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ON A LIFTING QUESTION OF BLACKADAR

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To Huaxin Lin on his sixtieth birthday

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ABSTRACT. Let A be the AF algebra whose scaled ordered group $K_0(A)$ is $(G\oplus H, (G_+\setminus\{0\})\oplus H\cup\{(0,0)\}, \tilde{g}\oplus 0)$, where $(G, G_+, \tilde{g})$ is the scaled ordered group $K_0(B)$ of a unital simple AF algebra B, and H is a countable torsion-free Abelian group. Let σ be an order 2 scaled ordered automorphism of $K_0(A)$, defined by $\sigma(g,h) = (g,-h)$, where $(g,h)\in G\oplus H$. We show that there is an order 2 automorphism α of A such that $\alpha_* = \sigma$. This gives a partial answer to a lifting question posed by Blackadar. Incidentally, the lift α we construct has the tracial Rokhlin property. Consequently, the crossed product $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple AH algebra with no dimension growth.

1. Introduction

An important recent problem has been to find and classify all order 2 automorphisms of an AF algebra. Historically, partly because of their intrinsic interest and partly because of their applications in C^*-dynamical systems, these kinds of problems have attracted considerable attention in the literature (see, e.g., [4], [6], [17]). Notable among these efforts is Blackadar's famous construction. Blackadar [4] constructed an action of $\mathbb{Z}_2$ on the 2^∞ *uniformly hyperfinite* (UHF) algebra such that the crossed product has nontrivial K_1-group, and hence gave a negative answer to one of two questions about AF algebras posed by him in [3, Section 10.11.3]. The other one is a lifting question, which is as follows.

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Question 1.1. Let A be an AF algebra, and let σ be an automorphism of the scaled ordered group $K_0(A)$ with $\sigma^n = 1$. Is there an automorphism α of A with $\alpha_* = \sigma$ and $\alpha^n = 1$?

Throughout this paper, for simplicity, we only consider the case where A is unital, simple, and where $n = 2$. Clearly, this question is important in studying the algebraic structure of $\mathrm{Aut}(A)$. However, this question appears to be open still, and even satisfactory partial answers to this question are very scarce. Until very recently, Barlak and Szabó [1, Corollary 2.13] showed that if A is a separable, unital, simple, and nuclear C^*-algebra with tracial rank zero, then any $\mathbb{Z}_2$-action on the invariant could be lifted to a Rokhlin action of $\mathbb{Z}_2$ on A, provided that A absorbs the 2^∞ UHF algebra.

In this article, we consider the following. Let A be the AF algebra whose scaled ordered group $K_0(A)$ is $(G \oplus H, (G_+ \setminus \{0\}) \oplus H \cup \{(0,0)\}, \tilde{g} \oplus \tilde{h})$, where $(G, G_+, \tilde{g})$ is the scaled ordered group $K_0(B)$ of a unital simple AF algebra B, H is a countable torsion-free Abelian group, and $\tilde{h} \in H$. Let σ be an order 2 scaled ordered automorphism of $K_0(A)$, defined by $\sigma(g,h) = (g, \eta(h))$, where $(g,h) \in G \oplus H$, η is an order 2 automorphism of H which is of type I (see Definition 2.3). Then there is an order 2 automorphism α of A such that $\alpha_* = \sigma$ (see Theorem 4.1). This provides a partial affirmative answer to Question 1.1. Moreover, we can choose that the action α has the tracial Rokhlin property. In this case, $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple AH algebra with no dimension growth (see Theorem 4.1).

Remark 1.2. Let $(G, G_+, \tilde{g})$ be the scaled ordered group $K_0(B)$ of a unital simple AF algebra B, let H be any countable torsion-free Abelian group, and let $\tilde{h} \in H$. Set

$$F = G \oplus H, \qquad F_+ = \big(G_+ \setminus \{0\}\big) \oplus H \cup \big\{(0,0)\big\}, \qquad u = \tilde{g} \oplus \tilde{h}.$$

Then it is easy to check that (F, F_+, u) is a scaled ordered unperforated simple group; moreover, (F, F_+, u) also has the Riesz interpolation property (see Remark 4.3 or [10, Theorem 8.1]). Then by the Effros–Handelman–Shen theorem (see [9, Theorem 2.2]), (F, F_+, u) is a dimension group.

