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SURJECTIVITY OF COERCIVE GRADIENT OPERATORS IN HILBERT SPACE AND NONLINEAR SPECTRAL THEORY

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Abstract. We consider continuous gradient operators F acting in a real Hilbert space H, and we study their surjectivity under the basic assumption that the corresponding functional $\langle F(x), x\rangle$—where $\langle\cdot\rangle$ is the scalar product in H—is coercive. While this condition is sufficient in the case of a linear operator (where one in fact deals with a bounded self-adjoint operator), in the general case we supplement it with a compactness condition involving the number $\omega(F)$ introduced by Furi, Martelli, and Vignoli, whose positivity indeed guarantees that F is proper on closed bounded sets of H. We then use Ekeland's variational principle to reach the desired conclusion. In the second part of this article, we apply the surjectivity result to give a perspective on the spectrum of these kinds of operators—ones not considered by Feng or the above authors—when they are further assumed to be sublinear and positively homogeneous.

1. Introduction and main result

Let H be a real Hilbert space with scalar product denoted $\langle\cdot,\cdot\rangle$ and corresponding norm $\|x\| = \sqrt{\langle x, x\rangle}$. Let $T : H \to H$ be a bounded linear operator, and suppose that

$$\langle Tx, x\rangle \geq c\|x\|^2 \tag{1.1}$$

for some constant $c > 0$ and for all $x \in H$. Then T is evidently injective, and by using for instance the Lax–Milgram lemma (see, e.g., [5, Corollary 5.8]), it

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follows from (1.1) that T is also surjective. Indeed, the equation $Tx = y$, for a given $y \in H$, is equivalent to the problem

$$\langle Tx, v\rangle = \langle y, v\rangle, \qquad \forall v \in H, \tag{1.2}$$

and the existence of an $x \in H$ satisfying (1.2) then follows on considering the continuous bilinear form a on $H \times H$ defined by putting $a(u,v) = \langle Tu, v\rangle$ for $u, v \in H$, which is coercive by virtue of (1.1).

It has to be noted for completeness that—aside from any consideration on surjectivity—(1.1) guarantees that T is not only injective, but is in fact *boundedly invertible*, that is, injective with continuous inverse T^{-1}. Indeed, (1.1) and the Cauchy–Schwarz inequality imply that $\|Tx\| \geq c\|x\|$ for all $x \in H$, or equivalently (putting $y = Tx$) that $\|Ty\| \leq c^{-1}\|y\|$, and thus ensure that T^{-1} is a bounded, that is, continuous, linear operator of the range $R(T)$ of T into H.

If we look for extensions to nonlinear operators of these "regularity" results under assumptions similar to (1.1), we easily find one which follows by the celebrated Minty–Browder theorem for monotone operators (see, e.g., [5, Theorem 5.16]). Indeed, suppose that $F : H \to H$ is continuous and such that

$$\big\langle F(x) - F(y), x - y\big\rangle \geq c\|x - y\|^2 \tag{1.3}$$

for some $c > 0$ and all $x, y \in H$. Then F is clearly injective with Lipschitz continuous inverse F^{-1}; moreover, F is also surjective, because by (1.3) F is monotone and satisfies the coercivity condition $\langle F(x), x\rangle/\|x\| \to +\infty$ as $\|x\| \to +\infty$.

