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DECOMPOSITIONS OF COMPLETELY BOUNDED MAPS INTO COMPLETELY POSITIVE MAPS INVOLVING TRACE CLASS OPERATORS

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Abstract. Let $K(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ be the sets of all compact operators and all bounded linear operators, respectively, on the Hilbert space $\mathcal{H}$. In this article, we mainly show that if $\Phi \in \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$, then there exist $\Phi_i \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$, for $i = 1, 2, 3, 4$, such that $\Phi = (\Phi_1 - \Phi_2) + \sqrt{-1}(\Phi_3 - \Phi_4)$. However, $\mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K})) \nsubseteq \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$, where $\mathrm{CB}(V, W)$ and $\mathrm{CP}(V, W)$ are the sets of all completely bounded maps and all completely positive maps from V into W, respectively.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional separable Hilbert spaces, and let $\mathcal{B}(\mathcal{H})$ ($\mathcal{B}(\mathcal{H}, \mathcal{K})$) be the set of all bounded linear operators on $\mathcal{H}$ (from $\mathcal{H}$ to $\mathcal{K}$). As usual, $|A|$ is the absolute value operator of $A \in \mathcal{B}(\mathcal{H})$, and A^+ and A^- are the positive and negative parts of the self-adjoint operator A. Let $T(\mathcal{H})$ and $K(\mathcal{H})$ be the sets of all trace class operators and all compact operators, respectively, on $\mathcal{H}$. As $T(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ are Banach spaces, we denote by $\mathcal{B}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$ the set of all bounded linear operators from $T(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$. For $x \in \mathcal{H}$ and $y \in \mathcal{K}$, $x \otimes y$ is the rank 1 linear operator $x \otimes y(z) := \langle z, y\rangle x$ ($z \in \mathcal{H}$). Also, we write $A \simeq B$ to mean that the operators A and B are unitarily equivalent and that $\|T\|_1 := \mathrm{tr}(|T|)$, for $T \in T(\mathcal{H})$.

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An involution on V is a conjugate linear map $*: V \to V$ given by $v \mapsto v^*$ such that $v^{**} = v$ and $(v + \lambda\mu)^* = v^* + \overline{\lambda}\mu^*$ for all $\lambda \in \mathbb{C}$ and $v, \mu \in V$. The complex vector space V together with the involution map is called a $*$-*vector space*. If V is a $*$-vector space, then we let $V_h = \{v \in V : v = v^*\}$ be the real vector space of self-adjoint elements of V. Note that V_h is a real vector space. An ordered $*$-vector space (V, V^+) is a pair consisting of a $*$-vector space V and a proper cone $V^+ \subseteq V_h$. Here, $V^+ \subseteq V_h$ is called a *proper cone* if the following three assumptions hold:

(a) $\lambda\nu \in V^+$ whenever $\lambda \in \mathbb{R}^+ := [0, \infty)$ and $\nu \in V^+$,
(b) $\mu + \nu \in V^+$ whenever $\mu, \nu \in V^+$,
(c) $V^+ \cap (-V^+) = \{0\}$.

In this case, V^+ is called the *cone of positive elements* of V. So $\mathcal{B}(\mathcal{H})^+$, $K(\mathcal{H})^+$, and $T(\mathcal{H})^+$ are the cones of positive elements of $\mathcal{B}(\mathcal{H})$, $K(\mathcal{H})$, and $T(\mathcal{H})$, respectively.

Let $M_{n,m}(V)$ denote the set of all $n \times m$ matrices with entries in V. The natural addition and scalar multiplication turn $M_{n,m}(V)$ into a complex vector space. We write $M_{n,m} = M_{n,m}(\mathbb{C})$ and use the identifications $M_{n,m}(V) = M_{n,m} \otimes V = V \otimes M_{n,m}$. Also, $\acute{t}$ is the usual transpose map for $M_{n,m}$. If $n = m$, then we use $M_n(V) = M_{n,n}(V)$. As usual, suppose that $\{e_{ij}\}$ is the canonical matrix unit system; then we denote $\sum_{i,j=1}^n a_{ij} \otimes e_{ij} = \sum_{i,j=1}^n e_{ij} \otimes a_{ij} := (a_{ij})_{i,j} \in M_n(V)$, where $a_{ij} \in V$, for $1 \leq i, j \leq n$. Furthermore, we define a $*$-operation on $M_{n,m}(V)$ by letting $(a_{ij})_{i,j}^* = (a_{ji}^*)_{i,j}$ and letting XA be the element of $M_{l,n}(V)$ whose (i,j) entry $(XA)_{i,j}$ equals $\sum_{k=1}^m x_{ik}a_{kj}$, for $X = (x_{ij})_{i,j} \in M_{l,m}$ and $A = (a_{ij})_{i,j} \in M_{m,n}(V)$. The definition of multiplication by scalar matrices on the right is obtained in a similar way. Let $n \geq 1$ be an integer. If V and W are vector spaces, and Φ: $V \longrightarrow W$ is a linear map, then the map Φ induces a linear map Φ_n: $M_n(V) \longrightarrow M_n(W)$ by the formula

$$\Phi_n\big((a_{i,j})_{i,j}\big) = \big(\Phi(a_{i,j})\big)_{i,j} \quad \text{for } (a_{i,j}) \in M_n(V).$$

Let V be a $*$-vector space. We say that $\{C_n\}_{n=1}^\infty$ is a *matrix ordering* on V if the following two conditions are satisfied:

(a) C_n is a proper cone in $M_n(V)_h$ for $n = 1, 2, \ldots$;
(b) $X^* C_n X \subseteq C_m$, for all positive integers n, m and every $X \in M_{n,m}$; in this case, we call $(V, \{C_n\}_{n=1}^\infty)$ a *matrix-ordered $*$-vector space* (see [12, Definition 2.3]).

