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PRODUCT OF QUASIHOMOGENEOUS TOEPLITZ OPERATORS ON THE PLURIHARMONIC BERGMAN SPACE OF THE POLYDISK

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Abstract. In this article, we first give an essential characterization of Toeplitz operators with quasihomogeneous symbols on the weighted pluriharmonic Bergman space of the unit polydisk. Then we completely characterize when the product of two Toeplitz operators with monomial-type symbols is a Toeplitz operator. As a result, some interesting higher-dimensional phenomena appear on the unit polydisk.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and let $\mathbb{D}^n$ be the unit polydisk in the complex vector space $\mathbb{C}^n$. For $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_j > -1$ and $z = (z_1, \dots, z_n) \in \mathbb{D}^n$, we write

$$dv_\lambda(z) = \prod_{j=1}^{n} (\lambda_j + 1)\big(1 - |z_j|^2\big)^{\lambda_j}\, dA(z_j),$$

where dA denotes the normalized area measure on $\mathbb{D}$. The weighted pluriharmonic Bergman space of the unit polydisk $b^2_\lambda(\mathbb{D}^n)$ is the closed subspace of $L^2(\mathbb{D}^n, dv_\lambda)$ consisting of all pluriharmonic functions on $\mathbb{D}^n$. For $f \in L^\infty(\mathbb{D}^n, dv_\lambda)$, the Toeplitz

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operator T_f with symbol f on $b^2_\lambda(\mathbb{D}^n)$ is defined by

$$T_f(h) = Q_\lambda(fh), \quad h \in b^2_\lambda(\mathbb{D}^n),$$

where Q_λ is the orthogonal projection from $L^2(\mathbb{D}^n, d\upsilon_\lambda)$ onto $b^2_\lambda(\mathbb{D}^n)$.

In this article, we are concerned with the problem of when the product of two quasihomogeneous Toeplitz operators on $b^2_\lambda(\mathbb{D}^n)$ is a Toeplitz operator. Recall that a bounded function f on $\mathbb{D}^n$ is called *quasihomogeneous* if it has the form

$$f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}) = e^{ip\cdot\theta}\varphi(r) := e^{i(p_1\theta_1+\cdots+p_n\theta_n)}\varphi(r_1, \dots, r_n),$$

where $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$ and $\varphi(r) = \varphi(r_1, \dots, r_n)$ is a separately radial function. In this case, the associated Toeplitz operator T_f is called a *quasihomogeneous Toeplitz operator* of degree p. Now, if $\mathbb{R}_+$ denotes the set of all nonnegative real numbers and $l = (l_1, \dots, l_n) \in \mathbb{R}^n_+$, then the quasihomogeneous function $r^l e^{ip\cdot\theta} = r_1^{l_1} \cdots r_n^{l_n} e^{i(p_1\theta_1+\cdots+p_n\theta_n)}$ is said to be of *monomial type*. Obviously, if both $l+p$ and $l-p \in (2\mathbb{N})^n$, then $r^l e^{ip\cdot\theta}$ is just the ordinary monomial $z^{(l+p)/2}\overline{z}^{(l-p)/2}$.

The problem of characterizing when the product of two Toeplitz operators is another Toeplitz operator has been studied for many years on various classical function spaces. In 1963, Brown and Halmos [2] proved that $T_f T_g$ is a Toeplitz operator on the Hardy space of the unit disk if and only if either $\overline{f}$ or g is analytic. For the Bergman space of the unit disk, Ahern and Čučković [1] showed that a Brown–Halmos-type result holds for Toeplitz operators with harmonic symbols. Later, Louhichi, Strouse, and Zakariasy [13] gave necessary and sufficient conditions for the product of two quasihomogeneous Toeplitz operators to be a Toeplitz operator. (For related results on the harmonic Bergman space, see [3], [6]–[8], [14].)

In the setting of the unit polydisk, the problem is known to be much more delicate and somewhat challenging. For example, $T_f T_g$ is a Toeplitz operator on the Hardy space of the unit polydisk if and only if $T_f T_g = T_{fg}$ and, for each $i \in \{1, 2, \dots, n\}$, either $\overline{f(z)}$ or $g(z)$ is analytic in z_i (see [5], [9]). Later, similar results were obtained on the Bergman space of the unit polydisk by Choe, Lee, Nam, and Zheng [4] with pluriharmonic symbols. Just recently, in [11], the current authors completely characterized finite-rank commutators and semicommutators of two monomial-type Toeplitz operators on the weighted Bergman space and weighted pluriharmonic Bergman space of the unit polydisk. In this article, we extend the results of [6] to the polydisk; namely, we first give an essential characterization of the quasihomogeneous Toeplitz operator on $b^2_\lambda(\mathbb{D}^n)$, and then we use it to study when the product of two quasihomogeneous Toeplitz operators is a Toeplitz operator.

For an n-tuple $k = (k_1, \dots, k_n) \in \mathbb{N}^n \cup (-\mathbb{N})^n$, we denote $|k| = (|k_1|, \dots, |k_n|)$. Note that $z^k = r^{|k|}e^{ik\cdot\theta}$ if $k \in \mathbb{N}^n$ and that $\overline{z}^{|k|} = r^{|k|}e^{ik\cdot\theta}$ if $k \in (-\mathbb{N})^n$, so

$$\Big\{\prod_{j=1}^n \sqrt{\frac{\Gamma(|k_j| + \lambda_j + 2)}{\Gamma(\lambda_j + 2)|k_j|!}}\, r^{|k|}e^{ik\cdot\theta}\Big\}_{k\in\mathbb{N}^n\cup(-\mathbb{N})^n} \tag{1.1}$$

is an orthonormal basis for $b^2_\lambda(\mathbb{D}^n)$. Thus Lemma 2.1 implies that quasihomogeneous Toeplitz operators are natural analogues of the classical bilateral shift operators. Moreover, quasihomogeneous Toeplitz operators on $b^2_\lambda(\mathbb{D}^n)$ have very interesting structure and enjoy some meaningful properties. We first give the following essential result.

Theorem 1.1. *Let $p \in \mathbb{Z}^n$, and let f be a bounded function on $\mathbb{D}^n$. Then f is a quasihomogeneous function of degree p if and only if the following equivalent conditions hold:*

(a) *there exist $(C_{p,k})_{k\in\mathbb{N}^n}$ such that $T_f(z^k) = C_{p,k} r^{|k+p|} e^{i(k+p)\cdot\theta}$ with $C_{p,k} = 0$ if $k + p \notin \mathbb{N}^n \cup (-\mathbb{N})^n$,*

(b) *there exist $(C'_{p,k})_{k\in\mathbb{N}^n}$ such that $T_f(\overline{z}^k) = C'_{p,k} r^{|k-p|} e^{i(-k+p)\cdot\theta}$ with $C'_{p,k} = 0$ if $-k + p \notin \mathbb{N}^n \cup (-\mathbb{N})^n$.*

Clearly, Theorem 1.1 shows that a Toeplitz operator is quasihomogeneous on $b^2_\lambda(\mathbb{D}^n)$ if and only if it is a weighted shift operator when applied to the holomorphic part or the conjugate holomorphic part of the orthonormal basis for $b^2_\lambda(\mathbb{D}^n)$. Next, we show a necessary condition for the product of two quasihomogeneous Toeplitz operators on $b^2_\lambda(\mathbb{D}^n)$ to be a Toeplitz operator.

