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NUCLEARITY AND TRACE FORMULAS OF INTEGRAL OPERATORS

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Abstract. We present some results on the nuclearity (or trace class) of integral operators acting on $L^2(X, \nu)$ under specific conditions. These results improve and adapt a number of methods found in references on this subject. Our discussions take place within the context of special subsets (and manifolds) of the Euclidean space (endowed with weighted Lebesgue measure), second-countable spaces, and Lusin and Souslin spaces (endowed with σ-finite Borel measure).

1. Introduction

Nuclearity and trace formulas frequently arise in a great variety of problems in several branches of mathematics like the asymptotic behavior of eigenvalues, regularity of pseudodifferential operators, Fredholm determinants, and learning theory. (For recent research on and applications of this subject, see, e.g., [4], [8], [11], and their references; for examples of classical authors and results on this subject, see [9] and [12].)

To clarify what we mean by *nuclearity*, we start with the basic setting of this article. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear and compact operator with adjoint $T^* : \mathcal{H}_2 \to \mathcal{H}_1$. The well-known singular value decomposition theorem implies the existence of (countable) orthonormal

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sets $\{\phi_j\} \subset \mathcal{H}_1$ and $\{\psi_j\} \subset \mathcal{H}_2$ and singular values $s_j \geq s_{j+1} \geq 0$ such that

$$T(f) = \sum_j s_j \langle f, \phi_j \rangle \psi_j, \quad f \in \mathcal{H}_1, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{H}_1$, $T(\phi_j) = s_j \psi_j$, $T^*(\psi_j) = s_j \phi_j$, and $T^*T(\phi_j) = s_j^2 \phi_j$. Note that

$$f = \sum_j \langle f, \phi_j \rangle \phi_j + u, \quad u \perp \phi_j, \qquad \text{and} \qquad T(u) = 0, \quad f \in \mathcal{H}_1.$$

Now, for each $p > 0$, we say that T is in the *Schatten class* $\mathcal{S}_p$ when

$$\|T\|_p^p = \sum_j s_j^p < \infty.$$

If $p \geq 1$, then $\mathcal{S}_p$ is a Banach space (see [9], [12, p. 12]).

The class $\mathcal{S}_1$ is the nuclear (or trace) class, and the class $\mathcal{S}_2$ is the Hilbert–Schmidt class. If T is in the trace class and $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$, then we can define its trace as

$$\operatorname{tr}(T) = \sum_{\phi \in B} \big\langle T(\phi), \phi \big\rangle,$$

for one (and any) orthonormal basis B of $\mathcal{H}$. It follows that $\operatorname{tr} : \mathcal{S}_1 \to \mathbb{C}$ is a bounded linear functional with norm 1. In particular, if $T \in \mathcal{S}_p$, then

$$\operatorname{tr}\big(|T|^p\big) = \|T\|_p^p = \sum_{\phi \in B} \big\langle |T|^p(\phi), \phi \big\rangle,$$

where B is an orthonormal basis for $\mathcal{H}$, and $|T| = (T^*T)^{1/2}$ is the only positive square root of T (see Theorem 3.1 in [12, p. 31]). Note that $\mathcal{S}_1 \subset \mathcal{S}_2$, and if $T \in \mathcal{S}_2$, then it is in $\mathcal{S}_1$ if and only if the Fredholm determinant

$$\det\big(I + |T|\big) = \prod_j (1 + s_j)$$

is convergent (see [1] for discussions on evaluation of Fredholm determinants).

In this article, we analyze the trace class operators when $\mathcal{H} = L^2(X, \nu)$ (see [7] for some measure theory). As such, in view of Lemma 2.4 below, we simply need to work with integral operators. Let $(X, \mathcal{A}, \nu)$ be a measure space, and let $k : X \times X \to \mathbb{C}$ be a measurable kernel for which the integral operator $K : L^2(X, \nu) \to L^2(X, \nu)$, given by

$$K(f)(x) = \int_X k(x, y) f(y) \, d\nu(y), \quad f \in L^2(X, \nu), \tag{1.2}$$

is well defined. Under conditions which we will state (as needed) throughout the paper, our plan is to show that if K is in the trace class, then

$$\operatorname{tr}(K) = \int_X k(x, x) \, d\nu(x). \tag{1.3}$$

(Readers are directed, e.g., to expressions (4.7) and (6.6) in [8] for details on the study of trace formulas like (1.3).)