Let $(V, \{C_n\}_{n=1}^\infty)$ and $(W, \{D_n\}_{n=1}^\infty)$ be matrix-ordered $*$-vector spaces. A linear map Φ is called *completely positive* if $\Phi_n(C_n) \subseteq D_n$, for every $n = 1, 2, \ldots$. In this case, we write $\Phi \in \mathrm{CP}(V, W)$. If we let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of $\mathcal{H}$, then there is a natural norm and inner product on $\mathcal{H}^{(n)}$ that makes it into a Hilbert space. Also, the identification $M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^{(n)})$ gives a norm and a positive cone $M_n(\mathcal{B}(\mathcal{H}))^+$ of $M_n(\mathcal{B}(\mathcal{H}))$. Clearly, $(\mathcal{B}(\mathcal{H}), \{M_n(\mathcal{B}(\mathcal{H}))^+\}_{n=1}^\infty)$, $(K(\mathcal{H}), \{M_n(K(\mathcal{H}))^+\}_{n=1}^\infty)$, and $(T(\mathcal{H}), \{M_n(T(\mathcal{H}))^+\}_{n=1}^\infty)$ are matrix-ordered $*$-vector spaces. Let $\mathrm{CP}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K}))$ denote the cone of all completely positive linear maps from $(\mathcal{B}(\mathcal{H}), \{M_n(\mathcal{B}(\mathcal{H}))^+\}_{n=1}^\infty)$ into $(\mathcal{B}(\mathcal{K}), \{M_n(\mathcal{B}(\mathcal{K}))^+\}_{n=1}^\infty)$. The definitions of $\mathrm{CP}(K(\mathcal{H}), M_n)$ and $\mathrm{CP}(T(\mathcal{H}), M_n)$ based on the above matrix-ordered $*$-vector spaces are similar.

It is well known that the dual space of $\mathcal{K}(\mathcal{H})$ is isometrically isomorphic to $T(\mathcal{H})$ $(K(\mathcal{H})^* \simeq T(\mathcal{H}))$ and that the dual space of $T(\mathcal{H})$ is isometrically isomorphic to $\mathcal{B}(\mathcal{H})$ $(T(\mathcal{H})^* \simeq \mathcal{B}(\mathcal{H}))$. We say that $\Phi : T(\mathcal{H})(\mathcal{B}(\mathcal{H})) \to T(\mathcal{K})(\mathcal{B}(\mathcal{K}))$ is *normal* if Φ is continuous with respect to the ultraweak (W^*) topology. Moreover, setting $M_n(K(\mathcal{H})^*)^+ := \mathrm{CP}(K(\mathcal{H}), M_n)$ and $M_n(T(\mathcal{H})^*)^+ := \mathrm{CP}(T(\mathcal{H}), M_n)$, we might verify that $(T(\mathcal{H}), \{M_n(K(\mathcal{H})^*)^+\}_{n=1}^{\infty})$ and $(\mathcal{B}(\mathcal{H}), M_n(T(\mathcal{H})^*)^+\}_{n=1}^{\infty})$ are matrix-ordered $*$-vector spaces. For a linear map Φ from $T(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$, we define that $\Phi \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ if $\Phi_n(M_n(K(\mathcal{H})^*)^+) \subseteq M_n(\mathcal{B}(\mathcal{K}))^+$ and $\Phi \in \mathrm{CP}(K(\mathcal{H})^*, T(\mathcal{K})^*)$ if $\Phi_n(M_n(K(\mathcal{H})^*)^+) \subseteq M_n(T(\mathcal{K})^*)^+$, for all $n = 1, 2, \ldots$. It is worth mentioning that the notation $T(\mathcal{H})$ means $T(\mathcal{H})$ with the usual operator ordering, while $K(\mathcal{H})^*$ carries a completely different matrix ordering.

Let $(A_{ij})_{i,j} \in M_n(T(\mathcal{H}))$. By the duality theory of operator spaces (see pp. 40–45 of [3]), there is a natural operator space structure for $M_n(T(\mathcal{H}))$. That is,

$$\|(A_{ij})_{i,j}\|_n' := \sup\bigl\{\bigl\|\bigl(\bigl(\mathrm{tr}(A_{ij}S_{kl})\bigr)_{ik,jl}\bigr)\bigr\| : \text{all } S_{kl} \in K(\mathcal{H}) \text{ with } \|(S_{kl})_{k,l}\| \le 1\bigr\}.$$

Then $\|\cdot\|_n'$ is a matrix norm for $T(\mathcal{H})$ (see p. 20 of [3] for the definition). We denote by $\mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ all completely bounded linear maps from $T(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ with respect to the above matrix norm $\|\cdot\|_n'$ and usual operator norm $\|\cdot\|$. That is, for a linear map Φ from $T(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$, we have

$$\Phi \in \mathrm{CB}\bigl(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K})\bigr) \quad \Longleftrightarrow \quad \sup\bigl\{\|\Phi_n\| : n = 1, 2, \ldots\bigr\} < \infty,$$

where

$$\|\Phi_n\| := \sup\bigl\{\bigl\|\bigl(\Phi(T_{ij})\bigr)_{i,j}\bigr\| : \text{all } T_{ij} \in T(\mathcal{H}) \text{ with } \|(T_{ij})_{i,j}\|_n' \le 1\bigr\}.$$

Also, we write

$$\|\Phi_n\|' := \sup\bigl\{\bigl\|\bigl(\Phi(T_{ij})\bigr)_{i,j}\bigr\| : \text{all } T_{ij} \in T(\mathcal{H}) \text{ with } \|(T_{ij})_{i,j}\|_1 \le 1\bigr\}$$

and

$$\begin{aligned}&\mathrm{CB}\bigl(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\bigr)\\ &\quad := \bigl\{\Phi : \Phi \in \mathcal{B}\bigl(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\bigr) \text{ and } \sup\bigl\{\|\Phi_n\|' : n = 1, 2, \ldots\bigr\} < \infty\bigr\}.\end{aligned}$$

For $\Phi \in \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ or $\Phi \in \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$, we denote by $\|\Phi\|_{\mathrm{cb}}$ the completely bounded norm of Φ.