Theorem 1.2. *Let $p, q \in \mathbb{Z}^n$, and let f and g be two bounded quasihomogeneous functions on $\mathbb{D}^n$ of degrees p and q, respectively. If there exists a bounded function h such that*

$$T_f T_g = T_h,$$

then h is quasihomogeneous of degree $p + q$.

To state our last result, we need the following notation and definition (see [11]). For two multi-indexes $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ and $q = (q_1, \ldots, q_n) \in \mathbb{Z}^n$, the notation $p \succeq q$ means that $p_j \geq q_j$, $j = 1, \ldots, n$. Let $l_i, m_i \in \mathbb{R}_+$ and let $p_i, q_i \in \mathbb{Z}$. Then we say that a tuple (l_i, p_i, m_i, q_i) satisfies [11, Introduction, Condition (J)] if at least one of the following conditions holds:

(i) $l_i = p_i = 0$ (i.e., the function $r^l e^{ip\cdot\theta}$ is a constant in z_i),
(ii) $m_i = q_i = 0$ (i.e., the function $r^m e^{iq\cdot\theta}$ is a constant in z_i),
(iii) $l_i = p_i$ and $m_i = q_i$ (i.e., both $r^l e^{ip\cdot\theta}$ and $r^m e^{iq\cdot\theta}$ are analytic in z_i),
(iv) $l_i = -p_i$ and $m_i = -q_i$ (i.e., both $r^l e^{ip\cdot\theta}$ and $r^m e^{iq\cdot\theta}$ are coanalytic in z_i),
(v) $p_i = q_i = 0$ (i.e., both $r^l e^{ip\cdot\theta}$ and $r^m e^{iq\cdot\theta}$ are radial in z_i),
(vi) $l_i = m_i$ and $p_i = q_i$ (i.e., $r^l e^{ip\cdot\theta}$ is identically equal to $r^m e^{iq\cdot\theta}$ in z_i).

The following theorem completely characterizes when the product of two monomial-type Toeplitz operators on $b^2_\lambda(\mathbb{D}^n)$ is a Toeplitz operator.

Theorem 1.3. *Let $l, m \in \mathbb{R}^n_+$, and let $p, q \in \mathbb{Z}^n$. Then the following statements are equivalent.*

(a) *The product $T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}}$ is equal to a Toeplitz operator T_h.*
(b) *The product $T_{r^m e^{iq\cdot\theta}} T_{r^l e^{ip\cdot\theta}}$ is equal to a Toeplitz operator T_h.*
(c) *The following two conditions hold.*
  (c1) *The symbol $h = e^{i(p+q)\cdot\theta} \prod_{i=1}^n \psi_i(r_i)$, where the ψ_i's are the (bounded) functions*

$$\psi_i(r_i) = \begin{cases} \frac{l_i+p_i}{l_i+p_i-m_i+q_i} r_i^{l_i+q_i} - \frac{m_i-q_i}{l_i+p_i-m_i+q_i} r_i^{m_i-p_i} & \text{if } l_i + p_i \neq m_i - q_i, \\ r_i^{l_i+q_i}(1 + (l_i + p_i)\log r_i) & \text{if } l_i + p_i = m_i - q_i. \end{cases}$$

(c2) *One of the following conditions holds.*
- *For each* $i \in \{1, 2, \dots, n\}$, *either* $p_i = l_i = 0$ *or* $p_i = q_i = 0$.
- *For each* $i \in \{1, 2, \dots, n\}$, *either* $q_i = m_i = 0$ *or* $p_i = q_i = 0$.
- *Neither* $p \preceq 0$, $q \preceq 0$ *nor* $p \succeq 0$, $q \succeq 0$, *and for each* $i \in \{1, 2, \dots, n\}$, (l_i, p_i, m_i, q_i) *satisfies [11, Introduction, Condition (J)].*

In the case when $n = 1$, Theorem 1.3 shows that the product of two monomial-type Toeplitz operators is a Toeplitz operator on the weighted harmonic Bergman space of the unit disk only when one of the following conditions holds:

(I) one of the two operators is the identity operator;
(II) both operators are diagonal (induced by radial symbols).

The two conditions above are often called the *trivial* (or *obvious*) *cases* (see [6] for this 1-dimensional result). However, our method, whose main idea is adapted from [11], is entirely different from that of [6].

When $n > 1$, Theorem 1.3 produces lots of nontrivial cases when the product of two monomial-type Toeplitz operators is a Toeplitz operator. For example, $T_{z_1|z_2|^2}T_{|z_2|^2} = T_{z_1|z_2|^2(1+2\log|z_2|)}$ on $b^2_\lambda(\mathbb{D}^n)$. (See Examples 3.3 and 3.4 for more complicated cases.) As a consequence, some interesting new phenomena appear in operator theory on the unit polydisk.

2. Basic results of quasihomogeneous Toeplitz operators

In order to prove our main results about quasihomogeneous Toeplitz operators on $b^2_\lambda(\mathbb{D}^n)$, we first recall some standard notation in this section. Let $A^2_\lambda(\mathbb{D}^n)$ be a Bergman space over the polydisk, which is the closed subspace consisting of all analytic functions in $L^2(\mathbb{D}^n, dv_\lambda)$. The reproducing kernel of $A^2_\lambda(\mathbb{D}^n)$ is

$$K_z^{(\lambda)}(w) = \prod_{j=1}^n \frac{1}{(1 - w_j\overline{z_j})^{\lambda_j+2}} = \prod_{j=1}^n \sum_{k_j=0}^\infty \frac{\Gamma(\lambda_j + k_j + 2)}{\Gamma(\lambda_j + 2)k_j!} w_j^{k_j}\overline{z_j}^{k_j}, \quad z, w \in \mathbb{D}^n,$$

where each component is the reproducing kernel of Bergman space over the unit disk (see [10]). Since $b^2_\lambda(\mathbb{D}^n) = \overline{A^2_\lambda(\mathbb{D}^n)} + A^2_\lambda(\mathbb{D}^n)$, the reproducing kernel $R_z^{(\lambda)}$ in $b^2_\lambda(\mathbb{D}^n)$ is given by

$$R_z^{(\lambda)} = K_z^{(\lambda)} + \overline{K_z^{(\lambda)}} - 1.$$

Therefore, the projection Q_λ can be represented by

$$Q_\lambda(f)(z) = \left\langle Q_\lambda(f), R_z^{(\lambda)} \right\rangle = \langle f, K_z^{(\lambda)} + \overline{K_z^{(\lambda)}} - 1 \rangle,$$

where $\langle \, , \, \rangle$ denotes the inner product in $L^2(\mathbb{D}^n, dv_\lambda)$.