This kind of result was motivated by the many versions of Mercer's theorem for positive definite kernels (see [5]), which is one of the basic tools for proving the results established in [6]. Recently, Jefferies [10] gave conditions to ensure the nuclearity of positive integral operators without the use of Mercer's theorem but using instead a perfect combination of procedures from [2], [3], and [6].

In this article, we extend or refine some of these results to ensure the nuclearity of integral operators. In Section 2, we present some basic results. Section 3 is devoted to providing technical results. Section 4 is the main section, where we give conditions to ensure the nuclearity of an integral operator and calculate their trace. Finally, in Section 5 we present our concluding remarks regarding the special contexts.

2. Auxiliary results

In this section, we present some technical results that we will use as the main tools of this work. Lemma 2.1 and Mercer's theorem are well known and were the basic tools used to prove the results of [6]. We provide an idea of the proof of this lemma for the sake of completeness.

Lemma 2.1. *Let $\{R_n\}$ and $\{S_n\}$ be sequences of bounded operators in $\mathcal{H}$ such that $R_n(f) \to R(f)$ and $S_n(f) \to S(f)$ for any $f \in \mathcal{H}$. If $p \geq 1$ and T is in $\mathcal{S}_p$, then for some $p \geq 1$,*

$$\lim_{n\to\infty} \|R_n T S_n^* - RTS^*\|_p = 0.$$

Proof. If $T_j(f) = s_j\langle f, \phi_j\rangle\psi_j$, where ϕ_j, ψ_j, and s_j come from (1.1), then direct calculation shows that

$$\begin{aligned}\|R_n T_j S_n^* - RT_jS^*\|_p &\leq \big\|R_nT_j(S_n^* - S^*)\big\|_p + \big\|(R-R_n)T_jS^*\big\|_p \\ &\leq s_j\big(\big\|(S_n - S)^*(\phi_j)\big\|\big\|R_n(\psi_j)\big\| \\ &\quad + \big\|S^*(\phi_j)\big\|\big\|(R-R_n)(\psi_j)\big\|\big)\end{aligned}$$

converges to zero as $n \to \infty$. This argument may be repeated to prove the same result for finite-rank operators. It follows from (1.1) that the set of finite-rank operators is dense in the compact operator class (and in the class $\mathcal{S}_p$). The proof follows from an ϵ approximation procedure. □

We consider the next lemma as the main functional analytic tool in this text. If $\mathcal{H}$ is separable, one can use another result like that used in [10, Proposition 3.1] as one of its main tools. Our version has a very simple proof.

Lemma 2.2. *Let $\mathcal{H}$ be a Hilbert space and let there be a sequence $T_n \in \mathcal{S}_p$, with $C = \sup_n \|T_n\|_p < \infty$, for some $p > 0$. If $|T_n|^p(f)$ converges to $|T|^p(f)$ for any $f \in \mathcal{H}$ and some bounded operator $T : \mathcal{H} \to \mathcal{H}$, then $T \in \mathcal{S}_p$ and $\|T\|_p \leq C$.*

Proof. Since

$$\|T_n\|_p^p = \sum_{f\in B}\big\langle |T_n|^p(\phi), \phi\big\rangle \leq C^p < \infty$$

for all $n \in \mathbb{N}$ and any orthonormal basis B of $\mathcal{H}$, it is easy to see that

$$\sum_{j=1}^{m} \big\langle |T|^p(\phi_j), \phi_j \big\rangle = \lim_{n\to\infty} \sum_{j=1}^{m} \big\langle |T_n|^p(\phi_j), \phi_j \big\rangle \le C^p < \infty$$

for each $m \in \mathbb{N}$ and any $\phi_1, \phi_2, \dots, \phi_m \in B$. Hence,

$$\|T\|_p \le \sup_D \sum_{\phi\in D} \big\langle |T|^p(\phi), \phi \big\rangle \le C^p < \infty,$$

where the supremum is taken over all finite sets $D \subset B$. It then follows that T is in $\mathcal{S}_p$. □

Remark 2.3. Note that the preceding lemmas still hold when n is replaced with u, where u is in a metric space V and $u \to \alpha$.

The next lemma gives a series representation for kernels of Hilbert–Schmidt operators.

Lemma 2.4. *Let $T : L^2(X,\nu) \to L^2(X,\nu)$ be a Hilbert–Schmidt operator. Then it is an integral operator with kernel*

$$k(x,y) = \sum_j s_j \eta_j(x)\overline{\xi_j(y)}, \quad x, y \in X \ (a.e.),$$

where $\{\xi_j\}$ and $\{\eta_j\}$ are orthonormal sets and s_j are singular values of T.