As usual, we set

$$l^2(\mathcal{H}) := \Bigl\{(h_1, h_2, \ldots) : h_i \in \mathcal{H} \text{ for all } i \text{ with } \sum_{i=1}^{\infty} \|h_i\|^2 < \infty\Bigr\}$$

and $A^t := \sum_{i,j=1}^{\infty} \mathrm{tr}(Af_{ij})f_{ij}$ the transpose of $A \in \mathcal{B}(\mathcal{K})$, where $A = \sum_{i,j=1}^{\infty} \mathrm{tr}(Af_{ji})f_{ij}$ is the matrix expression of A in the orthonormal basis $\{f_i\}_{i=1}^{\infty}$ and $f_{ij} := f_i \otimes f_j$. It is easy to show that $A \in \mathcal{B}(\mathcal{K})^+$ if and only if $A^t \in \mathcal{B}(\mathcal{K})^+$. Also, if t' is the transpose map on $\mathcal{B}(\mathcal{K})$ with respect to another orthonormal basis, then there is a unitary operator $U \in \mathcal{B}(\mathcal{K})$ such that $A^{t'} = UA^tU^*$ for any $A \in \mathcal{B}(\mathcal{K})$. Thus,

$$\|A\| = \|A^t\| = \|A^{t'}\|$$

and

$$A \in \mathcal{B}(\mathcal{K})^+ \iff A^t \in \mathcal{B}(\mathcal{K})^+ \iff A^{t'} \in \mathcal{B}(\mathcal{K})^+.$$

The study of completely bounded maps and completely positive maps is essential and useful in both mathematics and quantum theory. There are many interesting results in the literature focused on operator spaces, completely positive maps, and their applications (see [1]–[4], [9], [10], [14]). Recently, some delicate connections between completely bounded (positive) maps and quantum channel and entanglement theory were presented in [5], [11], [13], and [15]. Also, various complete order structures between dual spaces were given in [11], [16], and [17]. In particular, some complete order structures were established for a general $*$-vector space in [12].

Let $\mathcal{A}$ be a unital C*-algebra. An interesting decomposition theorem (Wittstock's decomposition theorem) for completely bounded maps (see [10, Theorem 8.5]) shows that $\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a completely bounded linear map if and only if there exist completely positive linear maps Φ_i from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ such that $\Phi = \Phi_1 - \Phi_2 + \sqrt{-1}(\Phi_3 - \Phi_4)$ for $i = 1, 2, 3, 4$. That is to say, the complete order structures and operator space structures for the linear maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$ are coordinated.

The purpose of this article is to consider decomposition properties of the $*$-vector spaces $\mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ into the positive cones $\mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$. Even though $T(\mathcal{H})$ is not a unital C*-algebra, there are partially analogous decompositions between them. The motivation of our choice $T(\mathcal{H})$ is based on the fact that there are dual relations between it and $K(\mathcal{H})$. Furthermore, $T(\mathcal{H})$ as target is important and useful in quantum information theory. We get that if $\Phi \in \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$, then there exist $\Phi_i \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$, for $i = 1, 2, 3, 4$, such that $\Phi = (\Phi_1 - \Phi_2) + \sqrt{-1}(\Phi_3 - \Phi_4)$. However, $\mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{H})) \not\subseteq \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))$. Our results show that the complete order structures and operator space structures for the linear maps from $T(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ are partially coordinated. Also, the proofs of our main results are completely different from that of Wittstock's decomposition theorem (see [10, Theorem 8.5]).

2. Main results

To show our main results, we need the following lemmas.

Lemma 2.1.

(a) *Let* $(B_{ij})_{n\times n} \in M_n(T(\mathcal{H}))^+$ *and let* $(T_{ij})_{i,j} \in M_n(K(\mathcal{H}))^+$. *Then* $(\mathrm{tr}(B_{ij}T_{ji}))_{i,j} \in M_n^+$.

(b) *Let* $(B_{ij})_{n\times n} \in M_n(T(\mathcal{H}))^+$ *and* $(A_{ij})_{i,j} \in M_n(\mathcal{B}(\mathcal{H}))^+$. *Then we have* $(\mathrm{tr}(B_{ij}A_{ji}))_{i,j} \in M_n^+$.

Proof. (a) As $\widetilde{T} := (T_{ij})_{i,j} \in M_n(K(\mathcal{H}))^+ \simeq K(\mathcal{H}^{(n)})^+$, we can then decompose $\widetilde{T} = \sum_{k=1}^{\infty} \lambda_k (x_k \otimes x_k)$, where $\lambda_k \geq 0$ and $\{x_k\}_{k=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}^{(n)}$, so there exists $x_{ki} \in \mathcal{H}$ (for $i = 1, 2, \ldots, n$) such that $x_k = (x_{k1}, x_{k2}, \ldots, x_{kn}) \in \mathcal{H}^{(n)}$. Thus

$$x_k \otimes x_k = (x_{ki} \otimes x_{kj})_{i,j} \in M_n\big(K(\mathcal{H})\big)^+, \tag{2.1}$$

which implies that $T_{ij} = \sum_{k=1}^{\infty} \lambda_k (x_{ki} \otimes x_{kj})$, for $i, j = 1, 2, \ldots, n$, and so

$$\begin{aligned}\big(\operatorname{tr}(B_{ij}T_{ji})\big)_{i,j} &= \Big(\operatorname{tr}\Big(B_{ij}\sum_{k=1}^{\infty}\lambda_k(x_{kj}\otimes x_{ki})\Big)\Big)_{i,j}\\ &= \sum_{k=1}^{\infty}\lambda_k\big(\operatorname{tr}(B_{ij}x_{kj}\otimes x_{ki})\big)_{i,j}\\ &= \sum_{k=1}^{\infty}\lambda_k\big(\langle B_{ij}x_{kj}, x_{ki}\rangle\big)_{i,j}. \end{aligned}\tag{2.2}$$

It is clear that $(\langle B_{ij}x_{kj}, x_{ki}\rangle)_{i,j} \in M_n^+$, for $k = 1, 2, \ldots,$ so $(\operatorname{tr}(B_{ij}T_{ji}))_{i,j} \in M_n^+$ follows from equation (2.2).