It is our aim in this article to prove a surjectivity result for F based on a condition like (1.1), supplemented by two more assumptions on F. As to the first of them, we observe that the surjectivity of a linear operator T satisfying (1.1) has a simple meaning, and a more direct proof, in the case in which T is *self-adjoint*, that is, such that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$. Indeed, in this case the bilinear form a induced by T is *symmetric*, and—as explained for instance in [5]—the unique solution x_0 of the equation $Tx = y$, for a given $y \in H$, is characterized as the unique point x_0 where the functional

$$\frac{\langle Tx, x\rangle}{2} - \langle x, y\rangle$$

attains its absolute minimum on H. Motivated by this variational interpretation, we restrict ourselves to considering the nonlinear analogues of self-adjoint operators, namely, the *gradient* operators (see below for the precise definition), and we apply Ekeland's variational principle to the corresponding functional (see, e.g., [7]) to prove that it reaches an absolute minimum also in this more general case. However, in order to obtain the desired conclusion we need to exploit a "numerical characteristic"—in the sense of [1]—of (linear and) nonlinear operators acting in a Banach space E which is related to the measure of noncompactness of bounded subsets of E, and whose introduction seems to be due to Furi, Martelli, and Vignoli's seminal paper [10] on nonlinear spectral theory. Here we briefly recall its definition and properties.

Measure of noncompactness. If A is a bounded subset of a Banach space E, let $\alpha(A)$ denote the (Kuratowski) *measure of noncompactness* of A (see, e.g., [2]) defined by

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ can be covered by finitely many subsets of diameter} \le \epsilon\}.$$

Here and henceforth, we will only consider maps $F : E \to E$ that are bounded on bounded sets, so that $\alpha(F(A))$ is defined whenever $A \subset E$ is bounded. We assume, moreover, that $\dim E = \infty$, so that there exist bounded sets $A \subset E$ with $\alpha(A) > 0$.

Definition 1.1. A map $F : E \to E$ is said to be *α-Lipschitz* if $\alpha(F(A)) \le k\alpha(A)$ for some $k \ge 0$ and all bounded subsets A of E; in this case we put

$$\alpha(F) = \inf\big\{k \ge 0 : \alpha\big(F(A)\big) \le k\alpha(A) \text{ for all bounded } A \subset E\big\}; \tag{1.4}$$

that is,

$$\alpha(F) = \sup\Big\{\frac{\alpha(F(A))}{\alpha(A)} : A \subset E, A \text{ bounded}, \alpha(A) > 0\Big\}. \tag{1.5}$$

Note that $\alpha(F) = 0$ if and only if F is *compact*, that is, such that $F(A)$ is relatively compact whenever $A \subset E$ is bounded.

Next, let $\omega(F)$ be defined as follows:

$$\omega(F) = \inf\Big\{\frac{\alpha(F(A))}{\alpha(A)} : A \subset E, A \text{ bounded}, \alpha(A) > 0\Big\}. \tag{1.6}$$

It follows by (1.6) that, for all bounded $A \subset E$,

$$\alpha\big(F(A)\big) \ge \omega(F)\alpha(A)$$

and that $\omega(F) \le \alpha(F)$ when F is α-Lipschitz. Among the several properties of $\omega(F)$ (see, e.g., [10, Proposition 3.1.3], [1, Proposition 2.4]), we select a few especially useful to us in the following statement. First, recall that

- $F : E \to E$ is said to be *proper on closed bounded sets* if, given any closed bounded set M of E, the set $M \cap F^{-1}(K)$ is compact whenever $K \subset E$ is compact;
- a linear map $F : E \to E$ is said to be *left semi-Fredholm* if it has finite-dimensional nullspace $N(F)$ and closed range $R(F)$.

Proposition 1.2. *Let $F : E \to E$ be continuous, and let $\omega(F)$ be as in (1.6). Then we have the following.*

(i) *If $\omega(F) > 0$, then F is proper on closed bounded sets.*

(ii) *Suppose that F is α-Lipschitz, and let I be the identity map in E. Then, for any $c \in \mathbb{R}$,*

$$|c| - \alpha(F) \le \omega(F - cI) \le |c| + \alpha(F). \tag{1.7}$$

(iii) *If F is linear, then $\omega(F) > 0$ if and only if F is left semi-Fredholm.*

(iv) *If F is linear, then*

$$\omega(F) \ge b(F) \equiv \inf_{x \ne 0} \frac{\|Fx\|}{\|x\|}. \tag{1.8}$$

Remark 1.3. As already noted, the condition $b(F) > 0$ holds if and only if F is boundedly invertible, that is, injective with *continuous* inverse F^{-1}. We recall, moreover, that if F is injective, the continuity property of F^{-1} is equivalent to the condition that the range $R(F)$ of F is a *closed* subspace of H. Therefore, in view of (iii), the inequality (1.8) can be seen as a quantitative (and most useful) version of the implication "F boundedly invertible $\Rightarrow$ F left semi-Fredholm."