Our K-theory setup, which is assumed for the AF algebra A in question, has a kind of "split" property (some concrete examples can be found in [3, Section 10.11.3], [6, Section 1], [17, Examples 4.1 and 4.5]). Roughly speaking, our strategy is to rewrite A as an AH algebra in a nonstandard way and then find an order 2 product-type action α with tracial Rokhlin property such that $\alpha_* = \sigma$. The construction is inspired by Blackadar's own in [4], and also by [17], [15], and [10]. The classification results for C^*-algebras, which have been obtained by many authors as a part of the Elliott program, play an important role here, illustrating the power of this theory.

This article is organized as follows. In Section 2, we review definitions, elementary facts, and important results which we need in later sections. Section 3 contains several technical constructions and existence results. In Section 4, we prove our main result (Theorem 4.1).

We use the notation $\mathbb{Z}_2$ for $\mathbb{Z}/2\mathbb{Z}$. We write $\tilde{K}^0(X)$ for the reduced K_0-group of a topological space X. If A is a C^*-algebra and $\alpha : A \to A$ is an automorphism of order 2, then we write $C^*(\mathbb{Z}_2, A, \alpha)$, A^α for the crossed product and the fixed point subalgebra of A by the action of $\mathbb{Z}_2$ generated by α, respectively. Also, RR and dr denote the *real rank* and the *decomposition rank*, respectively, for convenience. We take $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$.

2. Preliminaries

In this section, we recall basic definitions and properties which will be used throughout the article.

2.1. Here are some basic properties of Brouwer's notion of degree for maps $S_n \to S_n$.

(1) Let $f, g : S_n \to S_n$, if $f \simeq g$, that is, f is homotopic to g. Then $\deg(f) = \deg(g)$.
(2) Let $(x, y, z) \in S^2$, and let λ be the reflection map defined by $\lambda(x, y, z) = (x, y, -z)$. Then $\deg(\lambda) = -1$.
(3) If $f : S^n \to S^n$ has degree d, then $f^* : H^n(S_n, \mathbb{Z}) \to H^n(S_n, \mathbb{Z})$ is multiplication by d (see [12, p. 205, Exercise 9]).

2.2. Let X be a connected 3-dimensional finite CW complex. Then $\tilde{K}^0(X) \cong H^2(X, \mathbb{Z})$ (see [11, Section 3.12]).

Lemma 2.1 ([2, Lemma 2.1])**.** *Let T be an involutory matrix in $M_k(\mathbb{Z})$, that is, $T^2 = I$. Then there is an invertible matrix $S \in M_k(\mathbb{Z})$, and nonnegative integers p, q, r, such that*

$$T = S^{-1} \operatorname{diag}\Big\{\overbrace{1, \ldots, 1}^{p}; \overbrace{-1, \ldots, -1}^{q}; \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{r}\Big\} S.$$

Proposition 2.2. *Let H be a countable torsion-free Abelian group, and let η be an order 2 automorphism of H. Then there are a nondecreasing sequence of positive integers $\{n_k : k \in \mathbb{N}\}$ and monomorphisms $\beta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_{k+1}}$, order 2 automorphisms $\eta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_k}$ such that the following diagram commutes*

$$\begin{array}{ccccccc}
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots H \\
\downarrow{\eta_1} & & \downarrow{\eta_2} & & \downarrow{\eta_3} & & \\
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots H
\end{array}$$

and $H = \lim_{k\to\infty}(\mathbb{Z}^{n_k}, \beta_k)$, $\eta = \lim_{n\to\infty} \eta_k$. Moreover, it can be required that under the canonical basis of $\mathbb{Z}^{n_k}$, η_k has the form

$$\eta_k = \operatorname{diag}\Big\{\overbrace{1, \ldots, 1}^{p_k}; \overbrace{-1, \ldots, -1}^{q_k}; \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{r_k}\Big\}$$

for suitable nonnegative integers p_k, q_k, r_k such that $p_k + q_k + 2r_k = n_k$, $k \in \mathbb{N}$.