(b) Let P_m be the rank m orthogonal projection spanned by the orthonormal vectors $\{e_i\}_{i=1}^m$, for $m = 1, 2, \ldots,$ where $\{e_i\}_{i=1}^m$ is an orthonormal basis of $\mathcal{H}$. Then $(P_mA_{ij}P_m)_{i,j} \in M_n(K(\mathcal{H}))^+$, for $m = 1, 2, \ldots,$ so (a) yields $(\operatorname{tr}(B_{ij}P_mA_{ji}P_m))_{i,j} \in M_n^+$, for $m = 1, 2, \ldots.$ Thus, $(\operatorname{tr}(B_{ij}A_{ji}))_{i,j} \in M_n^+$ follows from the fact that

$$\lim_{m\longrightarrow\infty}\big\|\big(\operatorname{tr}(B_{ij}P_mA_{ji}P_m)\big)_{i,j} - \big(\operatorname{tr}(B_{ij}A_{ji})\big)_{i,j}\big\| = 0. \qquad \square$$

Lemma 2.2. *Let $(B_{ij})_{i,j} \in M_n(T(\mathcal{H}))^+$ and $(T_{ij})_{i,j} \in M_m(\mathcal{B}(\mathcal{H}))^+$. Then*

$$\Big(\big(\operatorname{tr}(B_{ij}T_{lk})\big)_{k,l}\Big)_{i,j} \in M_{mn}^+.$$

Proof. It is easy to see that $(B_{ij})_{i,j} \in M_n(T(\mathcal{H}))^+$ implies that

$$\sum_{k,l=1}^{m} e_{kl} \otimes \Big[\sum_{i,j=1}^{n} e_{ij} \otimes B_{ij}\Big] \in M_{mn}\big(T(\mathcal{H})\big)^+.$$

Also, $\sum_{k,l=1}^{m} e_{kl} \otimes T_{kl} \in M_m(\mathcal{B}(\mathcal{H}))^+$ implies that

$$\sum_{k,l=1}^{m} e_{kl} \otimes \Big[\sum_{i,j=1}^{n} e_{ij} \otimes T_{kl}\Big] \in M_{mn}\big(\mathcal{B}(\mathcal{H})\big)^+.$$

Then

$$\sum_{i,j=1}^{n}\sum_{k,l=1}^{m} \operatorname{tr}(B_{ij}T_{lk}) e_{kl} \otimes e_{ij} \in M_{mn}^+$$

follows from Lemma 2.1(b). $\square$

Lemma 2.3.

(a) *Let Φ_{ij} be bounded linear functions on $K(\mathcal{H})$ with $\Phi_{ij}(\cdot) = \operatorname{tr}(B_{ij}\cdot)$, for $i, j = 1, 2, \ldots, n$, and let $\widetilde{\Phi}(\cdot) := (\operatorname{tr}(B_{ij}\cdot))_{i,j}$ be the linear map from $K(\mathcal{H})$ into M_n. Then $\widetilde{\Phi} \in \mathrm{CP}(K(\mathcal{H}), M_n)$ if and only if $\sum_{i,j=1}^{n} e_{ij} \otimes B_{ji} \in M_n(T(\mathcal{H}))^+$.*

(b) *Let* Ψ_{ij} *be bounded linear functions on* $T(\mathcal{H})$ *with* $\Psi_{ij}(\cdot) = \mathrm{tr}(A_{ij}\cdot)$*, for* $i, j = 1, 2, \ldots, n$*, and let* $\widetilde{\Psi}(\cdot) := (\mathrm{tr}(A_{ij}\cdot))_{i,j}$ *be the linear map from* $T(\mathcal{H})$ *into* M_n*. Then* $\widetilde{\Psi} \in \mathrm{CP}(T(\mathcal{H}), M_n)$ *if and only if* $\sum_{i,j=1}^n e_{ij} \otimes A_{ji} \in M_n(\mathcal{B}(\mathcal{H}))^+$*.*

Proof. (a) Clearly, $B_{ij} \in T(\mathcal{H})$, for $i, j = 1, 2, \ldots, n$.

Sufficiency: Let $(T_{ij})_{i,j} \in M_m(K(\mathcal{H}))^+$. Then Lemma 2.2 implies that

$$\big(\big(\mathrm{tr}(B_{ij}T_{kl})\big)_{k,l}\big)_{i,j} \in M_{mn}^+,$$

which yields

$$\big(\widetilde{\Phi}(T_{kl})\big)_{k,l} = \big(\big(\mathrm{tr}(B_{ij}T_{kl})\big)_{i,j}\big)_{k,l} \simeq \sum_{i,j=1}^{n}\sum_{k,l=1}^{m} \mathrm{tr}(B_{ij}T_{kl})e_{kl} \otimes e_{ij} \in M_{mn}^+.$$

Thus, $\widetilde{\Phi} \in \mathrm{CP}(K(\mathcal{H}), M_n)$ follows from the arbitrariness of m.

Necessity: It is easy to see that $(B_{ji})_{i,j} \in M_n(T(\mathcal{H}))^+$ if and only if

$$\sum_{i,j=1}^{n} \mathrm{tr}(B_{ij}S_{ij}) = \mathrm{tr}\Big[\Big(\sum_{i,j=1}^{n} e_{ij} \otimes B_{ji}\Big)\sum_{i,j=1}^{n} e_{ij} \otimes S_{ij}\Big] \geq 0$$

for any $\sum_{i,j=1}^n e_{ij} \otimes S_{ij} \in M_n(K(\mathcal{H}))^+$.

Suppose that $(S_{kl})_{k,l} \in M_n(K(\mathcal{H}))^+$. Then

$$\sum_{i,j=1}^{n}\sum_{k,l=1}^{n} \mathrm{tr}(B_{ij}S_{kl})e_{kl} \otimes e_{ij} \simeq \big(\big(\mathrm{tr}(B_{ij}S_{kl})\big)_{i,j}\big)_{k,l} = \big(\widetilde{\Phi}(S_{kl})\big)_{k,l} \in M_{n^2}^+.$$

Setting $\alpha_i \in \mathbb{C}^n$ with $\alpha_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ni})$, where δ_{li} is the Kronecker delta for $l, i = 1, 2, \ldots, n$, we get that

$$\sum_{i,j=1}^{n} \mathrm{tr}(B_{ij}S_{ij}) = (\alpha_1, \alpha_2, \ldots, \alpha_n)\big(\big(\mathrm{tr}(B_{ij}S_{kl})\big)_{i,j}\big)_{k,l}\begin{pmatrix}\alpha_1^{\acute{t}}\\ \alpha_2^{\acute{t}}\\ \vdots\\ \alpha_n^{\acute{t}}\end{pmatrix} \geq 0.$$