Remark 1.4. In their original article, Furi, Martelli, and Vignoli [10] used the symbol $\beta(F)$ rather than $\omega(F)$. To the best of my knowledge, the latter was first used by Edmunds and Webb [8], then used again in the 1997 paper by Feng [9], and is now acknowledged by some (if not all) of the symbol's originators, as shown recently for instance in [3].

Surjectivity of gradient mappings. Before stating and proving our main result, we need to recall (see, e.g., [4]) that $F : H \to H$ is said to be a *gradient* (or *potential*) operator if there exists a differentiable functional $f : H \to \mathbb{R}$ such that

$$\big\langle F(x), y\big\rangle = f'(x)y \quad \text{for all } x, y \in H, \tag{1.9}$$

where $f'(x)$ denotes the (Fréchet) derivative of f at the point $x \in H$. When it is so, the functional f defined for $x \in H$ via the equation

$$f(x) = \int_0^1 \big\langle F(tx), x\big\rangle\, dt \tag{1.10}$$

coincides (up to a constant) with the function f previously mentioned and is called the *potential* of F. We also recall that a linear continuous operator is a gradient if and only if it is self-adjoint (see [4]).

Theorem 1.5. *Let H be a real Hilbert space, and let $F : H \to H$ be a continuous gradient operator. Suppose that*

(i) $\langle F(x), x\rangle \geq c\|x\|^2$ *for some* $c > 0$ *and for all* $x \in H$,
(ii) $\omega(F) > 0$.

Then F is surjective.

Proof. Let f be the potential of F, as given by the formula (1.10). Using the assumption (i) we then have, for $x \in H$ and $t \in \mathbb{R}$, $t > 0$,

$$\big\langle F(tx), x\big\rangle = \frac{\langle F(tx), tx\rangle}{t} \geq \frac{c\|tx\|^2}{t} = ct\|x\|^2,$$

whence

$$f(x) = \int_0^1 \big\langle F(tx), x\big\rangle\, dt \geq c\|x\|^2 \int_0^1 t\, dt = c'\|x\|^2 \tag{1.11}$$

with $c' = c/2 > 0$. To prove that F is surjective, we take $y \in H$ and look for an $x \in H$ such that $F(x) = y$; however, this equation is equivalent to

$$\big\langle F(x) - y, v\big\rangle = 0, \quad \forall v \in H,$$

and therefore is equivalent to the search of a critical point x (i.e., a point where the derivative vanishes) for the functional f_1 defined on H by putting

$$f_1(x) = f(x) - \langle y, x\rangle. \tag{1.12}$$

Using (1.11) and Cauchy–Schwarz, we get

$$f_1(x) \geq c'\|x\|^2 - \|y\|\|x\|, \tag{1.13}$$

whence, using the elementary inequality $at^2 - bt \geq -b^2/4a$ valid for fixed $a > 0$, $b \in \mathbb{R}$, and all $t \in \mathbb{R}$, it follows that

$$f_1(x) \geq -\frac{\|y\|^2}{4c'} \equiv K \quad (x \in H). \tag{1.14}$$

Therefore, f_1 is bounded below on H. As f_1 is of class C^1, Ekeland's variational principle (see, e.g., [7]) ensures the existence of a minimizing sequence along which the derivative of f_1 tends to zero, that is, a sequence $(x_n) \subset H$ such that

$$f_1(x_n) \to c_1 \equiv \inf_{x\in H} f_1(x) \qquad \text{and} \qquad f_1'(x_n) \to 0.$$