Proof. Since H is countable, we can write H as $H = \{e_1, e_2, \ldots\}$, and define

$$H_k = \mathbb{Z}\big[e_1, \ldots, e_k; \eta(e_1), \ldots, \eta(e_k)\big],$$

so for $k \in \mathbb{N}$, $\eta(H_k) = H_k$, $H = \lim_{k\to\infty}(H_k, \iota_k)$, where ι_k is the embedding map from H_k to H_{k+1}. Since H_k is finitely generated, without loss of generality, only replacing ι_k by some suitable monomorphism β'_k, we may assume that $H = \lim_{k\to\infty}(H_k, \beta'_k)$, where $H_k = \mathbb{Z}^{n_k}$, $k \in \mathbb{N}$. Moreover, there is a sequence of order 2 automorphisms η'_k, such that $\eta'_{k+1} \circ \beta'_k = \beta'_k \circ \eta'_k$, $k \in \mathbb{N}$, and $\eta = \lim_{k\to\infty} \eta'$. The proposition follows immediately from Lemma 2.1. □

Inspired by the preceding proposition, we introduce the following definition.

Definition 2.3. Let H be a countable torsion-free Abelian group, and let η be an order 2 automorphism of H. Then η is said to be of *type I* if there are a sequence of positive integers $\{n_k : k \in \mathbb{N}\}$ and monomorphisms $\beta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_{k+1}}$, order 2 automorphisms $\eta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_k}$ such that the following diagram commutes

$$\begin{array}{ccccccc}
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots H \\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \\
\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots H
\end{array}$$

and $H = \lim_{k\to\infty}(\mathbb{Z}^{n_k}, \beta_k)$, $\eta = \lim_{n\to\infty} \eta_k$, under the canonical basis of $\mathbb{Z}^{n_k}$, η_k has the form

$$\eta_k = \operatorname{diag}\{\overbrace{1, \ldots, 1}^{p_k}; \overbrace{-1, \ldots, -1}^{q_k}\},$$

for suitable nonnegative integers p_k, q_k such that $p_k + q_k = n_k$, $k \in \mathbb{N}$.

One special and important case is the *minus one map*, that is, $\eta(h) = -h$, for $h \in H$.

2.3. Let A_1, A_2 be unital C^*-algebras, and let φ be a unital monomorphism from A_1 to A_2. Let $\alpha_i : A_i \to A_i$ be an automorphism, $i = 1, 2$. Assume that $\alpha_2 \circ \varphi = \varphi \circ \alpha_1$. Then for any $a \in A_1^{\alpha_1}$, $\varphi(a) \in A_2^{\alpha_2}$.

By telescoping the Bratteli diagram, the following fact appears to be well known.

Lemma 2.4 ([13, Proposition 4.7.2, Lemma 4.7.3])**.** *Let A be a unital simple AF algebra, and let n be a positive integer. Let $a_1 < a_2 < \cdots$ be an increasing sequence of positive integers, and let $c_1 < c_2 < \cdots$ be another sequence of positive integers. Then $K_0(A)$ can be written so that*

$$\big(K_0(A), K_0(A)_+\big) = \lim_{k\to\infty}\big(G_k, (G_k)_+, \varphi_k\big),$$

where G_k is a finite direct sum of m_k copies of $\mathbb{Z}$ with $m_k \geq a_k$, and each partial map of φ_k has positive multiplicity at least c_k, $k \in \mathbb{N}$.

Theorem 2.5 ([11, Theorem 5.8]). *Let A and B be simple real rank zero inductive limits of direct sums of matrices over 3-dimensional CW complexes. Then $A \cong B$ if and only if*

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \cong \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

The following key idea is due to Lin.

Definition 2.6 ([13, Definition 3.6.2]). Let A be a unital simple C^*-algebra. Then the tracial rank of A is zero if the following holds: for any finite subset $\mathcal{F} \subset A$, $\varepsilon > 0$ and $c \in A_+ \setminus \{0\}$, there exist a nonzero projection $p \in A$ and a finite-dimensional C^*-subalgebra $B \subset A$ and with $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $\mathrm{dist}(pxp, B) \leq \varepsilon$ for all $x \in \mathcal{F}$,
(3) $1 - p$ is (Murray–von Neumann) equivalent to a projection in $\overline{cAc}$.

Lastly, we introduce a special case of a useful criterion for an action of $\mathbb{Z}_2$ to have the tracial Rokhlin property obtained by Phillips. The reader is referred to Phillips's seminal work [16] for details and more background information about the (tracial) Rokhlin property.