Let $\varphi \in L^1(\mathbb{D}^n, dv_\lambda)$ be a separately radial function. Then we define the Mellin transform $\widehat{\varphi}$ of the function φ by

$$\widehat{\varphi}(z) = \int_0^1 \cdots \int_0^1 \varphi(r_1, \dots, r_n) r_1^{z_1-1} \cdots r_n^{z_n-1} \prod_{j=1}^n (1 - r_j^2)^{\lambda_j}\, dr_j.$$

It is clear that $\widehat{\varphi}$ is well defined on $\{z \in \mathbb{C}^n : \operatorname{Re} z_j \geq 2, j = 1, \ldots, n\}$ and is holomorphic on $\{z \in \mathbb{C}^n : \operatorname{Re} z_j > 2, j = 1, \ldots, n\}$. For another separately radial function $\psi \in L^1(\mathbb{D}^n, dv_\lambda)$, the multiplicative convolution is defined by

$$(\varphi *_M \psi)(r_1, \ldots, r_n) = \int_{r_1}^1 \cdots \int_{r_n}^1 \varphi\Big(\frac{r_1}{t_1}, \ldots, \frac{r_n}{t_n}\Big)\psi(t_1, \ldots, t_n) \prod_{i=1}^n \frac{dt_i}{t_i}.$$

Then it is easy to see that

$$\widehat{\varphi *_M \psi}(z) = \widehat{\varphi}(z)\widehat{\psi}(z). \tag{2.1}$$

A direct calculation gives the following lemma (see [11, Lemma 3.2]).

Lemma 2.1. *Let $p \in \mathbb{Z}^n$, let $k \in \mathbb{N}^n \cup (-\mathbb{N})^n$, and let φ be a bounded separately radial function on $\mathbb{D}^n$. Then*

$$T_{e^{ip\cdot\theta}\varphi}(r^{|k|}e^{ik\cdot\theta}) = \begin{cases} \tau_{p,k} r^{|k+p|}e^{i(k+p)\cdot\theta} & \text{if } k+p \in \mathbb{N}^n \cup (-\mathbb{N})^n, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\tau_{p,k} = \begin{cases} \prod_{j=1}^n \frac{2\Gamma(k_j+p_j+\lambda_j+2)}{\Gamma(\lambda_j+1)(k_j+p_j)!}\widehat{\varphi}(2k+p+2) & \text{if } k \succeq 0 \text{ and } k+p \succeq 0, \\ \prod_{j=1}^n \frac{2\Gamma(-k_j-p_j+\lambda_j+2)}{\Gamma(\lambda_j+1)(-k_j-p_j)!}\widehat{\varphi}(-p+2) & \text{if } k \succeq 0 \text{ and } k+p \preceq 0, \\ \prod_{j=1}^n \frac{2\Gamma(-k_j-p_j+\lambda_j+2)}{\Gamma(\lambda_j+1)(-k_j-p_j)!}\widehat{\varphi}(-2k-p+2) & \text{if } k \preceq 0 \text{ and } k+p \preceq 0, \\ \prod_{j=1}^n \frac{2\Gamma(k_j+p_j+\lambda_j+2)}{\Gamma(\lambda_j+1)(k_j+p_j)!}\widehat{\varphi}(p+2) & \text{if } k \preceq 0 \text{ and } k+p \succeq 0. \end{cases}$$

We are now ready to prove Theorem 1.1, which characterizes all quasihomogeneous Toeplitz operators on the weighted pluriharmonic Bergman space of the unit polydisk.

Proof of Theorem 1.1. We fix a $p \in \mathbb{Z}^n$ and a bounded function f. First we prove the equivalence of conditions (a) and (b). Let us assume that condition (a) holds. Fix $k \in \mathbb{N}^n$. Then for any $\beta \in \mathbb{N}^n$, by condition (a) we have

$$\langle T_f\overline{z}^k, z^\beta\rangle = \langle T_f z^0, z^{k+\beta}\rangle = C_{p,0}\langle r^{|p|}e^{ip\cdot\theta}, z^{k+\beta}\rangle \tag{2.2}$$

with $C_{p,0} = 0$ if $p \notin \mathbb{N}^n \cup (-\mathbb{N})^n$, and we have

$$\langle T_f\overline{z}^k, \overline{z}^\beta\rangle = \langle T_f z^\beta, z^k\rangle = C_{p,\beta}\langle r^{|\beta+p|}e^{i(\beta+p)\cdot\theta}, z^k\rangle \tag{2.3}$$

with $C_{p,\beta} = 0$ if $\beta + p \notin \mathbb{N}^n \cup (-\mathbb{N})^n$.

If $-k + p \notin \mathbb{N}^n \cup (-\mathbb{N})^n$, then it follows from (2.2) and (2.3) that $\langle T_f\overline{z}^k, z^\beta\rangle = \langle T_f\overline{z}^k, \overline{z}^\beta\rangle = 0$ for any $\beta \in \mathbb{N}^n$, and hence that $T_f\overline{z}^k = 0$.

If $-k + p \in \mathbb{N}^n \cup (-\mathbb{N})^n$, then by (2.2) and (2.3) we obtain

$$\langle T_f\overline{z}^k, r^{|k'|}e^{ik'\cdot\theta}\rangle = 0$$

for any $k' \in \mathbb{N}^n \cup (-\mathbb{N})^n$ and $k' \neq -k + p$, which implies that $T_f(\overline{z}^k)$ is orthogonal to every element of the basis (1.1) except the one with index $-k + p$, and hence condition (b) holds. The proof that (b) implies (a) follows from the same arguments (we leave the details to the reader).

Now we turn to the proof that f is a quasihomogeneous function of degree p if and only if the equivalent conditions (a) and (b) hold. If f is a quasihomogeneous function of degree p, then $f = e^{ip\cdot\theta}\varphi$ for some separately radial function φ on $\mathbb{D}^n$. As a consequence of Lemma 2.1, we see that both (a) and (b) hold.

Conversely, assuming that conditions (a) and (b) hold, we will show that f is a quasihomogeneous function of degree p. We first prove the result for the special case $p = 0$. For any unitary transformation U of $\mathbb{C}^n$ with a diagonal matrix, that is, $U = \operatorname{diag}\{e^{i\theta_1}, \dots, e^{i\theta_n}\}$, by condition (b) we have

$$\begin{aligned}
T_{f\circ U}(\overline{w}^\alpha)(z) &= T_f(\overline{U^{-1}w})^\alpha(Uz)\\
&= e^{i\alpha_1\theta_1}\cdots e^{i\alpha_n\theta_n}T_f(\overline{w}^\alpha)(Uz)\\
&= e^{i\alpha_1\theta_1}\cdots e^{i\alpha_n\theta_n}C'_{0,\alpha}\overline{w}^\alpha(Uz)\\
&= C'_{0,\alpha}\overline{z}^\alpha = T_f(\overline{w}^\alpha)(z).
\end{aligned}$$

Similarly, it follows from condition (a) that $T_{f\circ U}(w^\alpha)(z) = T_f(w^\alpha)(z)$, and so $T_{f\circ U} = T_f$. Thus $f\circ U = f$, which implies that f is separately radial. This proves the desired results for $p = 0$.