Proof. Let $\{\xi_j\}$, $\{\eta_j\}$, and s_j be given by the singular value decomposition theorem applied to T. It follows that $\{\eta_j \otimes \overline{\xi_j}\}$ is an orthonormal set in $L^2(X\times X, \nu\otimes\nu)$, where $\eta_j \otimes \overline{\xi_j}(x,y) = \eta_j(x)\overline{\xi_j(y)}$, $x, y \in X$. Now take

$$k = \sum_j s_j \eta_j \otimes \overline{\xi_j},$$

which is in $L^2(X \times X, \nu \otimes \nu)$. A direct calculation shows that

$$\|k\|^2 = \sum s_j^2 < \infty$$

and that k is the kernel of T. □

If T is a (self-adjoint) positive definite operator, that is, if

$$\big\langle T(f), f \big\rangle \ge 0, \quad f \in L^2(X,\nu), \tag{2.1}$$

then a quite general version of Mercer's theorem along the lines of those proved in [5] may hold for T. Precisely, T has spectral representation

$$T(f) = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n, \quad f \in L^2(X,\nu),$$

where $\{\phi_n\}$ is orthonormal in $L^2(X,\nu)$ and $\lambda_n \ge \lambda_{n+1} \ge 0$ are the eigenvalues of T. Also, if Mercer's theorem holds true, then

$$k(x,y) = \sum_{n=1}^{\infty} \lambda_n \phi_n(x)\overline{\phi_n(y)}, \quad x, y \in X,$$

with at least absolute convergence on X. It is now clear that one simply needs to put $x = y$ in this series and integrate it to obtain the trace of T as in (1.3).

3. Filters and approximations

We begin the discussion in this section as generally as possible, gradually adding hypotheses and providing examples as needed (given the context). (Readers are directed to [2], [3], [6], and [10] for a number of special and more general contexts in which the conditions that we will be adding hold.)

Let $(X, \mathcal{A}, \nu)$ be a measure space. We say that a (countable) family $F = \{F_1, F_2, \dots\}$ of sub-σ-algebras of $\mathcal{A}$ is a *filter* when $F_n \subset F_{n+1}$ and each F_n is generated by a (countable) partition $\mathcal{P}_n^F$ of $X_n \subset X$, with elements (of finite and positive measure) in $\mathcal{A}$.

Definition 3.1. Let F be a filter. The *conditional expectation operator* $P_n : L^2(X,\nu) \to L^2(X,\nu)$ is given by

$$P_n(f) = \mathbb{E}(f|F_n) = \sum_{U \in \mathcal{P}_n^F} \frac{\chi_U}{\nu(U)} \int_U f\, d\nu, \quad f \in L^2(X,\nu), \tag{3.1}$$

where χ_U is the characteristic function of U.

Note that if x is in some $U_n(x) \in \mathcal{P}_n^F$, then

$$P_n(f)(x) = \frac{1}{\nu(U_n(x))} \int_{U_n(x)} f\, d\nu.$$

Definition 3.2. Let k be a kernel with finite integral in $U \times V$ for each U, V in $\mathcal{P}_n^F$ and each $n = 1, 2, \dots$: The conditional expectation $\mathbb{E}(k|F_n \times F_n)$ is given by

$$k_n = \mathbb{E}(k|F_n \times F_n) = \sum_{U,V \in \mathcal{P}_n^F} \frac{\chi_{U\times V}}{\nu(U \times V)} \int_{U \times V} k\, d(\nu \times \nu). \tag{3.2}$$

Again, if x is in some $U_n(x)$ and y is in some $U_n(y)$, both in $\mathcal{P}_n^F$, then

$$k_n(x,y) = \frac{1}{\nu(U_n(x))\nu(U_n(y))} \int_{U_n(x)} \int_{U_n(y)} k(s,t)\, d\nu(s)\, d\nu(t).$$

An important tool related to the boundedness and convergence analysis of P_n is the maximal operator given by

$$H_F(f)(x) = \sup_{n \in \mathbb{N}} P_n\big(|f|\big)(x), \quad x \in X.$$

We also use the notation

$$H_{F^2}(k)(x,y) = \sup_{n \in \mathbb{N}} \mathbb{E}\big(|k||F_n \times F_n\big)(x,y), \quad x, y \in X.$$

Remark 3.3. In [6], concepts are used similar to (3.1) and (3.2), with the notation $\mathcal{D}_t$ and k^t, respectively, where $t > 0$, but they are not exactly the same. In particular, $\mathcal{D}_t$ did not come from a partition in that work; nevertheless, the concepts we discuss here make sense there.