Using the expression (1.12) of f_1 and (1.9), we see that the condition $f_1'(x_n) \to 0$ is equivalent to

$$F(x_n) \to y.$$

Here the assumption $\omega(F) > 0$ comes into play, for the sequence (x_n) is bounded by virtue of (1.13), and since F is proper on closed bounded sets by Proposition 1.2(i), it follows that (x_n) contains a convergent subsequence. Letting (x_{n_k}) denote this subsequence and letting $x = \lim_{k\to\infty} x_{n_k}$, we then see immediately by the continuity of f_1 and F that $f_1(x) = c_1$ and $F(x) = y$. □

Remark 1.6. For a linear operator, Theorem 1.5(ii) is *redundant* because it is a consequence of (i). Indeed, as has already been remarked, (i) implies that $\|Fx\| \geq c\|x\|$ for all $x \in H$, whence

$$b(F) = \inf_{x\neq 0} \frac{\|Fx\|}{\|x\|} \geq c > 0.$$

It follows from (1.8) that $\omega(F) > 0$. Therefore, when considered for linear operators, Theorem 1.5 reduces to the statement of surjectivity for coercive self-adjoint operators recalled and commented in the Introduction.

Remark 1.7. To exhibit examples of mappings F having $\omega(F) > 0$, just take $F = I - G$ with G compact; indeed, it follows from (1.7) that $\omega(F) = 1$ for such an F.

Remark 1.8. The conclusion of Theorem 1.5 continues to hold if the assumption (i) on F is relaxed to

$$\big\langle F(x), x\big\rangle \geq c\|x\|^2 + \langle p, x\rangle \tag{1.15}$$

for some $c > 0$, some $p \in H$, and all $x \in H$. Indeed in this case, (1.11) is modified as

$$f(x) \geq c'\|x\|^2 + \langle p, x\rangle \tag{1.16}$$

so that from (1.12) we have

$$f_1(x) = f(x) - \langle y, x\rangle \geq c'\|x\|^2 + \langle p - y, x\rangle \geq c'\|x\|^2 - \|p - y\|\|x\|. \tag{1.17}$$

The proof now proceeds as before.

Remark 1.9. Of course, it would be interesting to know if the conclusion of Theorem 1.5 continues to hold if one drops the assumption that F is a gradient.

2. Consequences on (linear and) nonlinear spectral theory

For a bounded linear operator T acting in a real Banach space E, denote by $\sigma(T)$ the *spectrum* of T:

$$\sigma(T) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is not a homeomorphism of } E \text{ onto } E\}.$$

In a Hilbert space H, the spectrum of T is to a good extent determined by its quadratic form $\langle Tx, x\rangle$. We report in particular the following statement.

Proposition 2.1 ([5, Proposition 6.9])**.** *Let $T : H \to H$ be a self-adjoint bounded linear operator. Set*

$$m = \inf_{\|x\|=1} \langle Tx, x\rangle, \qquad M = \sup_{\|x\|=1} \langle Tx, x\rangle. \tag{2.1}$$

Then $\sigma(T) \subset [m, M]$ and $m, M \in \sigma(T)$.

As to the inclusion $\sigma(T) \subset [m, M]$ stated in Proposition 2.1, we first observe that the self-adjointness of T should be strictly required in the case of a *complex* Hilbert space (see, e.g., [12, Theorem 6.2-B]), while in the context of real Hilbert spaces—considered here and in [5]—the inclusion holds in fact for *any* bounded linear operator. Indeed, it follows from (2.1) that $\langle Tx, x\rangle \geq m\|x\|^2$ for every $x \in H$, and therefore that

$$\langle Tx - \lambda x, x\rangle = \langle Tx, x\rangle - \lambda\|x\|^2 \geq (m - \lambda)\|x\|^2. \tag{2.2}$$