In general, if $p = s - t$, where $s, t \in \mathbb{N}^n$ with $s_1t_1 + \cdots + s_nt_n = 0$, then we consider the function $\varphi(w) = \overline{w}^s w^t f(w)$. Clearly,

$$\begin{aligned}
T_\varphi(\overline{w}^\alpha)(z) &= T_{\overline{w}^s w^t f}(\overline{w}^\alpha)(z)\\
&= \langle f\overline{w}^{\alpha+s}w^t, K_z^{(\lambda)} + \overline{K_z^{(\lambda)}} - 1\rangle\\
&= \langle f\overline{w}^{\alpha+s}, \overline{w}^t\overline{K_z^{(\lambda)}}\rangle + \big\langle w^t, \overline{f}w^{\alpha+s}(K_z^{(\lambda)} - 1)\big\rangle.
\end{aligned}$$

Note that $\overline{w}^t\overline{K_z^{(\lambda)}} \in b^2_\lambda(\mathbb{D}^n)$ and that

$$-\alpha - s + p = -\alpha - t \in (-\mathbb{N})^n,$$

so by condition (b) we have

$$\begin{aligned}
\langle f\overline{w}^{\alpha+s}, \overline{w}^t\overline{K_z^{(\lambda)}}\rangle &= \big\langle T_f(\overline{w}^{\alpha+s}), \overline{w}^t\overline{K_z^{(\lambda)}}\big\rangle\\
&= \Big\langle C'_{p,\alpha+s}\overline{w}^{\alpha+t}, \overline{w}^t\prod_{j=1}^n\sum_{k_j=0}^\infty\frac{\Gamma(\lambda_j + k_j + 2)}{\Gamma(\lambda_j + 2)k_j!}\overline{w_j}^{k_j}z_j^{k_j}\Big\rangle.
\end{aligned}$$

Note also that the above sum converges uniformly in w for each fixed $z \in \mathbb{D}^n$, and hence we can interchange the inner product and the sum. Then as a consequence of the orthogonality of the conjugate holomorphic monomials, we have

$$\begin{aligned}
\langle f\overline{w}^{\alpha+s}, \overline{w}^t\overline{K_z^{(\lambda)}}\rangle &= C'_{p,\alpha+s}\prod_{j=1}^n\frac{\Gamma(\lambda_j + \alpha_j + 2)}{\Gamma(\lambda_j + 2)\alpha_j!}\langle\overline{w}^{\alpha+t}, \overline{w}^{\alpha+t}z^\alpha\rangle\\
&= C'_{p,\alpha+s}\prod_{j=1}^n\frac{\Gamma(\lambda_j + \alpha_j + 2)(\alpha_j + t_j)!}{\Gamma(\lambda_j + \alpha_j + t_j + 2)\alpha_j!}\overline{z}^\alpha.
\end{aligned}$$

Observe that $\alpha + \beta + t \neq t$ for any $\beta \in \mathbb{N}^n$ with $\beta \neq 0$. Then we have

$$\langle w^t, \overline{f}w^{\alpha+s+\beta}\rangle = \big\langle T_f(\overline{w}^{\alpha+s+\beta}), \overline{w}^t\big\rangle = \langle C'_{p,\alpha+s+\beta}\overline{w}^{\alpha+\beta+t}, \overline{w}^t\rangle = 0,$$

which implies that

$$\left\langle w^t, \overline{f} w^{\alpha+s}(K_z^{(\lambda)} - 1)\right\rangle = 0.$$

Therefore

$$T_\varphi \overline{z}^\alpha = \delta'_{0,\alpha} \overline{z}^\alpha, \quad \alpha \in \mathbb{N}^n,$$

for constants $(\delta'_{0,\alpha})_{\alpha \in \mathbb{N}^n}$. Since we have proved the equivalence of (a) and (b) and the result for $p = 0$, we know that the function φ is separately radial, which implies that f is a quasihomogeneous function of degree p. This completes the proof. □

Remark 2.2. By a simple calculation, we obtain from (2.2) and (2.3) that

$$C'_{p,k} = \begin{cases} \frac{C_{p,k-p}\langle z^k, z^k\rangle}{\langle \overline{z}^{k-p}, \overline{z}^{k-p}\rangle} & \text{if } k \succeq p, \\ \frac{C_{p,0}\langle z^p, z^p\rangle}{\langle z^{p-k}, z^{p-k}\rangle} & \text{if } k \preceq p. \end{cases}$$

In particular, $C'_{0,k} = C_{0,k}$. In fact, the above equations can be verified directly by elementary calculations using Lemma 2.1, provided that T_f is a quasihomogeneous Toeplitz operator.

With Theorem 1.1 at hand, it is easy to prove Theorem 1.2.

Proof of Theorem 1.2. Let $f = e^{ip\cdot\theta}\varphi_1$ and let $g = e^{iq\cdot\theta}\varphi_2$, where φ_1 and φ_2 are two separately radial functions. By Lemma 2.1, for each $k \in \mathbb{N}^n$ we have

$$T_f T_g(z^k) = C_{p+q,k} r^{|k+p+q|} e^{i(k+p+q)\cdot\theta},$$

where $C_{p+q,k} = \tau_{q,k}\tau_{p,k+q}$ if both $k+q$ and $k+p+q$ belong to $\mathbb{N}^n \cup (-\mathbb{N})^n$, and $C_{p+q,k} = 0$ if $k+q \notin \mathbb{N}^n \cup (-\mathbb{N})^n$ or $k+p+q \notin \mathbb{N}^n \cup (-\mathbb{N})^n$. Thus by Theorem 1.1, we see that h is a quasihomogeneous function of degree $p+q$, which completes the proof. □

The following corollary gives a complete description of when the product of two Toeplitz operators with separately radial symbols is a Toeplitz operator.

Corollary 2.3. *Let φ_1 and φ_2 be two bounded separately radial functions on $\mathbb{D}^n$. Then $T_{\varphi_1}T_{\varphi_2}$ is equal to a Toeplitz operator if and only if there exists a bounded separately radial function ψ such that ψ is a solution to the multiplicative convolution equation*

$$\mathbb{I} *_M \psi = \varphi_1 *_M \varphi_2,$$

where $\mathbb{I}$ denotes the constant function with value 1. In this case, $T_{\varphi_1}T_{\varphi_2} = T_\psi$.

