omega^{(n-1)^2} \end{bmatrix},$$

where $\omega = \exp(2\pi\sqrt{-1}/n)$. The (k, ℓ)-entry $b_{k\ell}$ of the Fourier transform $B = \tilde{A}$ of an $n \times n$ matrix $A = (a_{pq})$ is given by

$$\tilde{b}_{k\ell} = n b_{k\ell} = \sum_{p,q=1}^{n} \omega^{-(k-1)(p-1)} \omega^{(\ell-1)(q-1)} a_{p,q}.$$

We present our main theorem.

Theorem 2.1. *Let $A = (a_{pq})$ be an $n \times n$ complex matrix. Then the Fourier transform $B = U^*AU$ of A is a complex symmetric matrix if and only if $a_{1,k+1} = a_{n-k+1,1}$ and $a_{1+k,1+\ell} = a_{n+1-\ell,n+1-k}$ for all $k, \ell = 1, 2, \ldots, n-1$. That is,*

$$A = \left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{1n} & & & \\ \vdots & & \tilde{A} & \\ a_{12} & & & \end{array}\right],$$

where $\tilde{A}$ is an $(n-1) \times (n-1)$ complex matrix which is symmetric with respect to the main skew-diagonal line.

For the 5×5 case, A is of the following form:

$$A = \left[\begin{array}{c|cccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \hline a_{15} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{14} & a_{32} & a_{33} & a_{34} & a_{24} \\ a_{13} & a_{42} & a_{43} & a_{33} & a_{23} \\ a_{12} & a_{52} & a_{42} & a_{32} & a_{22} \end{array}\right].$$

Proof. Suppose that $B = U^*AU$ is a complex symmetric matrix. Note that $A = (a_{pq})$ can be divided by the following:

$$\sum_{\substack{q-p\equiv 1\\ \bmod n}} (a_{pq}) + \sum_{\substack{q-p\equiv 2\\ \bmod n}} (a_{pq}) + \cdots + \sum_{\substack{q-p\equiv n\\ \bmod n}} (a_{pq}).$$

For instance, when $n = 4$, A can be divided by

$$\begin{bmatrix} 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & a_{34} \\ a_{41} & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ a_{13} & 0 & 0 & 0 \\ 0 & a_{24} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & a_{14} \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$

Fix $m = 1, \ldots, n$, and let $b_{ij}^{(m)}$ be the (i,j)-entry of (a_{pq}) under the discrete Fourier transform, where m satisfies $q - p \equiv m \pmod n$. Therefore,

$$\begin{aligned} b_{ij}^{(m)} &= \frac{1}{n}(U^*)_{i*} \\ &\quad\times \left[\begin{array}{cccc|cccc} & & & & a_{1,m+1} & & & \\ & & & & & a_{2,m+2} & & \\ & & & & & & \ddots & \\ & & & & & & & a_{n-m,n} \\ \hline a_{n-m+1,1} & & & & & & & \\ & a_{n-m+2,2} & & & & & & \\ & & \ddots & & & & & \\ & & & a_{n,m} & & & & \end{array}\right] \\ &\quad\times U_{*j} \\ &= \frac{\omega^{m(j-1)}}{n}\big(a_{1,m+1} + a_{2,m+2}\omega^{(j-i)} + \cdots + a_{n-m,m}\omega^{(n-m+1)(j-i)} \\ &\quad + a_{n-m+1,1}\omega^{(n-m)(j-i)} + a_{n-m+2,2}\omega^{(n-m-1)(j-i)} + \cdots + a_{n-m,n}\omega^{(n-1)(j-i)}\big) \\ &= \left(\frac{\omega^{m(j-1)}}{n}U \begin{bmatrix} a_{1,m+1} \\ a_{2,m+2} \\ \vdots \\ a_{n-m,m} \\ a_{n-m+1,1} \\ a_{n-m+2,2} \\ \vdots \\ a_{n-m,n} \end{bmatrix}\right)_{j-i+1}, \end{aligned}$$

and we have the $(j-i+1)$th component of the above vector, where

$$(U^*)_{i*} = [1, \omega^{-(i-1)}, \omega^{-2(i-1)}, \dots, \omega^{-(n-1)(i-1)}]$$

and

$$U_{*j} = \begin{bmatrix} 1 \\ \omega^{j-1} \\ \vdots \\ \omega^{(n-1)(j-1)} \end{bmatrix}.$$