The proof of (b) is similar to (a). □

We denote by $M_n(T(\mathcal{H}))_+$ and $M_n(\mathcal{B}(\mathcal{H}))_+$ the partial transpose of $M_n(T(\mathcal{H}))^+$ and $M_n(\mathcal{B}(\mathcal{H}))^+$, respectively. That is,

$$M_n\big(T(\mathcal{H})\big)_+ := \big\{(T_{ij})_{i,j} : (T_{ji})_{i,j} \in M_n\big(T(\mathcal{H})\big)^+\big\}$$

and

$$M_n\big(\mathcal{B}(\mathcal{H})\big)_+ := \big\{(S_{ij})_{i,j} : (S_{ji})_{i,j} \in M_n\big(\mathcal{B}(\mathcal{H})\big)^+\big\}.$$

Lemma 2.3(a) shows that $\widetilde{\Phi} \in \mathrm{CP}(K(\mathcal{H}), M_n)$ if and only if $\sum_{i,j=1}^n e_{ij} \otimes B_{ij} \in M_n(T(\mathcal{H}))_+$, where $\widetilde{\Phi}(\cdot) = (\mathrm{tr}(B_{ij}\cdot))_{i,j}$. Suppose that Φ is a linear map from

$T(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$. Then Lemma 2.3 says that $\Phi \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ if and only if $\Phi_n(M_n(T(\mathcal{H}))_+) \subseteq M_n(\mathcal{B}(\mathcal{K}))^+$ for all $n = 1, 2, \ldots$. Indeed,

$$\begin{aligned} \Phi \in \mathrm{CP}\big(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K})\big) \quad &\Longleftrightarrow \quad \Phi_n\big(\mathrm{CP}\big(K(\mathcal{H}), M_n\big)\big) \subseteq M_n\big(\mathcal{B}(\mathcal{K})\big)^+ \\ &\Longleftrightarrow \quad \Phi_n\big(M_n\big(T(\mathcal{H})\big)_+\big) \subseteq M_n\big(\mathcal{B}(\mathcal{K})\big)^+. \end{aligned}$$

Similarly,

$$\Phi \in \mathrm{CP}\big(K(\mathcal{H})^*, T(\mathcal{K})^*\big) \quad \Longleftrightarrow \quad \Phi_n\big(M_n\big(T(\mathcal{H})\big)_+\big) \subseteq M_n\big(\mathcal{B}(\mathcal{K})\big)_+$$

for all $n = 1, 2, \ldots$.

Lemma 2.4. *Let Φ be a linear map from $T(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$. Then we have the following:*

(a) *$\Phi \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(K))$ if and only if $t \circ \Phi \in \mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$ for the transpose map t of $\mathcal{B}(\mathcal{K})$ in any orthonormal basis of $\mathcal{K}$;*

(b) *$\Phi \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(K))$ if and only if $\Phi \circ t' \in \mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$, where t' is a transpose map for $\mathcal{B}(\mathcal{H})$.*

Proof. (a) For necessity, let $\Phi \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ and $(B_{ij})_{i,j} \in M_n(T(\mathcal{H}))^+$. We need to show that

$$\sum_{i,j=1}^{n} e_{ij} \otimes \big(\Phi(B_{ij})\big)^t = \big(t \circ \Phi(B_{ij})\big)_{i,j} \in M_n\big(\mathcal{B}(\mathcal{K})\big)^+, \tag{2.3}$$

which is equivalent to

$$\sum_{i,j=1}^{n} e_{ji} \otimes \Phi(B_{ij}) = (\acute{t} \otimes t)\Big[\sum_{i,j=1}^{n} e_{ij} \otimes \big(\Phi(B_{ij})\big)^t\Big] \in M_n\big(\mathcal{B}(\mathcal{K})\big)^+. \tag{2.4}$$

According to Lemma 2.3, we get that $(B_{ij})_{i,j} \in M_n(T(\mathcal{H}))^+$ implies that

$$\sum_{i,j=1}^{n} e_{ji} \otimes B_{ij} \in M_n\big(T(\mathcal{H})\big)_+,$$

which yields

$$\sum_{i,j=1}^{n} e_{ji} \otimes \Phi(B_{ij}) \in M_n\big(\mathcal{B}(\mathcal{K})\big)^+.$$

Then by the arbitrariness of n, we get

$$t \circ \Phi \in \mathrm{CP}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big).$$

Sufficiency is clear from the converse direction of the above proof. The proof of (b) is similar to (a). $\square$

The following lemma (see [7]) seems to be natural and its proof is direct.

Lemma 2.5 ([7, Lemma 2.1])**.** *Let*

$$\widetilde{A} := \sum_{i,j=1}^{\infty} e_{ij} \otimes A_{ij} := \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} & \cdots \\ A_{21} & A_{22} & \cdots & A_{2m} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}$$

and let $M > 0$, where all $A_{ij} \in \mathcal{B}(\mathcal{H})$. If $n \times n$ operator matrices $\|(A_{ij})_{i,j}\| \leq M$ for all positive integers n, then $\widetilde{A}$ is densely defined on $l^2(\mathcal{H})$ and can be uniquely extended to an operator $\widetilde{B} \in \mathcal{B}(l^2(\mathcal{H}))$ with $\|\widetilde{B}\| \leq M$.