Lemma 2.7 ([17, Lemma 1.8]). *Let A be a finite infinite-dimensional simple unital C^*-algebra with tracial rank zero. Let $\alpha \in \mathrm{Aut}(A)$ satisfy $\alpha^2 = \mathrm{id}_A$. Suppose that for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections e_0, e_1 such that*

(1) $\|\alpha(e_0) - e_1\| < \varepsilon$,
(2) $\|e_j a - a e_j\| < \varepsilon$ *for all* $a \in F$ *and* $j = 0, 1$,
(3) *with* $e = e_0 + e_1$, $\tau(1 - e) < \varepsilon$ *for each tracial state* τ *on* A.

The action of $\mathbb{Z}_2$ generated by α has the tracial Rokhlin property.

3. Existence results

We start with the following construction.

3.1. Let $S^2 = \{(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) : \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}], \theta \in [-\pi, \pi]\}$ be an oriented unit 2-dimensional sphere. Define $\chi : S^2 \to S^2$ by

$$\chi(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \begin{cases} (\cos\varphi, \sin\varphi, 0), & \theta \in [0, \pi], \\ (\sin(2\theta + \frac{\pi}{2})\cos\varphi, \sin(2\theta + \frac{\pi}{2})\sin\varphi\cos(2\theta + \frac{\pi}{2})), & \theta \in [-\pi, 0]. \end{cases}$$

Then χ is a continuous surjective map from S^2 to S^2, and $\chi \simeq \mathrm{id}_{S^2}$. Moreover, χ is symmetric with respect to the xy-plane; that is, if $x_1, x_2 \in S^2$ are symmetric with respect to the xy-plane, then $\chi(x_1), \chi(x_2)$ are also symmetric with respect to the xy-plane. That is to say, χ and λ do commute, where λ is defined as in the preceding Section 2.1(2) above.

3.2. Let $S^2 = \{(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) : \varphi \in [0, 2\pi], \theta \in [0, \pi]\}$ be an oriented unit 2-dimensional sphere. For $0 \le \varphi_1 < \varphi_2 \le 2\pi$, consider the following subset of S^2:

$$I_{[\varphi_1,\varphi_2]} = \big\{(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) : \varphi \in [\varphi_1, \varphi_2], \theta \in [0, \pi]\big\}.$$

For $k \ne 0$, define

$$\begin{aligned}&\hat{g}_{[\varphi_1,\varphi_2],k}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\\ &\quad = \Big(\sin\theta\cos\Big(2k\pi\frac{\varphi-\varphi_1}{\varphi_2-\varphi_1}\Big), \sin\theta\sin\Big(2k\pi\frac{\varphi-\varphi_1}{\varphi_2-\varphi_1}\Big), \cos\theta\Big).\end{aligned}$$

Then for each integer $k \ne 0$, $\hat{g}_{[\varphi_1,\varphi_2],k}$ is a continuous surjective map from $I_{[\varphi_1,\varphi_2]}$ to S^2. Let $g_{[\varphi_1,\varphi_2],k} = \chi \circ \hat{g}_{[\varphi_1,\varphi_2],k}$. Then $g_{[\varphi_1,\varphi_2],k}$ is also a continuous surjective map from $I_{[\varphi_1,\varphi_2]}$ to S^2 which satisfies the following:

(1) $g_{[\varphi_1,\varphi_2],k}$ maps the boundary of $I_{[\varphi_1,\varphi_2]}$ to the point $(1, 0, 0) \in S^2$;
(2) $g_{[\varphi_1,\varphi_2],k}$ is symmetric with respect to the xy-plane, that is, if $x_1, x_2 \in I_{[\varphi_1,\varphi_2]}$ are symmetric with respect to the xy-plane, then $g_{[\varphi_1,\varphi_2],k}(x_1)$, $g_{[\varphi_1,\varphi_2],k}(x_2)$ are also symmetric with respect to the xy-plane (i.e., $g_{[\varphi_1,\varphi_2],k}$ and λ do commute).

Inspired by [15, Definition 3.3], we have the following constructions.