Thus if $\lambda < m$, then $T_\lambda \equiv T - \lambda I$ satisfies the condition (1.1), ensuring as explained at the beginning of this article that T_λ is a homeomorphism of H onto itself, and therefore that $\lambda \notin \sigma(T)$. A similar conclusion holds if $\lambda > M$, whence one concludes that $\sigma(T) \subset [m, M]$. Second, we emphasize the fact that the inclusion $\sigma(T) \subset [m, M]$, valid for bounded linear operators in a real Hilbert space, improves the inclusion

$$\sigma(T) \subset \big[-\|T\|, \|T\|\big]$$

holding for a bounded linear operator acting in any Banach space (see e.g. Proposition 6.7 of [5]). Indeed by the Schwarz inequality, if $\|x\| = 1$, then

$$\big|\langle Tx, x\rangle\big| \leq \|T\|$$

so that

$$-\|T\| \leq m \leq M \leq \|T\|. \tag{2.3}$$

Essentials on the nonlinear spectrum. Let E be a real, infinite-dimensional Banach space and let $F : E \to E$ be continuous and bounded on bounded subsets of E. To recall very briefly the definition of the spectrum of F along the lines of [9] and [10] (see also [1]), consider first the quantities

$$|F| = \limsup_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|}, \qquad d(F) = \liminf_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|}. \tag{2.4}$$

Note that $|F|$ can be ∞. However, $|F|$ is finite if (and only if) F is *linearly bounded*—that is, it satisfies an inequality of the form $\|F(x)\| \le A\|x\| + B$ for some $A, B \ge 0$ and all $x \in E$.

For the statements and definitions which follow, we refer the reader to [10]; some of them hold in greater generality than given here. A map $F : E \to E$ is said to be *stably solvable* if the equation $F(x) = H(x)$ has a solution $x \in E$ for any $H : E \to E$ such that H is compact and $|H| = 0$. A stably solvable map is clearly surjective, and vice versa (a (continuous) *linear* surjective map is stably solvable).

A map $F : E \to E$ is said to be *FMV-regular* (Furi–Martelli–Vignoli regular) if it is stably solvable and moreover $d(F) > 0$, $\omega(F) > 0$ ($\omega(F)$ was defined in (1.6)). It follows that an FMV-regular map is surjective and proper, and that a linear map is FMV-regular if and only if it is a homeomorphism.

The *spectrum* of F, denoted $\sigma_{\mathrm{FMV}}(F)$, is defined as

$$\sigma_{\mathrm{FMV}}(F) = \{\lambda \in \mathbb{R} : F - \lambda I \text{ is not FMV-regular}\}.$$

If $|F| < \infty$ and F is α-Lipschitz, then $\sigma_{\mathrm{FMV}}(F)$ is compact, and moreover,

$$\sigma_{\mathrm{FMV}}(F) \subset \{\lambda \in \mathbb{R} : |\lambda| \le \max\big\{\alpha(F), |F|\big\} \tag{2.5}$$

with $\alpha(F)$ as in (1.4).

It is often useful to also consider the partial spectra

$$\begin{aligned} \sigma_\omega(F) &= \big\{\lambda \in \mathbb{R} : \omega(F - \lambda I) = 0\big\}, \\ \Sigma(F) &= \big\{\lambda \in \mathbb{R} : d(F - \lambda I) = 0\big\}, \\ \sigma_\delta(F) &= \{\lambda \in \mathbb{R} : F - \lambda I \text{ is not stably solvable}\}. \end{aligned}$$

We then have by definition

$$\sigma_{\mathrm{FMV}}(F) = \sigma_\omega(F) \cup \Sigma(F) \cup \sigma_\delta(F). \tag{2.6}$$