The conditional expectation has the following important properties.

Lemma 3.4. *We have that P_n is an orthogonal projector from $L^2(X,\nu)$ to W_n, where $W_n = \operatorname{span}\{\chi_U, U \in \mathcal{P}_n^F\}$. Also, P_n is an integral operator with (positive definite) kernel*

$$e_n(x,y) = \sum_{U\in\mathcal{P}_n^F} \frac{\chi_U(x)\chi_U(y)}{\nu(U)}, \quad x,y \in X.$$

Proof. It suffices to show that $P_n^2 = P_n = P_n^*$. In fact, it follows from (3.1) that

$$P_n(\alpha\chi_U) = \alpha\chi_U, \quad U \in \mathcal{P}_n^F, \alpha \in \mathbb{C},$$

and that $\{\frac{\chi_U}{\sqrt{\nu(U)}}, U \in \mathcal{P}_n^F\}$ is an orthonormal basis for W_n. Also, if we write $\mathcal{P}_n^F = \{U_{nj}\}$ and $\xi_j = \frac{\chi_{U_{nj}}}{\sqrt{\nu(U_{nj})}}$, we can see that

$$P_n(f) = \sum_j \langle f, \xi_j\rangle \xi_j, \quad f \in L^2(X,\nu). \tag{3.3}$$

Hence, $P_n = P_n^2$ is self-adjoint and positive definite, with $\|P_n\| = 1$. Since

$$\begin{aligned}\int_X e_n(x,y)f(y)\,d\nu(y) &= \int_X \sum_{U\in\mathcal{P}_n^F} \frac{\chi_U(x)\chi_U(y)}{\nu(U)} f(y)\,d\nu(y)\\ &= \sum_{U\in\mathcal{P}_n^F} \frac{\chi_U(x)}{\nu(U)} \int_U f\,d\nu(y)\\ &= P_n(f)(x), \quad x \in X,\end{aligned}$$

it follows that P_n is an integral operator with kernel e_n. □

To prove the next lemma, we add boundedness to the integral operator $K : L^2(X,\nu) \to L^2(X,\nu)$ given by (1.2).

Lemma 3.5. *Let K be a bounded integral operator given by (1.2), with kernel k. The integral operator K_n, with kernel k_n given by (3.2), is also bounded and*

$$(P_nKP_n)(f) = K_n(f), \quad f \in L^2(X,\nu).$$

Proof. Since each element of $\mathcal{P}_n^F$ has finite measure, it follows from Fubini's/Tonelli's theorem and Lemma 3.4 that, for each $n = 1, 2, \ldots,$

$$\begin{aligned}(P_nKP_n)(f) &= \sum_i \langle f, \xi_i\rangle P_n(K)(\xi_i)\\ &= \sum_i \langle f, \xi_i\rangle \sum_j \big\langle K(\xi_i), \xi_j\big\rangle \xi_j\\ &= \sum_{i,j} \frac{\chi_{U_{nj}}}{\nu(U_{ni})\nu(U_{nj})} \int_{U_{ni}} f\,d\nu \int_{U_{nj}}\int_{U_{ni}} K\,d\nu\,d\nu\\ &= \sum_{U,V\in\mathcal{P}_n^F} \frac{\chi_U}{\nu(U)\nu(V)} \int_U\int_V k(u,v)\,d\nu(u)\,d\nu(v) \int_U f\,d\nu\\ &= (K_nf), \quad f \in L^2(X,\nu).\end{aligned}$$

The result follows. In particular, $\|K_n\| \le \|P_nKP_n\| = \|K\|$. □

We can easily verify that when X is a Lebesgue measurable subset of the Euclidean space, the Lebesgue differentiation theorem and boundedness of the Hardy–Littlewood maximal operator hold (see [6]).

We are now able to add another hypothesis to $(X, \mathcal{A}, \nu)$ to ensure that similar versions to this result hold in a more general setting. The contexts within which we will work include those where X is in the Euclidean sphere or a similar manifold. The results are related to the concept of martingale and Lusin filtration and extend topics covered in [2], [3], and [10].

Assumption 3.6. For each $f \in L^2(X, \nu)$, the sequence $\mathbb{E}(f|F_n) = P_n(f)$ converges to f (almost everywhere).

Note that this and the next assumption are (weak) versions of the Lebesgue differentiation theorem and are related to the martingale convergence theorem (see [2], [10]). It is clear that

$$\big|P_n(f)(x)\big| \le H_F(f)(x), \quad x \in X.$$

To bound the maximal operator we also need to add the next assumption.