Proof. Using Theorem 1.2, one sees that if $T_{\varphi_1}T_{\varphi_2}$ is a Toeplitz operator, then this operator is quasihomogeneous of degree 0; namely, $T_{\varphi_1}T_{\varphi_2} = T_\psi$ for some bounded separately radial function ψ. Hence by Lemma 2.1, the equation

$$T_{\varphi_1}T_{\varphi_2}(z^\alpha) = T_\psi(z^\alpha)$$

implies that

$$\Big(\prod_{i=1}^{n} \frac{\Gamma(\alpha_j+1)\Gamma(\lambda_j+1)}{2\Gamma(\alpha_j+\lambda_j+2)}\Big)\widehat{\psi}(2\alpha+2) = \widehat{\varphi_1}(2\alpha+2)\widehat{\varphi_2}(2\alpha+2) \tag{2.4}$$

for all $\alpha \in \mathbb{N}^n$. A direct calculation gives

$$\widehat{\mathbb{I}}(2\alpha+2) = \prod_{i=1}^{n} \frac{\Gamma(\alpha_j+1)\Gamma(\lambda_j+1)}{2\Gamma(\alpha_j+\lambda_j+2)},$$

so the desired result is an immediate consequence of (2.1) and (2.4). Conversely, it is clear that $T_{\varphi_1}T_{\varphi_2}(z^\alpha) = T_\psi(z^\alpha)$ for all $\alpha \in \mathbb{N}^n$. Since φ_1, φ_2 and ψ are quasi-homogeneous functions of degree 0, it follows from Remark 2.2 that $T_{\varphi_1}T_{\varphi_2}(\overline{z}^\alpha) = T_\psi(\overline{z}^\alpha)$, which completes the proof. $\square$

The following corollary shows that there are no idempotent Toeplitz operators with bounded quasihomogeneous symbols other than the obvious ones.

Corollary 2.4. *Let f be a bounded quasihomogeneous function on $\mathbb{D}^n$. Then $T_f^2 = T_f$ if and only if either $f = 0$ or $f = 1$.*

Proof. If $T_f^2 = T_f$, then Theorem 1.2 implies that f is separately radial. It follows from (2.4) that $\widehat{f}(2\alpha+2)[\widehat{f}(2\alpha+2)-\widehat{\mathbb{I}}(2\alpha+2)] = 0$. Then by [12, Proposition 3.2], we have $\widehat{f}(z) = 0$ or $\widehat{f}(z) = \widehat{\mathbb{I}}(z)$ on $\{z \in \mathbb{C}^n : \operatorname{Re} z_j > 2, j = 1, \ldots, n\}$, thus $f = 0$ or $f = 1$. The converse implication is clear. This completes the proof. $\square$

3. Product of monomial-type Toeplitz operators

In this section, we consider monomial-type Toeplitz operators on $b^2_\lambda(\mathbb{D}^n)$ and we study the problem of when the product of two such operators is a Toeplitz operator. For $p, q \in \mathbb{Z}^n$, in what follows we will employ the notation

$$\delta_i = \max\{0, -q_i, -p_i - q_i\}$$

and

$$\delta_i' = \min\{0, -q_i, -p_i - q_i\}$$

for each $i \in \{1, 2, \ldots, n\}$. Obviously, $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ with $\delta + q \succeq 0$ and $\delta + p + q \succeq 0$, and $\delta' = (\delta_1', \ldots, \delta_n') \in (-\mathbb{N})^n$.

Lemma 3.1. *Let $l_i, m_i \in \mathbb{R}_+$, let $p_i, q_i \in \mathbb{Z}$, and assume that (l_i, p_i, m_i, q_i) satisfies [11, Introduction, Condition (J)]. Then we can get the following properties.*

(a) *If $\delta_i > 0$, then $\delta_i = -p_i - q_i$ and $\delta_i' = 0$.*
(b) *If $\delta_i' < 0$, then $\delta_i' = -p_i - q_i$ and $\delta_i = 0$.*
(c) *If $\delta_i = 0$, then $p_i \geq 0$ and $q_i \geq 0$.*
(d) *If $\delta_i' = 0$, then $p_i \leq 0$ and $q_i \leq 0$.*

Proof. Since (l_i, p_i, m_i, q_i) satisfies [11, Introduction, Condition (J)], it follows that

$$p_i = 0 \qquad \text{or} \qquad q_i = 0 \qquad \text{or} \qquad p_i q_i > 0.$$

It is easy to obtain the desired results in each case. This completes the proof. $\square$

In sharp contrast to the Hardy and Bergman space cases, the algebraic properties of Toeplitz operators on the weighted pluriharmonic Bergman space are quite different. Next, we will give a very interesting lemma.

Lemma 3.2. *Let f_1, f_2, and f be bounded functions on $\mathbb{D}^n$. If $T_{f_1}T_{f_2} = T_f$ on $b^2_\lambda(\mathbb{D}^n)$, then $T_{f_2}T_{f_1} = T_f$, and hence the operators T_{f_1} and T_{f_2} commute on $b^2_\lambda(\mathbb{D}^n)$.*

Proof. The proof follows the same arguments as those found in [8, Theorem 1] for $\mathbb{D}$, and we present them for the sake of completeness. Since $R_z^{(\lambda)}$ is real-valued for each $z \in \mathbb{D}^n$, we note that $\overline{Q_\lambda(g)} = Q_\lambda(\overline{g})$ for every $g \in L^2(\mathbb{D}^n, d\upsilon_\lambda)$, and hence

$$\begin{aligned}\overline{(T_{f_2}T_{f_1} - T_f)(h)} &= Q_\lambda\big(\overline{f_2}Q_\lambda(\overline{f_1 h})\big) - Q_\lambda(\overline{fh})\\ &= (T_{\overline{f_2}}T_{\overline{f_1}} - T_{\overline{f}})(\overline{h}) = (T_{f_1}T_{f_2} - T_f)^*(\overline{h})\end{aligned}$$

for any $h \in b^2_\lambda(\mathbb{D}^n)$, where $(T_{f_1}T_{f_2} - T_f)^*$ denotes the adjoint operator of $T_{f_1}T_{f_2} - T_f$. Since $T_{f_1}T_{f_2} = T_f$ on $b^2_\lambda(\mathbb{D}^n)$, it follows that $T_{f_2}T_{f_1} = T_f$ and that

$$T_{f_1}T_{f_2} - T_{f_2}T_{f_1} = (T_{f_1}T_{f_2} - T_f) - (T_{f_2}T_{f_1} - T_f) = 0.$$

This completes the proof. □

Essentially, $T_{f_1}T_{f_2} - T_f$ is a transpose of $T_{f_2}T_{f_1} - T_f$ on $b^2_\lambda(\mathbb{D}^n)$; namely, $T_{f_1}T_{f_2} - T_f = C(T_{f_2}T_{f_1} - T_f)^*C$ for the ordinary complex conjugation operator $C : Cf = \overline{f}$ on $L^2(\mathbb{D}^n, d\upsilon_\lambda)$, and they must have the same rank (see [11] for more details).

We are now ready to prove Theorem 1.3 stated in the Introduction, which completely characterizes when the product of two monomial-type Toeplitz operators is a Toeplitz operator on $b^2_\lambda(\mathbb{D}^n)$.