In general, $\sigma_{\mathrm{FMV}}(F)$ need not contain the *point spectrum* $\sigma_p(F)$,

$$\sigma_p(F) = \big\{\lambda \in \mathbb{R} : F(x) - \lambda x = 0 \text{ for some } x \neq 0\big\},$$

that is, the set of *eigenvalues* of F (see, e.g., the discussion in [8]). This is one of the main reasons motivating Feng's extension of $\sigma_{\mathrm{FMV}}(F)$ (see [9]). Essentially, Feng replaces the quantities $|F|$ and $d(F)$ defined in (2.4) with the following ones:

$$\|F\| = \sup_{\|x\| \neq 0} \frac{\|F(x)\|}{\|x\|}, \qquad b(F) = \inf_{\|x\| \neq 0} \frac{\|F(x)\|}{\|x\|}. \tag{2.7}$$

Note that, of course,

$$b(F) \le d(F) \le |F| \le \|F\|. \tag{2.8}$$

Again, we remark that $\|F\|$ can be ∞ unless we assume that F is *sublinear*; that is, it satisfies an inequality of the form

$$\big\|F(x)\big\| \le A\|x\|$$

for some $A \ge 0$ and all $x \in E$. Moreover, Feng [9] replaces the property of being stably solvable with a stronger requirement based on the concept of *p-epi* *mapping* introduced in [11]. The spectrum is defined consequently along the same lines sketched above and will be denoted $\sigma_F(F)$. We do not go into further details, but simply note the following:

(i) $\sigma_F(F) \supset \sigma_p(F)$, since $b(F - \lambda I) = 0$ if λ is an eigenvalue of F;
(ii) what we have just said about stable solvability, together with the inequality $b(F) \leq d(F)$ shown in (2.8), yields the inclusion $\sigma_{\text{FMV}}(F) \subset \sigma_F(F)$, which in general may be strict.

Positively homogeneous operators. The situation becomes more definite when F is *positively homogeneous* (of degree 1), that is, such that $F(tx) = tF(x)$ for $x \in E$ and $t > 0$. We first note that in this case F is linearly bounded if and only if it is sublinear; for if F satisfies the inequality $\|F(x)\| \leq A\|x\| + B$ for some $A, B \geq 0$ and all $x \in E$, then writing this for tx ($t > 0$) we obtain at once that $\|F(x)\| \leq A\|x\| + B/t$, whence letting $t \to \infty$ it follows that $\|F(x)\| \leq A\|x\|$. Using similar remarks, one can easily check the equalities

$$|F| = \|F\| = \sup_{\|x\|=1} \big\|F(x)\big\|, \qquad d(F) = b(F) = \inf_{\|x\|=1} \big\|F(x)\big\| \tag{2.9}$$

to hold for a positively homogeneous F. More importantly, Theorem 8.11 in [1] proves that in this case

$$\sigma_{\text{FMV}}(F) = \sigma_F(F) \tag{2.10}$$

and allows us to rewrite (2.6) as

$$\sigma(F) = \sigma_\omega(F) \cup \Sigma(F) \cup \sigma_\delta(F) \tag{2.11}$$

with $\sigma(F)$ denoting any of the two spectra defined above.

To see the effect of Theorem 1.5 in this spectral framework, we focus now on the case (not considered in [9] and [10]) that F is a gradient operator acting in a real Hilbert space H.

Theorem 2.2. *Let $F : H \to H$ be a continuous gradient operator. Suppose further that F is sublinear and positively homogeneous, and let $m(F)$, $M(F)$ be defined by*

$$m(F) = \inf_{\|x\|=1} \big\langle F(x), x \big\rangle, \qquad M(F) = \sup_{\|x\|=1} \big\langle F(x), x \big\rangle. \tag{2.12}$$

Then the following conclusions hold true:

(1) *$m(F)$ and $M(F)$ belong to the spectrum $\sigma(F)$ of F; more precisely,*

$$m(F), M(F) \in \Sigma(F) = \big\{\lambda \in \mathbb{R} : b(F - \lambda I) = 0\big\};$$

(2) *$\Sigma(F) \subset [m(F), M(F)]$;*
(3) *If $\lambda \notin \sigma_\omega(F) \cup [m(F), M(F)]$, then $F_\lambda \equiv F - \lambda I$ is surjective.*

Proof.