Assumption 3.7. The maximal operator H_F is bounded in $L^2(X, \nu)$, which means the existence of a constant $C > 0$ such that

$$\big\|H_F(f)\big\|_2 \le C\|f\|_2, \quad f \in L^2(X, \nu). \tag{3.4}$$

This enables us to obtain a convergence result related to the lemmas of Section 2.

Proposition 3.8. *Let K be bounded, and suppose that Assumptions 3.6 and 3.7 hold. If $f \in L^2(X, \nu)$, then $K_n(f)$ converges to $K(f)$.*

Proof. Since $|P_n(f)(x) - f(x)| \le H_F(f)(x) + |f(x)|$, for $x \in X$, it follows that $P_n(f)$ converges to f in $L^2(X, \nu)$. Hence,

$$\begin{aligned}\big\|P_n K P_n(f) - K(f)\big\| &\le \|P_n\|\big\|K P_n(f) - K(f)\big\| + \big\|P_n K(f) - K(f)\big\| \\ &\le \|P_n\|\|K\|\big\|P_n(f) - f\big\| + \big\|P_n\big(K(f)\big) - K(f)\big\|,\end{aligned}$$

and the result follows. □

Now we use (3.3) to define another orthogonal projector $P_{n,m} : L^2(X, \nu) \to L^2(X, \nu)$ as

$$P_{n,m}(f) = \sum_{j=1}^{m} \langle f, \xi_j\rangle \xi_j, \quad f \in L^2(X, \nu).$$

It follows from Bessel's inequality that

$$\big\|P_n(f) - P_{n,m}(f)\big\|_2^2 = \sum_{j>m} \big|\langle f, \xi_j\rangle\big|^2, \quad f \in L^2(X, \nu), \tag{3.5}$$

and that $P_{n,m}(f)$ converges to $P_n(f)$ as $m \to \infty$. We can also use (3.2) to define

$$K_{n,m}(x,y) = \sum_{U,V \in \mathcal{P}^F_{n,m}} \frac{\chi_U(x)\chi_V(y)}{\nu(U)\nu(V)} \int_{U \times V} k\, d(\nu \times \nu), \quad x, y \in X, \tag{3.6}$$

where $P^F_{n,m} = \{U_{n1}, U_{n2}, \dots, U_{nm}\}$.

4. The main results

In this section, we use Assumptions 3.6 and 3.7 to prove results on the nuclearity of integral operators. We note that these assumptions are not needed in the next result.

Lemma 4.1. *Let k be as in Definition 3.2. Then $P_{n,m}KP_{n,m}$ is a nuclear operator with trace*

$$\operatorname{tr}(P_{n,m}KP_{n,m}) = \int_X k_{n,m}(x,x)\, d\nu(x).$$

Proof. It is clear that each operator $P_{n,m}KP_{n,m}$ has finite rank, and hence, it is nuclear and

$$\operatorname{tr}(P_{n,m}KP_{n,m}) = \sum \langle KP_{n,m}u, P_{n,m}u\rangle, \quad u \in B,$$

to any orthonormal basis B of $L^2(X,\nu)$. If one chooses the basis

$$B = \left\{ \frac{\chi_{U_{n_j}}}{\sqrt{\nu(U_{n_j})}} \right\} \cup B_1,$$

it follows that

$$\begin{aligned}
\operatorname{tr}(P_{n,m}KP_{n,m}) &= \sum_{j=1}^{m} \big\langle K(\xi_j), \xi_j \big\rangle \\
&= \sum_{j=1}^{m} \int_X \Big(\int_{U_{nj}} \frac{k(u,v)\, d\nu(v)}{\sqrt{\nu(U_{nj})}} \frac{\chi_{U_{n_j}}(u)}{\sqrt{\nu(U_{nj})}} \Big)\, d\nu(u) \\
&= \sum_{U \in \mathcal{P}^F_{n,m}} \frac{1}{\nu(U)} \int_{U \times U} k(u,v)\, d\nu(u)\, d\nu(v) \\
&= \int_X k_{n,m}(x,x)\, d\nu(x).
\end{aligned}$$

□

The last result is used to prove one of the main results of this article. Note that the calculations ahead are similar to those in [2].