Lemma 2.6. *Let $(A_{ij})_{i,j} \in M_n(T(\mathcal{H}))$ and $(B_{ij})_{i,j} \in M_n(\mathcal{B}(\mathcal{H}))$. Then*

(a) *$\|(A_{ij})_{i,j}\|'_n \leq \|(A_{ij})_{i,j}\|_1$;*
(b) *the following holds:*

$$\big\|(B_{ij})_{i,j}\big\| = \sup\Big\{\Big\|\Big(\big(\operatorname{tr}(B_{ij}T_{kl})\big)_{ij}\Big)_{kl}\Big\| : \text{all } T_{kl} \in T(\mathcal{H}) \text{ with } \big\|(T_{kl})_{k,l}\big\|'_n \leq 1\Big\}. \tag{2.5}$$

Proof. (a) As $K(\mathcal{H})$ with operator norm $\|\cdot\|$ is an operator space, [3, Lemma 4.1.1] and [3, (4.1.6)] imply that the first inequality $\|(A_{ij})_{n\times n}\|'_n \leq \|(A_{ij})_{i,j}\|_1$. Also, (b) follows directly from [3, Theorem 3.2.3]. ☐

Lemma 2.7. *Let $X \in T(\mathcal{H})$ and $B_{ij} \in \mathcal{B}(\mathcal{H})$, for $i, j = 1, 2, \ldots, n$. Then*

$$\big\|\big(\operatorname{tr}(XB_{ij})\big)_{i,j}\big\| \leq \|X\|_1 \big\|(B_{ij})_{i,j}\big\|.$$

Proof. By defining the linear function for $\mathcal{B}(\mathcal{H})$ by $f(Y) := \operatorname{tr}(XY)$, we conclude from [3, Corollary 2.2.3] that f is completely bounded with $\|f\|_{\text{cb}} = \|f\| = \|X\|_1$. Thus,

$$\big\|\big(\operatorname{tr}(B_{ij}X)\big)_{i,j}\big\| \leq \|f\|_{\text{cb}}\big\|(B_{ij})_{i,j}\big\| = \|X\|_1\big\|(B_{ij})_{i,j}\big\|.$$ ☐

Lemma 2.8 ([8, Lemma 3.3])**.** *Let $P_n, A \in \mathcal{B}(\mathcal{H})$, $n = 1, 2, \ldots$. If P_n is an orthogonal projection such that P_n converges to the unit operator I in the weak operator topology, then $A \geq 0$ if and only if $P_nAP_n \geq 0$ for all $n = 1, 2, \ldots$.*

Theorem 2.9. *Let $\Phi \in \operatorname{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$. Then there exist completely positive maps $\Phi_i \in \operatorname{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ for $i = 1, 2, 3, 4$ such that $\Phi = (\Phi_1 - \Phi_2) + \sqrt{-1}(\Phi_3 - \Phi_4)$.*

Proof. Let $\Phi \in \operatorname{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$, and let $\{f_i\}_{i=1}^{\infty}$ be the orthonormal basis of $\mathcal{K}$. Denote

$$\Phi_{ij}(X) = \big\langle \Phi(X)f_j, f_i\big\rangle \quad \text{for } X \in T(\mathcal{H}), i, j = 1, 2, \ldots. \tag{2.6}$$

Then Φ_{ij} is a bounded linear functional on $T(\mathcal{H})$, so there exist operators $T_{ij} \in \mathcal{B}(\mathcal{H})$ such that

$$\Phi_{ij}(X) = \operatorname{tr}(T_{ij}X) \quad \text{for } i, j = 1, 2, \ldots.$$

Define

$$T=\sum_{i,j=1}^{\infty} f_{ij}\otimes T_{ij}=\begin{pmatrix} T_{11} & \cdots & T_{1m} & \cdots \\ \vdots & \ddots & \vdots & \cdots \\ T_{m1} & \cdots & T_{mm} & \cdots \\ \vdots & \cdots & \vdots & \ddots \end{pmatrix}. \tag{2.7}$$

Suppose that P_n is the rank n orthogonal projection spanned by the orthonormal vectors $\{f_i\}_{i=1}^n$, for $n=1,2,\ldots$. Then $P_n\Phi P_n\in \mathrm{CB}(K(\mathcal{H})^*,\mathcal{B}(\mathcal{K}))$ and for $X\in T(\mathcal{H})$,

$$P_n\Phi(X)P_n=\sum_{i,j=1}^{n} f_{ij}\otimes \Phi_{ij}(X)\simeq \begin{pmatrix} \mathrm{tr}(T_{11}X) & \cdots & \mathrm{tr}(T_{1n}X) \\ \vdots & \ddots & \vdots \\ \mathrm{tr}(T_{n1}X) & \cdots & \mathrm{tr}(T_{nn}X) \end{pmatrix}.$$

Thus Lemma 2.6(b) implies that for $n=1,2,\ldots,$

$$\begin{aligned} &\big\|(T_{i,j})_{i,j}\big\| \\ &\quad=\sup\Big\{\Big\|\Big(\big(\mathrm{tr}(T_{ij}S_{kl})\big)_{i,j}\Big)_{k,l}\Big\| : \text{all } S_{kl}\in T(\mathcal{H}) \text{ with } \big\|(S_{kl})_{k,l}\big\|_n'\le 1\Big\} \\ &\quad=\sup\Big\{\Big\|\big(P_n\Phi(S_{kl})P_n\big)_{k,l}\Big\| : \text{all } S_{kl}\in T(\mathcal{H}) \text{ with } \big\|(S_{kl})_{k,l}\big\|_n'\le 1\Big\} \\ &\quad\le \big\|P_n\Phi(\cdot)P_n\big\|_{\mathrm{cb}} \\ &\quad\le \|\Phi\|_{\mathrm{cb}}. \end{aligned} \tag{2.8}$$

Using Lemma 2.5, we get that T is densely defined on $\mathcal{K}\otimes\mathcal{H}$ and can be uniquely extended to $\widetilde{T}\in\mathcal{B}(\mathcal{K}\otimes\mathcal{H})$. Clearly, $\widetilde{T}=(\widetilde{B_1}-\widetilde{B_2})+\sqrt{-1}(\widetilde{B_3}-\widetilde{B_4})$, where $\widetilde{B_s}\in\mathcal{B}(\mathcal{K}\otimes\mathcal{H})^+$. Also, we write $\widetilde{B_s}=\sum_{i,j=1}^{\infty} f_{ij}\otimes B_{ij}^{(s)}$, for $s=1,2,3,4$, where $\{f_{ij}\}_{i,j=1}^{\infty}$ are the matrix units corresponding to the orthonormal basis $\{f_i\}_{i=1}^{\infty}$, so

$$\begin{aligned} \Phi_{ij}(X)&=\mathrm{tr}(T_{ij}X) \\ &=\mathrm{tr}(B_{ij}^{(1)}X)-\mathrm{tr}(B_{ij}^{(2)}X)+\sqrt{-1}\big[\mathrm{tr}(B_{ij}^{(3)}X)-\mathrm{tr}(B_{ij}^{(4)}X)\big], \end{aligned} \tag{2.9}$$

where $X\in T(\mathcal{H})$ and $i,j=1,2,\ldots$.