Proof of Theorem 1.3. By Lemma 3.2, the equivalence of (a) and (b) is obvious. So we just need to show the equivalence of (a) and (c). First, we suppose condition (a) holds. Then by Theorem 1.2, we have $h = e^{i(p+q)\cdot\theta}\psi$ for some bounded separately radial function ψ. For each $\alpha \in \mathbb{N}^n$ with $\alpha \succeq \delta$, using Lemma 2.1, we get

$$\begin{aligned}&(T_{r^l e^{ip\cdot\theta}}T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(z^\alpha) = 0\\ &\iff \widehat{r^l}(2\alpha + 2q + p + 2)\widehat{r^m}(2\alpha + q + 2)\\ &\qquad = \widehat{\mathbb{I}}(2\alpha + 2q + 2)\widehat{\psi}(2\alpha + p + q + 2).\end{aligned} \tag{3.1}$$

Let $\gamma \in \mathbb{Z}^n$ with

$$\gamma_i = \max\{0, -p_i, -q_i, -p_i - q_i\}.$$

Then (3.1) can be rewritten as

$$(\widehat{r^{l+p+q+\gamma} *_M r^{m+\gamma}})(2\alpha + q - \gamma + 2) = (\widehat{r^{q+\gamma} *_M r^{p+\gamma}\psi})(2\alpha + q - \gamma + 2),$$

and hence

$$r^{l+p+q+\gamma} *_M r^{m+\gamma} = r^{q+\gamma} *_M r^{p+\gamma}\psi.$$

Then by the definition of convolution, we obtain

$$\int_{r_1}^1 \cdots \int_{r_n}^1 \psi(t_1, \ldots, t_n) \prod_{i=1}^n t_i^{p_i - q_i - 1}\, dt_i = r^{l+p} \int_{r_1}^1 \cdots \int_{r_n}^1 \prod_{i=1}^n t_i^{m_i - l_i - p_i - q_i - 1}\, dt_i.$$

Differentiating both sides of the equation, we get $\psi(r) = \prod_{i=1}^{n} \psi_i(r_i)$ with

$$\psi_i(r_i) = \begin{cases} \frac{l_i+p_i}{l_i+p_i-m_i+q_i} r_i^{l_i+q_i} - \frac{m_i-q_i}{l_i+p_i-m_i+q_i} r_i^{m_i-p_i} & \text{if } l_i+p_i \neq m_i - q_i, \\ r_i^{l_i+q_i}(1+(l_i+p_i)\log r_i) & \text{if } l_i+p_i = m_i-q_i, \end{cases}$$

which gives condition (c1). Moreover, a direct calculation shows that

$$\widehat{r^l}(z) = \prod_{j=1}^{n} \frac{\Gamma(\frac{z_j+l_j}{2})\Gamma(\lambda_j+1)}{2\Gamma(\frac{z_j+l_j}{2}+\lambda_j+1)}.$$

For simplicity, we define

$$L_{\lambda_j}(z_j) = \frac{\Gamma(z_j+1)}{\Gamma(z_j+\lambda_j+2)}.$$

Then by (3.1), we have

$$\begin{aligned} &(T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(z^\alpha) = 0 \\ \iff &\prod_{j=1}^{n} \frac{L_{\lambda_j}(\alpha_j + \frac{m_j+q_j}{2}) L_{\lambda_j}(\alpha_j + q_j + \frac{l_j+p_j}{2})}{L_{\lambda_j}(\alpha_j+q_j)} \\ &= \Big[\prod_{j=1}^{n} \frac{2}{\Gamma(\lambda_j+1)}\Big] \widehat{\psi}(2\alpha+p+q+2) \end{aligned} \tag{3.2}$$

for each $\alpha \in \mathbb{N}^n$ with $\alpha \succeq \delta$.

We are going to prove that (a) implies (c2). First from condition (a) and Lemma 3.2, it follows that the operators $T_{r^l e^{ip\cdot\theta}}$ and $T_{r^m e^{iq\cdot\theta}}$ commute on $b^2_\lambda(\mathbb{D}^n)$. Recall from [11, Theorem 1.3] that $T_{r^l e^{ip\cdot\theta}}$ commutes with $T_{r^m e^{iq\cdot\theta}}$ if and only if at least one of the following statements holds.

1. (1) For each $i \in \{1,2,\dots,n\}$, either $p_i = l_i = 0$ or $p_i = q_i = 0$.
2. (2) For each $i \in \{1,2,\dots,n\}$, either $q_i = m_i = 0$ or $p_i = q_i = 0$.
3. (3) For each $i \in \{1,2,\dots,n\}$, either $p_i = q_i$, $l_i = m_i$ or $p_i = q_i = 0$.
4. (4) Neither $p \preceq 0, q \preceq 0$ nor $p \succeq 0, q \succeq 0$, and for each $i \in \{1,2,\dots,n\}$, (l_i, p_i, m_i, q_i) satisfies [11, Introduction, Condition (J)].

To show that condition (c2) holds, it suffices to show that conditions (3) and (a) together imply that at least one of the conditions (1), (2), and (4) hold. First, we assume that $p_i = q_i = 0$ for all $i \in \{1,2,\dots,n\}$. Then conditions (1) and (2) both hold, and we are done.

Next, suppose that for each $j \in \{1,2,\dots,n\}$ either $p_j = q_j, l_j = m_j$, or $p_j = q_j = 0$ and, additionally, $p_i = q_i \neq 0$ for some $i \in \{1,2,\dots,n\}$. Since (1) and (2) are automatically false, we need to show that (4) holds. Note that (l_j, p_j, m_j, q_j) satisfies [11, Introduction, Condition (J)] for each $j \in \{1,2,\dots,n\}$, and hence what remains is to show that assuming that either $p \preceq 0$, $q \preceq 0$ or that $p \succeq 0, q \succeq 0$ leads to a contradiction.

Without loss of generality, we can further assume that $p_i = q_i > 0$, for otherwise we could take the adjoints. So we have $p = q \succneqq 0$, and we consider

$$T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}}(\overline{z}^\alpha) = T_{e^{i(p+q)\cdot\theta}\psi}(\overline{z}^\alpha)$$

with $\alpha = (2p_1, \ldots, 2p_{i-1}, p_i, 2p_{i+1}, \ldots, 2p_n) \in \mathbb{N}^n$. Then a calculation from Lemma 2.1 gives that

$$\Big[\prod_{j\neq i} \frac{L_{\lambda_j}(2p_j + \frac{m_j - p_j}{2})}{L_{\lambda_j}(p_j)}\Big] \frac{L_{\lambda_i}(\frac{m_i+p_i}{2})}{L_{\lambda_i}(0)} \times \Big[\prod_{j\neq i} \frac{L_{\lambda_j}(\frac{l_j+p_j}{2})}{L_{\lambda_j}(0)}\Big] \frac{L_{\lambda_i}(\frac{l_i+p_i}{2})}{L_{\lambda_i}(p_i)}$$
$$= \Big[\prod_{j\neq i} 2(\lambda_j+1)\Big] \frac{2\widehat{\psi}(2p+2)}{\Gamma(\lambda_i+1)L_{\lambda_i}(p_i)}.$$