Theorem 4.2. *Let $K : L^2(X,\nu) \to L^2(X,\nu)$ be an integral operator, and suppose that Assumptions 3.6 and 3.7 hold. If K is in the trace class, then*

$$\int_X H_{F^2}(k)(x,x)\, d\nu(x) < C^2 \|K\|_1,$$

where C comes from (3.4). In particular,

$$\operatorname{tr}(K) = \int_X \tilde{k}(x,x)\, d\nu(x), \tag{4.1}$$

where

$$\tilde{k} = \lim_{n\to\infty} \mathbb{E}(k|F_n \times F_n).$$

Proof. By using Lemma 2.4, one can define

$$g(x,y) = \sum_j s_j \big|\eta_j(x)\big|\big|\xi_j(y)\big|, \quad x,y \in X.$$

It follows that

$$\int_{X\times X} |g|^2 \nu\times\nu \le \Big(\sum_j s_j\Big)^2 < +\infty,$$

and g is in $L^2(X\times X, \nu\otimes\nu)$. Since

$$\big|k(x,y)\big|^2 \le \big|g(x,y)\big|^2 \le \sum_j s_j \big|\eta_j(x)\big|^2 \sum_l s_l \big|\xi_l(y)\big|^2, \quad x,y\in X \text{ (a.e.)},$$

(3.2) implies that

$$\mathbb{E}\big(|k||F_n\times F_n\big) \le \mathbb{E}(g|F_n\times F_n), \quad n\in\mathbb{N},$$

and that

$$H_{F^2}(k) \le H_{F^2}(g).$$

Otherwise, the equalities

$$\begin{aligned}
\mathbb{E}(g|F_n\times F_n)(x,y) &= \sum_{U,V\in\mathcal{P}_n} \frac{\chi_{U\times V}(x,y)}{\nu(U\times V)} \int_{U\times V} g\,d\nu\times\nu \\
&= \sum_{U,V\in\mathcal{P}_n} \frac{\chi_{U\times V}(x,y)}{\nu(U)\nu(V)} \int_U\int_V \sum_j s_j\big|\eta_j(u)\big|\big|\xi_j(v)\big|\,d\nu\times\nu(u,v) \\
&= \sum_j s_j \sum_{U\in\mathcal{P}_n} \frac{\chi_U(x)}{\nu(U)} \int_U |\eta_j|\,d\nu \sum_{U\in\mathcal{P}_n} \frac{\chi_U(y)}{\nu(V)} \int_V |\xi_j|\,d\nu \\
&= \sum_j s_j \mathbb{E}\big(|\eta_j||F_n\big)(x)\mathbb{E}\big(|\xi_j||F_n\big)(y), \quad x,y\in X,
\end{aligned}$$

help us to see that

$$H_{F^2}(k)(x,y) \le \sum_j s_j H_{F^2}(\xi_j)(y) H_{F^2}(\eta_j)(x), \quad x,y\in X.$$

Assumption 3.7 and the Cauchy–Schwarz inequality imply the inequalities

$$\begin{aligned}
\int_X H_{F^2}(k)(x,x)\,d\nu(x) &\le \int_X \sum_j s_j H_{F^2}(\xi_j)(x) H_{F^2}(\eta_j)(x)\,d\nu(x) \\
&\le \sum_j s_j \big\|H_{F^2}(\xi_j)\big\|\big\|H_{F^2}(\eta_j)\big\| \\
&= C^2 \sum_j s_j \\
&= C^2 \|K\|_1.
\end{aligned}$$

The end result stated follows from applications of Lemmas 2.1 and 4.1, which produce (4.1). First, note that

$$\operatorname{tr}(K_{n,m}) = \int_X k_{n,m}(x,x)\,d\nu(x)$$

and that

$$\left|k_{n,m}(x,y)\right| \le H_{F^2}(k)(x,y), \qquad \left|k_n(x,y)\right| \le H_{F^2}(k)(x,y)$$

for all $x, y \in X$. The dominated convergence theorem produces the equalities

$$\operatorname{tr}(K_n) = \lim_{m\to\infty} \operatorname{tr}(K_{n,m}) = \int_X k_n(x,x)\,d\nu(x) = \int_X \mathbb{E}(k|F_n \times F_n)(x,x)\,d\nu(x).$$

The same arguments show that

$$\operatorname{tr}(K) = \lim_{n\to\infty} tr(K_n) = \int_X \tilde{k}(x,x)\,d\nu(x). \qquad \square$$

As we can see, it is generally not an easy task to find the kernel $\tilde{k}$ explicitly and, even if we can do it, we should evaluate an integral. In some cases, we can easily estimate the trace.