By Lemma 2.7, for $n=1,2,\ldots$ and $s=1,2,3,4$, we have

$$\Big\|\sum_{i,j=1}^{n} f_{ij}\otimes \mathrm{tr}(B_{ij}^{(s)}X)\Big\|\le \big\|(B_{ij}^{(s)})_{i,j}\big\|\|X\|_1\le \|\widetilde{B_s}\|\|X\|_1,$$

and so $\sum_{i,j=1}^{\infty} f_{ij}\otimes \mathrm{tr}(B_{ij}^{(s)}X)\in\mathcal{B}(\mathcal{K})$ for all $X\in T(\mathcal{H})$. That is, for $s=1,2,3,4$, the map

$$\Phi_s(X):=\sum_{i,j=1}^{\infty} f_{ij}\otimes \mathrm{tr}(B_{ij}^{(s)}X) \tag{2.10}$$

is well defined from $T(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$.

Let $(A_{ij})_{i,j} \in M_m(T(\mathcal{H}))^+$. Then Lemma 2.2 and the fact that $(B_{ij}^{(s)})_{i,j} \in M_n(\mathcal{B}(\mathcal{H}))^+$ for $s = 1, 2, 3, 4$ imply that

$$\begin{aligned}\sum_{k,l=1}^{m} \big(P_n \big[\Phi_s(A_{lk})\big]^t P_n\big) \otimes e_{lk} &= \sum_{k,l=1}^{m} \big[P_n \Phi_s(A_{lk}) P_n\big]^t \otimes e_{lk} \\ &= \sum_{i,j=1}^{n} \sum_{k,l=1}^{m} \operatorname{tr}(B_{ij}^{(s)} A_{lk}) f_{ji} \otimes e_{lk} \\ &\simeq \sum_{i,j=1}^{n} \sum_{k,l=1}^{m} \operatorname{tr}(B_{ij}^{(s)} A_{lk}) e_{ji} \otimes e_{lk} \in M_{mn}^+, \quad (2.11)\end{aligned}$$

for all $n = 1, 2, \ldots$. Thus $P_n(t \circ \Phi_s)P_n \in \mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$, which yields $t \circ \Phi_s \in \mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$ from Lemma 2.8. Then by Lemma 2.4, we can directly get $\Phi_s \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ for $s = 1, 2, 3, 4$. Obviously, equations (2.6), (2.9), and (2.10) induce $\Phi = (\Phi_1 - \Phi_2) + \sqrt{-1}(\Phi_3 - \Phi_4)$. □

Theorem 2.10.

(a) *We have* $\mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{H})) \nsubseteq \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))$.
(b) *We have* $\mathrm{CP}(K(\mathcal{H})^*, T(\mathcal{K})^*) = \mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{K})) \subseteq \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$.
(c) *We have* $\mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K})) \subseteq \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K})) \cap t \circ \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$, *where t is the transpose map in an orthonormal basis of $\mathcal{K}$ and* $t \circ \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K})) = \{t \circ \Phi : \Phi \in \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))\}$.

Proof. (a) Let t' be the transpose map in the orthonormal basis $\{f_i\}_{i=1}^{\infty}$ of $\mathcal{H}$. Then Lemma 2.4 implies that $t' \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))$, as the identity $i = t' \circ t' \in \mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{H}))$.

For any positive integer m, setting $\widetilde{B} := \sum_{i,j=1}^{m} e_{ij} \otimes f_{ji} \in M_m(T(\mathcal{H}))$, we claim that $\|\widetilde{B}\|'_m \leq 1$. Indeed, suppose that $(S_{ij})_{i,j} \in M_m(K(\mathcal{H}))$ with $\|(S_{ij})_{i,j}\| \leq 1$. Then we get

$$\begin{aligned}\Big\|\sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \operatorname{tr}(f_{ji} S_{kl}) e_{ij} \otimes e_{kl}\Big\| &= \Big\|\sum_{k,l=1}^{n} \Big(\sum_{i,j=1}^{n} \operatorname{tr}(S_{kl} f_{ji}) e_{ij}\Big) \otimes e_{kl}\Big\| \\ &= \Big\|\sum_{k,l=1}^{n} \Big(\sum_{i,j=1}^{n} \operatorname{tr}(S_{kl} f_{ji}) f_{ij}\Big) \otimes e_{kl}\Big\| \\ &\leq \big\|(S_{ij})_{i,j}\big\| \leq 1,\end{aligned}$$

which yields $\|\widetilde{B}\|'_m \leq 1$. However,

$$\big\|t'_m(\widetilde{B})\big\| = \big\|(f_{ij})_{i,j}\big\| = m,$$

so $\|t'_m\| \geq m$, which implies that $t' \notin \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))$.

(b) It is easy to see that $(\Phi(B_{ij}))_{i,j} \in M_n(\mathcal{B}(\mathcal{H}))^+$ for all $(B_{ij})_{i,j} \in M_n(T(\mathcal{H}))^+$ if and only if $(\Phi(C_{ij}))_{i,j} \in M_n(\mathcal{B}(\mathcal{H}))_+$ for all $(C_{ij})_{i,j} \in M_n(T(\mathcal{H}))_+$. Thus

$$\mathrm{CP}\big(K(\mathcal{H})^*, T(\mathcal{K})^*\big) = \mathrm{CP}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big).$$

The inclusion relation is in [7, Lemma 2.5]. For the reader's convenience, we give the proof here.