Similarly,

$$\begin{aligned} b_{ji}^{(m)} &= \frac{\omega^{m(j-1)}}{n}(a_{n-m+1,1} + a_{n-m,n}\omega^{(j-i)} + \dots + a_{2,m+2}\omega^{(n-m+1)(j-i)} \\ &\quad + a_{1,m+1}\omega^{(n-m)(j-i)} + a_{n,m}\omega^{(n-m-1)(j-i)} + \dots + a_{n-m+2,2}\omega^{(n-1)(j-i)}) \\ &= \left(\frac{\omega^{m(j-1)}}{n} U \begin{bmatrix} a_{n-m+1,1} \\ a_{n-m,n} \\ \vdots \\ a_{2,m+2} \\ a_{1,m+1} \\ a_{n,m} \\ \vdots \\ a_{n-m+2,2} \end{bmatrix} \right)_{j-i+1}. \end{aligned}$$

Let A_m be the following column vector, and let $A_m(j)$ be the jth component of this vector. We have

$$A_m = U \left(\begin{bmatrix} a_{1,m+1} \\ a_{2,m+2} \\ \vdots \\ a_{n-m,m} \\ a_{n-m+1,1} \\ a_{n-m+2,2} \\ \vdots \\ a_{n-m,n} \end{bmatrix} - \begin{bmatrix} a_{n-m+1,1} \\ a_{n-m,n} \\ \vdots \\ a_{2,m+2} \\ a_{1,m+1} \\ a_{n,m} \\ \vdots \\ a_{n-m+2,2} \end{bmatrix} \right) = U \begin{bmatrix} a_{1,m+1} - a_{n-m+1,1} \\ a_{2,m+2} - a_{n-m,n} \\ \vdots \\ a_{n-m,m} - a_{2,m+2} \\ a_{n-m+1,1} - a_{1,m+1} \\ a_{n-m+2,2} - a_{n,m} \\ \vdots \\ a_{n-m,n} - a_{n-m+2,2} \end{bmatrix}. \tag{2.1}$$

Note that if $j-i+1 < 0$, then we can choose $j-i+1$ to be k, where $k \in \{1, 2, \dots, n\}$ which satisfies $j-i+1 \equiv k \pmod{n}$. Applying the above argument, we have

$$b_{ij} - b_{ji} = \sum_{m=1}^{n} \frac{\omega^{m(j-1)}}{n} A_m(j-i+1). \tag{2.2}$$

Hence, if $a_{1,k+1} = a_{n-k+1,1}$ and $a_{1+k,1+\ell} = a_{n+1-\ell,n+1-k}$ for all $k, \ell = 1, 2, \dots, n-1$, then $A_m(j) = 0$ for all $j, m = 1, 2, \dots, n$. So $b_{ij} - b_{ji} = 0$, and this establishes the "if" part.

On the other hand, if B is a complex symmetric matrix, then, since $\omega^{j-1} \neq$ and $n \neq 0$, (2.2) becomes

$$0 = \sum_{m=1}^{n} \frac{\omega^{(m-1)(j-1)}}{\sqrt{n}} A_m(j-i+1). \tag{2.3}$$