Corollary 4.3. *Let $K : L^2(X,\nu) \to L^2(X,\nu)$ be an integral operator, and suppose that Assumptions 3.6 and 3.7 hold. If K is in the trace class, with $k \in L^\infty(X,\nu)$ and $\nu(X) < \infty$, then*

$$\left|\operatorname{tr}(K)\right| \le \nu(X)\|k\|_{L^\infty}.$$

Proof. It follows from the proof of Theorem 4.2 that

$$\left|\operatorname{tr}(K)\right| = \left|\lim_{n\to\infty} \operatorname{tr}(K_n)\right| = \left|\lim_{n\to\infty}\left(\lim_{m\to\infty}\int_X k_{n,m}(x,x)\,d\nu(x)\right)\right|.$$

Now, we can use (3.6) to see that

$$\left|k_{n,m}(x,y)\right| \le \|k\|_{L^\infty}, \quad x, y \in X.$$

The result follows. $\square$

Note that [10] uses the positivity of K as a hypothesis to prove a version of the last theorem, and we do not. On the other hand, a careful read of Jefferies's proof will make it clear that the author does not use positivity there. If we add positive definiteness (2.1) to K, then we can rewrite the preceding theorem as an equivalence. These results are extensions of those given in [2], [3], [6], and [10].

Theorem 4.4. *Let $K : L^2(X,\nu) \to L^2(X,\nu)$ be a positive integral operator, and suppose that Assumptions 3.6 and 3.7 hold. Then K is in the trace class if and only if*

$$\int_X H_{F^2}(k)(x,x)\,d\nu(x) < \infty. \tag{4.2}$$

Proof. We need only do the first part; the other follows from Theorem 4.2. As such, let us assume that (4.2) holds. Since K is a positive operator, the inequalities

$$\begin{aligned} k_{n,m}(x,x) &= \sum_{U\in\mathcal{P}^F_{n,m}} \frac{\chi_U(x)}{\nu(U)^2}\int_U\int_U k(u,v)\,d\nu(u)\,d\nu(v) \\ &= \sum_{U\in\mathcal{P}^F_{n,m}} \frac{\chi_U(x)}{\nu(U)^2}\int_X\int_X k(u,v)\chi_U(v)\,d\nu(v)\overline{\chi_U(u)}\,d\nu(u) \\ &= \sum_{U\in\mathcal{P}^F_{n,m}} \frac{\chi_U(x)}{\nu(U)^2}\langle K\chi_U,\chi_U\rangle \\ &\geq 0, \quad x\in X, \end{aligned}$$

clearly show that $0\leq k_{n,m}(x,x)\leq k_n(x,x)\leq H_{F^2}(k)(x,x)$ for $x\in X$. Applying Lemma 4.1, we find that

$$\operatorname{tr}(P_{n,m}KP_{n,m})\leq \int_X H_{F^2}(k)(x,x)\,d\nu(x),\quad n,m=1,2,\ldots.$$

It follows from (3.5) that $P_{n,m}(f)$ converges to $P_n(f)$ for each $f\in L^2(X,\nu)$. Since $\|K_{n,m}\|_1=\operatorname{tr}(K_{n,m})$ and $\|K_n\|_1=\operatorname{tr}(K_n)$, because both are positive definite, Lemma 2.2 may be used to ensure that the K_n's are in the trace class, with

$$\operatorname{tr}(K_n)\leq \int_X H_{F^2}(k)(x,x)\,d\nu(x).$$

To finish the proof, note that Proposition 3.8 implies that $K_n(f)$ converges to $K(f)$ for all $f\in L^2(X,\nu)$. Another application of Lemma 2.2 shows that K is in the trace class and that

$$\|K\|_1=\operatorname{tr}(K)\leq \int_X H_{F^2}(k)(x,x)\,d\nu(x).$$

□

5. Final results and remarks

We start this final section with the next result, which completes the statements of Theorem 4.4. Its proof is based on the spectral decomposition of K, as given in Lemma 2.4, and it can be found in [6, Lemma 5.1].

Lemma 5.1. *If K is a Hilbert–Schmidt operator in $L^2(X,\nu)$, then the equality*

$$K=\alpha_1K_1+\alpha_2K_2+\alpha_3K_3+\alpha_4K_4$$

holds, where the operators $K_i: L^2(X,\nu)\to L^2(X,\nu)$ are positive definite and each α_i is a complex number.

To better manipulate $\tilde{k}$ in Theorem 4.2, we use the following new assumption from now on.