Let $\Phi \in \mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$ and $\widetilde{T} := (T_{ij})_{i,j} \in M_n(T(\mathcal{H}))^+$. Then by a similar way as [8, Lemma 2.3], we get

$$\|\Phi\| := \sup\big\{\big\|\Phi(A)\big\| : A \in T(\mathcal{H}) \text{ with } \|A\|_1 \leq 1\big\} < \infty.$$

Furthermore, [6, Theorem 2.3] and the fact that $(\Phi(T_{ij}))_{i,j} \in M_n(\mathcal{B}(\mathcal{K}))^+$ imply that

$$\big\|\big(\Phi(T_{ij})\big)_{i,j}\big\| \leq \sum_{i=1}^{n}\big\|\Phi(T_{ii})\big\| \leq \|\Phi\| \sum_{i=1}^{n} \|T_{ii}\|_1 = \|\Phi\|\|\widetilde{T}\|_1. \tag{2.12}$$

For a general $\widetilde{S} \in M_n(T(\mathcal{H}))$, we also write $\widetilde{S} = S_1 - S_2 + \sqrt{-1}(S_3 - S_4)$, where $S_i \in M_n(T(\mathcal{H}))^+$ for $i = 1, 2, 3, 4$ satisfy that

$$\|S_1\|_1 + \|S_2\|_1 = \|S_1 - S_2\|_1 \leq \|\widetilde{S}\|_1$$

and

$$\|S_3\|_1 + \|S_4\|_1 = \|S_3 - S_4\|_1 \leq \|\widetilde{S}\|_1.$$

Thus inequality (2.12) implies that

$$\begin{aligned} \big\|\Phi_n(\widetilde{S})\big\| &\leq \sum_{k=1}^{4}\big\|\Phi_n(S_k)\big\| \\ &\leq \sum_{k=1}^{4} \|S_k\|_1\|\Phi\| \\ &\leq 2\|\Phi\|\|\widetilde{S}\|_1, \end{aligned} \tag{2.13}$$

which yields

$$\|\Phi\|_{\mathrm{cb}} = \sup\big\{\|\Phi_n\| : n = 1, 2, \ldots\big\} \leq 2\|\Phi\|, \tag{2.14}$$

and so $\mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{K})) \subseteq \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$.

(c) Let $\Phi \in \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$. Then for any positive integer n and $(T_{ij})_{i,j} \in M_n(T(\mathcal{H}))$, we conclude from Lemma 2.6(a) that

$$\big\|\big(\Phi(T_{ij})\big)_{i,j}\big\| \leq \|\Phi\|_{\mathrm{cb}}\big\|(T_{ij})_{i,j}\big\|_n' \leq \|\Phi\|_{\mathrm{cb}}\big\|(T_{ij})_{i,j}\big\|_1,$$

and so

$$\Phi \in \mathrm{CB}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big).$$

Using Theorem 2.9, we get that

$$\Phi = (\Phi_1 - \Phi_2) + \sqrt{-1}(\Phi_3 - \Phi_4),$$

where $\Phi_i \in \mathrm{CP}(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K}))$ for $i = 1, 2, 3, 4$. Thus Lemma 2.4 and (b) imply that

$$t \circ \Phi_i \in \mathrm{CP}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big) \subseteq \mathrm{CB}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big) \quad \text{for } i = 1, 2, 3, 4.$$

Then

$$t \circ \Phi = (t \circ \Phi_1 - t \circ \Phi_2) + \sqrt{-1}(t \circ \Phi_3 - t \circ \Phi_4) \in \mathrm{CB}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big)$$

yields $\Phi \in t \circ \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$. $\square$

Remark 2.11. (a) Lemma 2.4 and Theorem 2.10(c) show that

$$\Phi \in \mathrm{CP}\big(K(\mathcal{H})^*, \mathcal{B}(K)\big) \quad \Longleftrightarrow \quad t \circ \Phi \in \mathrm{CP}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big)$$

and that

$$\Phi \in \mathrm{CB}\big(K(\mathcal{H})^*, \mathcal{B}(K)\big) \quad \Longrightarrow \quad t \circ \Phi \in \mathrm{CB}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{H})\big).$$

We claim that $t \circ \Phi \in \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ does not imply that $\Phi \in \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(H))$. Indeed, the identity map i from $T(\mathcal{H})$ onto $T(\mathcal{H})$ belongs to $\mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{H}))$, so $t \circ t = i \in \mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{H}))$. However, Theorem 2.10 says that $t \notin \mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(H))$.

(b) Using the advanced operator space theory (the projective tensor product and normal spatial tensor product), there is a completely isometric isomorphism

$$\mathrm{CB}\big(K(\mathcal{H})^*, \mathcal{B}(K)\big) = \big(K(\mathcal{H})^* \mathbin{\widehat{\otimes}} K(\mathcal{K})^*\big)^* = \mathcal{B}(\mathcal{H}) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{K}) = \mathcal{B}(\mathcal{H} \otimes \mathcal{K}),$$

where $K(\mathcal{H})^* \mathbin{\widehat{\otimes}} K(\mathcal{K})^*$ is the projective tensor product of operator spaces $K(\mathcal{H})^*$ and $K(\mathcal{K})^*$, and where $\mathcal{B}(\mathcal{H}) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{K})$ is the usual spatial von Neumann algebraic tensor product of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$. Let Λ be this completely isometric isomorphism from $\mathrm{CB}(K(\mathcal{H})^*, \mathcal{B}(K))$ onto $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. The inclusion relation

$$\Lambda\big(\mathrm{CP}\big(K(\mathcal{H})^*, \mathcal{B}(K)\big)\big) \supseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{K})^+$$

seems true. However,

$$\Lambda\big(\mathrm{CP}\big(K(\mathcal{H})^*, \mathcal{B}(K)\big)\big) \nsubseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{K})^+,$$

since

$$\mathrm{CP}\big(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K})\big) \nsubseteq \mathrm{CB}\big(K(\mathcal{H})^*, \mathcal{B}(\mathcal{K})\big).$$

(c) It seems difficult for us to determine whether a similar decomposition between $\mathrm{CP}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$ and $\mathrm{CB}(T(\mathcal{H}), \mathcal{B}(\mathcal{K}))$ exists, although

$$\mathrm{CP}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big) \subseteq \mathrm{CB}\big(T(\mathcal{H}), \mathcal{B}(\mathcal{K})\big).$$

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