We fix $k \in \{1, 2, \ldots, n\}$ with $j - i + 1 \equiv k \pmod{n}$ for all $i, j = 1, 2, \ldots, n$. Using both that j varies from 1 to n and (2.3), we have that

$$U \begin{bmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_n(k) \end{bmatrix}$$

is a zero vector. This implies that $A_m(k) = 0$ for all $k, m = 1, 2, \ldots, n$ as U is invertible. Again, using the invertibility of U in (2.1), we have that

$$\begin{bmatrix} a_{1,m+1} \\ a_{2,m+2} \\ \vdots \\ a_{n-m,m} \\ a_{n-m+1,1} \\ a_{n-m+2,2} \\ \vdots \\ a_{n-m,n} \end{bmatrix} - \begin{bmatrix} a_{n-m+1,1} \\ a_{n-m,n} \\ \vdots \\ a_{2,m+2} \\ a_{1,m+1} \\ a_{n,m} \\ \vdots \\ a_{n-m+2,2} \end{bmatrix}$$

is a zero vector for all $k, m = 1, 2, \ldots, n$. So $a_{1,k+1} = a_{n-k+1,1}$ and $a_{1+k,1+\ell} = a_{n+1-\ell,n+1-k}$ for all $k, \ell = 1, 2, \ldots, n-1$. This establishes the "only if" part and completes the proof. □

The following result can be deduced easily from Theorem 2.1.

Corollary 2.2. *A weighted shift matrix with reversible weights is unitarily similar to a complex symmetric matrix.*

We provide some examples of the matrix $A = (a_{pq})$ satisfying Theorem 2.1, where m satisfies $q - p \equiv m \pmod{n}$.

Example 2.3. When $n = 6$, $m = 2$,

$$A = \begin{bmatrix} 0 & 0 & w_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_2 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2.4. When $n = 6$, $m = 3$,

$$A = \begin{bmatrix} 0 & 0 & 0 & w_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_2 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_3 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2.5. When $n = 6$, $m = 1$,

$$A = \begin{bmatrix} 0 & w_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_2 \\ w_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2.6. When $n = 7$, $m = 1$,

$$A = \begin{bmatrix} 0 & w_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w_2 \\ w_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2.7. When $n = 5$, $m = 5$,

$$A = \begin{bmatrix} w_1 & 0 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ 0 & 0 & w_3 & 0 & 0 \\ 0 & 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & 0 & w_2 \end{bmatrix}.$$

The authors wonder if weighted shift matrices are essentially determined by the ternary form $F_W(x, y, z)$. Such a hypothesis is related with the inverse problem of the construction of a matrix W from the $F_W(x, y, z)$. The formula obtained by Helton and Vinnikov [10] and by Plaumann, Sturmfels, and Vinzant [12] provides a strong tool to treat this subject (see also [3]). The following result would be the first step of our study along this line.

Corollary 2.8. *Let W be an $n \times n$ weighted cyclic shift matrix with reversible weight $\omega_1, \omega_2, \ldots, \omega_2, \omega_1$, and let n be odd. Suppose that the curve $F_W(x, y, z) = 0$ has no singular points and that $\Im(W)$ has n distinct nonzero eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$. Then there exists a real symmetric matrix S_1 satisfying*

$$\det\bigl(xS_1 + y\,\mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n) + zI_n\bigr) = F_W(x, y, z),$$

where S_1 is provided by the Helton–Vinnikov theorem (see [10, Theorem 4]) and $S_1 + i\,\mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$ is unitarily similar to W.

Proof. By Theorem 2.1, the matrix W is unitarily similar to a complex symmetric matrix. Under this condition and the assumption that the curve $F_W(x,y,z) = 0$ has no singular points, Theorem 7 of [12] guarantees that there is one pair of real symmetric matrices S_1 and S_2 satisfying

$$\det(xS_1 + yS_2 + zI_n) = \det\big(x\Re(W) + y\Im(W) + zI_n\big)$$

and that $S_1 + iS_2$ is unitarily similar to W. To apply this theorem, we assume that one standard condition $\Im(W)$ has n distinct nonzero roots. □

Remark 2.9. The condition that "n be odd" in the above corollary is crucial. In the case where n is even, the curve $F_W(x,y,z)$ has singular points provided that the weights of W are reversible (see [2]).

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