Assumption 5.2. The measure space $(X,\mathcal{A},\nu)$ and the filter F are such that

$$\lim_{n\to\infty} k_n(x,x)=k(x,x),\quad x\in X \text{ (a.e.)}.$$

Remark 5.3. Note that Assumption 3.7 implies that

$$\tilde{k}(x,y) = k(x,y), \quad x,y \in X \text{ (a.e.)},$$

and that the diagonal of $X \times X$ has null measure and enables that $\tilde{k}(x,x) \neq k(x,x)$, for almost all $x \in X$. On the other hand, Assumption 5.2 clearly holds when k is a continuous kernel and X is in a context where Lebesgue's differentiation theorem may be applied. This is the case when X is a Lebesgue measurable subset (with boundary $\partial(X)$ with null measure) of the Euclidean space $\mathbb{R}^m$, or a subset of Euclidean manifolds like the sphere $\mathbb{S}^m$ or the torus $\mathbb{T}^m$. These previous settings include, for instance, those in which $d\nu(x) = \rho(x)\, d\mu(x)$, where μ is the usual Lebesgue measure and the weight $\rho : \mathbb{R}^m \to (0,\infty)$ is a continuous function.

In the case in which $X \subset \mathbb{R}^m$, a filter F is constructed by using cubes of sides 2^{-n} for each partition related to F_n. In this way, the $U_n(x)$'s shrink nicely to x, if x is in the interior of X (see [7, p. 98]). A similar process can be applied to other special surfaces like spheres, tori, and similar manifolds. See Theorem 2.4 in [2] and [3] to guarantee that our assumptions 3.6, 3.7, and 5.2 still hold on a σ-finite Borel measure on a second countable space.

The last assumption enables us to obtain the next corollaries of the main results. Note that these are different ways to prove and extend the results of Sections 4 and 5 of [6], and include some results of [2], [6], and [10] as special contexts.

Corollary 5.4. *Let $K : L^2(X,\nu) \to L^2(X,\nu)$ be an integral operator, and suppose that Assumptions 3.6, 3.7, and 5.2 hold. If K is in the trace class, then*

$$\operatorname{tr}(K) = \int_X k(x,x)\, d\nu(x).$$

Proof. The proof follows from Theorem 4.2. □

We conclude with a result summarizing the previous one.

Corollary 5.5. *Let $K : L^2(X,\nu) \to L^2(X,\nu)$ be a positive integral operator, and suppose that Assumptions 3.6, 3.7, and 5.2 hold. We have that K is in the trace class if and only if the function $x \in X \to K(x,x)$ is in $L^1(X,\nu)$. If K is in the trace class, then*

$$\operatorname{tr}(K) = \int_X k(x,x)\, d\nu(x).$$

Proof. The proof follows from Theorems 4.2 and 4.4. □

We finish this article with a classical and beautiful example (of a positive integral operator) and a counterexample (with a nonpositive integral operator) to this kind of result, when $X = [0,1]$ (with the usual Lebesgue measure). The first one is related to Poisson's equation

$$u'' = -\lambda u, \quad u(0) = u(1) = 0.$$

This boundary value problem is related to the positive definite integral operator K with kernel

$$k(x,y) = \min(x,y) - xy, \quad x,y \in [0,1].$$

It is straightforward to find the eigensystem of K as

$$s_n = \frac{1}{n^2\pi^2}, \qquad \phi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \dots .$$

Corollary 5.5 helps us find a solution to the Basel problem as

$$\mathrm{Tr}(K) = \frac{1}{\pi^2}\Big(\sum_{n=1}^{\infty}\frac{1}{n^2}\Big) = \int_0^1 (x - x^2)\,dx = \frac{1}{6}.$$

For a counterexample, we consider the Volterra integral operator

$$T(f)(x) = \int_0^x f(t)\,dt, \quad f \in L^2[0,1],$$

with nonpositive definite kernel

$$G(x,y) = \chi_{[0,x]}(y), \quad x, y \in [0,1].$$

A direct calculation shows that

$$T^* \circ T(f)(x) = \int_x^1 \int_0^s f(t)\,dt\,ds, \quad f \in L^2[0,1],$$

has eigensystem

$$\phi_n(x) = c_n \cos\Big(\frac{(2n-1)}{2}\pi x\Big), \qquad s_n^2 = \Big(\frac{2}{(2n-1)\pi}\Big)^2.$$

This means that

$$\sum_{n=1}^{\infty} s_n = +\infty$$

and that T is not in the trace class, but rather

$$\int_0^1 G(x,x)\,dx = 1.$$

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