According to (3.2) with $p = q \gneq 0$ and $\alpha = 0$, we have

$$\widehat{\psi}(2p+2) = \prod_{j=1}^{n} \frac{\Gamma(\lambda_j+1)L_{\lambda_j}(\frac{m_j+p_j}{2})L_{\lambda_j}(p_j + \frac{l_j+p_j}{2})}{2L_{\lambda_j}(p_j)},$$

and hence

$$\Big[\prod_{j\neq i} L_{\lambda_j}\Big(2p_j + \frac{m_j - p_j}{2}\Big) L_{\lambda_j}\Big(\frac{l_j+p_j}{2}\Big)\Big] \frac{L_{\lambda_i}(\frac{m_i+p_i}{2})L_{\lambda_i}(\frac{l_i+p_i}{2})}{L_{\lambda_i}(0)}$$
$$= \frac{1}{L_{\lambda_i}(p_i)} \prod_{j=1}^{n} L_{\lambda_j}\Big(\frac{m_j+p_j}{2}\Big) L_{\lambda_j}\Big(p_j + \frac{l_j+p_j}{2}\Big).$$

Since either $p_j = q_j$, $l_j = m_j$, or $p_j = q_j = 0$ for all $j \in \{1, 2, \ldots, n\}$, we have

$$\frac{L_{\lambda_i}(\frac{l_i+p_i}{2})}{L_{\lambda_i}(0)} = \frac{L_{\lambda_i}(p_i + \frac{l_i+p_i}{2})}{L_{\lambda_i}(p_i)},$$

which is equivalent to

$$\frac{\Gamma(\lambda_i+2)\Gamma(\frac{l_i+p_i}{2}+1)}{\Gamma(\frac{l_i+p_i}{2}+\lambda_i+2)} = \frac{\Gamma(p_i+\lambda_i+2)\Gamma(p_i+\frac{l_i+p_i}{2}+1)}{\Gamma(p_i+1)\Gamma(p_i+\frac{l_i+p_i}{2}+\lambda_i+2)}. \tag{3.3}$$

Let $a = \lambda_i + 1 > 0$ and $b = (l_i + p_i)/2 > 0$, and denote

$$F(x) = \frac{\Gamma(x)\Gamma(x+b-a)}{\Gamma(x-a)\Gamma(x+b)}.$$

Then (3.3) can be rewritten as

$$F(p_i + \lambda_i + 2) = F(\lambda_i + 2),$$

but [8, Lemma 5] shows that $F(x)$ is strictly monotone increasing on $(a, +\infty)$, which is a contradiction, as desired.

Conversely, we suppose that condition (c) holds. To prove that condition (a) holds, it suffices to show that

$$(T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(r^{|k|}e^{ik\cdot\theta}) = 0 \tag{3.4}$$

holds for any $k \in \mathbb{N}^n \cup (-\mathbb{N})^n$. We break the discussion into four cases.

First, it is clear from condition (c1) that (3.1) holds, and hence (3.4) holds for any $k \in \mathbb{N}^n$ with $k \succeq \delta$.

Second, we denote $\delta'' \in \mathbb{N}^n$ with

$$\delta''_i = \max\{0, -p_i, -p_i - q_i\}.$$

Then for each $\alpha \in \mathbb{N}^n$ with $\alpha \succeq -\delta'$, we have $\alpha = \beta + p + q$ with $\beta \succeq \delta''$. Since condition (c2) holds, we have

$$T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} = T_{r^m e^{iq\cdot\theta}} T_{r^l e^{ip\cdot\theta}}.$$

Hence by Lemma 2.1, we get

$$\begin{aligned}
&(T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(\overline{z}^{\alpha}) = 0 \\
&\quad\Longleftrightarrow (T_{r^m e^{iq\cdot\theta}} T_{r^l e^{ip\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(\overline{z}^{\beta+p+q}) = 0 \\
&\quad\Longleftrightarrow \widehat{r^l}(2\beta + 2q + p + 2)\widehat{r^m}(2\beta + q + 2) \\
&\qquad\qquad = \widehat{\mathbb{I}}(2\beta + 2q + 2)\widehat{\psi}(2\beta + p + q + 2)
\end{aligned} \tag{3.5}$$

for any $\beta \succeq \delta''$. It is clear from condition (c2) and the proof of Lemma 3.1 that

$$\delta''_i = \delta_i = \max\{0, -p_i - q_i\}$$

for all $i \in \{1, 2, \ldots, n\}$. Thus (3.5) is the same as (3.1), and so

$$(T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(\overline{z}^{\alpha}) = 0,$$

which implies that (3.4) holds for any $k \in (-\mathbb{N})^n$ with $k \preceq \delta'$.

Third, we consider $\alpha \in \mathbb{N}^n$ with $\alpha \not\succeq \delta$ and $\alpha \not\preceq \delta$. Then by Lemma 3.1 and the definition of δ_j, we get that $\alpha_i < -p_i - q_i$ and $\alpha_j > -p_j - q_j$ for some $i, j \in \{1, \ldots, n\}$. Consequently, it follows from Lemma 2.1 that

$$T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}}(z^{\alpha}) = T_{e^{i(p+q)\cdot\theta}\psi}(z^{\alpha}) = 0,$$

and hence (3.4) holds for any $k \in \mathbb{N}^n$ with $k \not\succeq \delta$ and $k \not\preceq \delta$. Similarly, for each $\alpha \in \mathbb{N}^n$ with $\alpha \not\succeq -\delta'$ and $\alpha \not\preceq -\delta'$, one can get $(T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(\overline{z}^{\alpha}) = 0$, and thus (3.4) holds for any $k \in (-\mathbb{N})^n$ with $k \not\succeq \delta'$ and $k \not\preceq \delta'$.

Fourth and finally, we will consider $k \in \mathbb{N}^n \cup (-\mathbb{N})^n$ with $\delta' \precneqq k \precneqq \delta$. It is clear from Lemma 3.1 that

$$\delta = 0 \Longleftrightarrow p \succeq 0, q \succeq 0$$

and that

$$\delta' = 0 \Longleftrightarrow p \preceq 0, q \preceq 0.$$

Now we consider two cases.

Case 1: *Neither* $p \preceq 0, q \preceq 0$ *nor* $p \succeq 0, q \succeq 0$. Then $\delta'_i \neq 0$ and $\delta_j \neq 0$ for some $i, j \in \{1, \ldots, n\}$. Furthermore, conditions (a) and (b) of Lemma 3.1 imply that

$$\delta_i = 0, \qquad \delta'_i = -p_i - q_i < 0, \qquad \delta_j = -p_j - q_j > 0, \qquad \delta'_j = 0.$$

We first consider $k \in \mathbb{N}^n$ with $\delta' \precneqq k \precneqq \delta$. On the one hand, $k_i = \delta_i = 0$ since $k \in \mathbb{N}^n$ and $k_i \leq \delta_i = 0$. Therefore, $k_i + p_i + q_i > 0$. On the other hand, for those j such that $\delta_j \neq 0$, at least one of k_j satisfies $0 \leq k_j < \delta_j$ since $k \neq \delta$. Thus, $k_j + p_j + q_j < 0$. As a consequence of Lemma 2.1, we obtain that (3.4) holds. Next, we consider $k \in (-\mathbb{N})^n$ with $\delta' \precneqq k \precneqq \delta$. Similarly, we get $k_j = \delta'_j = 0$ and

$k_j + p_j + q_j < 0$, and $\delta'_i < k_i \leq 0$ and $k_i + p_i + q_i > 0$. Therefore, we also have that (3.4) holds.