(1) For the proof of the first assertion, see Theorem 1.1 of [6].

(2) In order to prove the second assertion, observe that the definitions (2.12) and the positive homogeneity of F imply—exactly as for linear operators—that, for every $x \in H$,

$$m(F)\|x\|^2 \leq \big\langle F(x), x\big\rangle \leq M(F)\|x\|^2, \tag{2.13}$$

whence, as in Proposition 2.1,

$$\big\langle F(x) - \lambda x, x\big\rangle = \big\langle F(x), x\big\rangle - \lambda\|x\|^2 \geq \big(m(F) - \lambda\big)\|x\|^2. \tag{2.14}$$

Equation (2.14) implies once again that $b(F_\lambda) > 0$ if $\lambda < m(F)$. The same conclusion holds if $\lambda > M(F)$. Therefore, no such λ belongs to $\Sigma(F)$.

(3) What is more, (2.14) shows that F_λ satisfies Theorem 1.5(i) if $\lambda < m(F)$. For these λ's—and in fact for every $\lambda \notin [m(F), M(F)]$—the surjectivity of F_λ under the additional condition that $\omega(F_\lambda) > 0$ thus follows from Theorem 1.5. □

Remark 2.3. If to the assumptions of Theorem 2.2 one adds that F is α-Lipschitz, then the first conclusion can be strengthened as follows (see [6]). If $M(F) > \alpha(F)$, then $M(F)$ is an eigenvalue of F; moreover it is (clearly) the largest eigenvalue of F, and finally it is a *compact* eigenvalue of F in the sense that the set of corresponding normalized eigenvectors

$$\big\{x \in H : F(x) - Mx = 0, \|x\| = 1\big\}$$

is compact. Similar conclusions hold for $m(F)$ in the case when $m(F) < -\alpha(F)$.

Remark 2.4. The content of Theorem 2.2 becomes clearer when considered for a compact operator F. In this case, from (1.7) we have $\omega(F_\lambda) = |\lambda|$, so that $\sigma_\omega = \{0\}$. Therefore, the last two statements in Theorem 2.2 imply that

$$\lambda < m(F)\ \big(\text{or } \lambda > M(F)\big), \lambda \neq 0 \Rightarrow b(F_\lambda) > 0 \text{ and } F_\lambda \text{ is surjective.} \tag{2.15}$$

Moreover, we know from (2.5) and (2.9) that

$$\sigma(F) \subset \big[-\|F\|, \|F\|\big]. \tag{2.16}$$

Assume further that $m(F) \leq 0 \leq M(F)$ (this is necessarily the case when F is linear, for if for example $m(F) > 0$, then F would be boundedly invertible and $I = F^{-1}F$ would be compact, which is impossible since $\dim H = \infty$). When at least one of the strict inequalities (see (2.3)) $-\|F\| < m(F)$ or $M(F) < \|F\|$ holds, we then have the following situation:

- if $-\|F\| \leq \lambda < m(F)$ or $M(F) < \lambda \leq \|F\|$, then $b(F_\lambda) > 0$ and F_λ is surjective;
- while for λ outside $[-\|F\|, \|F\|]$, F_λ is regular.

In this rough picture, the two intervals $[-\|F\|, m(F)[,\ ]M(F), \|F\|]$ seem to be a sort of ambiguous zone between the core of the spectrum, contained in $[m(F), M(F)]$, and the set $\{\lambda \in \mathbb{R} : |\lambda| > \|F\|\}$ on which F_λ is definitely regular.

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