Case 2: *Either* $p \preceq 0, q \preceq 0$ *or* $p \succeq 0, q \succeq 0$. Furthermore, condition (c2) shows that either $p = 0$ or $q = 0$. If $p = q = 0$, then $\delta' = \delta = 0$, and there is nothing to prove. Without loss of generality, we may assume that $p = 0$ and $q \precneqq 0$. Then $\delta' = 0$ and $\delta = -q$. Since (3.4) holds for any $k \in \mathbb{N}^n$ with $k \succeq \delta$, we have that $(T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(z^{-q}) = 0$. It follows from (3.2) that

$$\widehat{\psi}(-q+2) = \prod_{j=1}^{n} \frac{\Gamma(\lambda_j+1)}{2L_{\lambda_j}(0)} L_{\lambda_j}\Big(\frac{l_j}{2}\Big) L_{\lambda_j}\Big(\frac{m_j - q_j}{2}\Big). \tag{3.6}$$

Then for any $\alpha \in \mathbb{N}^n$ with $0 \precneqq \alpha \precneqq \delta = -q$, it follows from Lemma 2.1 and (3.6) that

$$\begin{aligned}
&(T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(z^\alpha) \\
&\quad = T_{r^l} T_{r^m e^{iq\cdot\theta}}(z^\alpha) - T_{e^{iq\cdot\theta}\psi}(z^\alpha) \\
&\quad = \prod_{j=1}^{n} \Big[\frac{L_{\lambda_j}(-\alpha_j - q_j + \frac{l_j}{2})}{L_{\lambda_j}(-\alpha_j - q_j)} - \frac{L_{\lambda_j}(\frac{l_j}{2})}{L_{\lambda_j}(0)} \Big] \times \prod_{j=1}^{n} \frac{L_{\lambda_j}(\frac{m_j - q_j}{2})}{L_{\lambda_j}(-\alpha_j - q_j)} \overline{z}^{\delta - \alpha}.
\end{aligned}$$

Recall that $p = 0$, $q \precneqq 0$. Combining this with condition (c2), we find a $j \in \{1, \ldots, n\}$ such that $l_j = 0$. Then it follows that $(T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} - T_{e^{i(p+q)\cdot\theta}\psi})(z^\alpha) = 0$, as desired. Thus we have derived that $T_{r^l e^{ip\cdot\theta}} T_{r^m e^{iq\cdot\theta}} = T_{e^{i(p+q)\cdot\theta}\psi}$, and hence condition (a) holds. This completes the proof. $\square$

We close the article with some interesting applications of Theorem 1.3. We first present three specific examples.

Example 3.3. It is easy to check that $T_{z_1} T_{z_2} \neq T_{z_1 z_2}$ on $b^2_\lambda(\mathbb{D}^n)$. However, $T_{z_1} T_{z_2 \overline{z}_2} = T_{z_1 z_2 \overline{z}_2}$, $T_{z_1} T_{z_2 \overline{z}_3} = T_{z_1 z_2 \overline{z}_3}$, and even

$$T_{z_1} T_{z_2 \prod_{i=3}^n r_i^{m_i} e^{iq_i\theta}} = T_{z_1 z_2 \prod_{i=3}^n r_i^{m_i} e^{iq_i\theta}}$$

for some $q_i < 0$ on $b^2_\lambda(\mathbb{D}^n)$. Although monomial-type Toeplitz operators may look like a class of very simple operators, they provide many meaningful examples in operator theory.

Example 3.4. Consider the symbols $r^l e^{ip\cdot\theta} = z_2|z_2|z_3\overline{z}_4|z_5|z_6|z_6|^2$ and $r^m e^{iq\cdot\theta} = z_1|z_1|^2 z_3 \overline{z}_4^2 |z_5|^2 z_6 |z_6|^2$, which correspond to each case of (i)–(vi) in [11, Introduction, Condition (J)] for the tuple (l_i, p_i, m_i, q_i), $i = \{1, \ldots, 6\}$. Then on $b^2_\lambda(\mathbb{D}^n)$, we have

$$T_{z_2|z_2|z_3\overline{z}_4^2|z_5|z_6|z_6|^2} T_{z_1|z_1|^2 z_3^2 \overline{z}_4 |z_5|^2 z_6 |z_6|^2} = T_{z_1|z_1|^2 z_2 |z_2| z_3^3 \overline{z}_4^3 (2|z_5|^2 - |z_5|) z_6^2 (2|z_6|^2 - 1)}.$$

Example 3.5. Lemma 3.2 shows that two Toeplitz operators must be commutative on $b^2_\lambda(\mathbb{D}^n)$ if their product is a Toeplitz operator. But the converse is false. For example, $T_{z_1|z_2|^2}$ commutes with T_{z_1} on $b^2_\lambda(\mathbb{D}^n)$. However, $T_{z_1|z_2|^2} T_{z_1}(\overline{z_1}) \neq T_{z_1^2|z_2|^2}(\overline{z_1})$, and hence the product $T_{z_1|z_2|^2} T_{z_1}$ is not a Toeplitz operator on $b^2_\lambda(\mathbb{D}^n)$.

Recall that the semicommutator $(T_{f_1}, T_{f_2}]$ of two Toeplitz operators T_{f_1} and T_{f_2} is defined by $(T_{f_1}, T_{f_2}] = T_{f_1}T_{f_2} - T_{f_1 f_2}$. As a direct consequence of Theorem 1.3, we have the following corollary which completely characterizes when the semicommutator of two monomial-type Toeplitz operators is zero on $b^2_\lambda(\mathbb{D}^n)$. This can be found in [11, Theorem 1.4].

Corollary 3.6. *Let $l, m \in \mathbb{R}^n_+$, $p, q \in \mathbb{Z}^n$. Then the following statements are equivalent.*

(a) *The semicommutator $(T_{r^l e^{ip\cdot\theta}}, T_{r^m e^{iq\cdot\theta}}]$ is zero on $b^2_\lambda(\mathbb{D}^n)$.*
(b) *The semicommutator $(T_{r^m e^{iq\cdot\theta}}, T_{r^l e^{ip\cdot\theta}}]$ is zero on $b^2_\lambda(\mathbb{D}^n)$.*
(c) *One of the following statements holds.*
- *For each $i \in \{1, 2, \ldots, n\}$, either $p_i = l_i = 0$ or $p_i = q_i = m_i = 0$.*
- *For each $i \in \{1, 2, \ldots, n\}$, either $q_i = m_i = 0$ or $q_i = p_i = l_i = 0$.*
- *Neither $p \preceq 0, q \preceq 0$ nor $p \succeq 0, q \succeq 0$, and for each $i \in \{1, 2, \ldots, n\}$, (l_i, p_i, m_i, q_i) satisfies one of (i)–(iv) of [11, Introduction, Condition (J)].*

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