3.3. Let

$$\beta = \begin{bmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n_1} \\ t_{2,1} & t_{2,2} & \cdots & t_{2,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n_2,1} & t_{n_2,2} & \cdots & t_{n_2,n_1} \end{bmatrix}$$

be a monomorphism from $\mathbb{Z}^{n_1}$ to $\mathbb{Z}^{n_2}$. Let η_k be an order 2 automorphism of $\mathbb{Z}^{n_k}$ which has the form

$$\eta_k = \mathrm{diag}\{\overbrace{1, \dots, 1}^{p_k}; \overbrace{-1, \dots, -1}^{q_k}\}$$

under the canonical basis of $\mathbb{Z}^{n_k}$, such that $\beta \circ \eta_1 = \eta_2 \circ \beta$, $k = 1, 2$. It is straightforward to check that $t_{j,i} = 0$ when $\eta_{1i}\eta_{2j} = -1$, where $1 \le i \le n_1$, $1 \le j \le n_2$.

For $k = 1, 2$, let $X_{k,l} = \{(x, y, z, w) : x^2 + y^2 + z^2 = 1, w = l\} \subset \mathbb{R}^4, \text{\o}_{k,l} = (1, 0, 0, l)$. Then $X_{k,l} = S^2$, $l = 1, 2, \dots, n_k$. We define $X_k = \bigvee_{1\le l\le n_k} X_{k,l}$ to be the quotient of $\bigsqcup_{1\le i\le n_k} X_{k,i}$ obtained by identifying $\text{\o}_{k,1}, \dots, \text{\o}_{k,n_k}$ to a single point $\text{\o}_1$, and we define $\pi_k : \bigsqcup_{1\le l\le n_k} X_{k,l} \to X_k$ to be the corresponding quotient map. As we know, $\tilde{K}^0(X_1) = \mathbb{Z}^{n_1}$ and $\tilde{K}^0(X_2) = \mathbb{Z}^{n_2}$.

First, we will define an order 2 homeomorphism ω_k of X_k such that the induced map $\omega_k^* : \tilde{K}^0(X_k) \to \tilde{K}^0(X_k)$ is exactly η_k, $k = 1, 2$. Fix $k \in \{1, 2\}$. For $l \in \{1, \dots, p_k\}$, define $\omega_k^l : X_{k,l} \to X_{k,l}$ by

$$\omega_k^l(\xi) = \xi;$$

for $l \in \{p_k + 1, \dots, p_k + q_k = n_k\}$, define $\omega_k^l : X_{k,l} \to X_{k,l}$ by

$$\omega_k^l(x, y, z, l) = \big(\lambda(x, y, z), l\big) = (x, y, -z, l), \quad \text{where } (x, y, z, l) = \xi \in X_{k,l}.$$

Note that $\pi_k(\omega_k^l(\text{\o}_{k,l})) = \text{\o}_k$.

For $\zeta \in X_k$, if $\zeta = \varnothing_k$, define $\omega_k(\zeta) = \varnothing_k$; if $\zeta \neq \varnothing_k$, there exists a unique $\xi \in \bigsqcup_{1\leq l\leq n_k} X_{k,l}$ such that $\pi_k(\xi) = \zeta$. Let $l_1 = P_4(\xi)$, where P_4 is the projection onto the fourth coordinate axis, and define $\omega_k(\zeta) = \pi_k(\omega_k^{l_1}(\xi))$. Then by Section 2.1, it is straightforward to check that ω_k is an order 2 homeomorphism ω_i of X_k such that the induced map $\omega_k^* : \tilde{K}^0(X_k) \to \tilde{K}^0(X_k)$ is exactly η_k, and $\varnothing_k$ is a fixed point of ω_k, $k = 1, 2$.

Lemma 3.1. *There exists a map $s : X_2 \to X_1$ such that the induced map $s^* : \tilde{K}^0(X_1) \to \tilde{K}^0(X_2)$ is exactly β; moreover, $s \circ \omega_2 = \omega_1 \circ s$.*

Proof. For each $1 \leq j \leq n_2$, set

$$I_{j,i} = \Big\{(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta, w) : \varphi \in \Big[\frac{2\pi(i-1)}{n_1}, \frac{2\pi i}{n_1}\Big], \theta \in [0,\pi], w = j\Big\} \subset X_{2,j},$$

$i = 1, 2, \ldots, n_1$. For each pair $(j, i) \in \{1, 2, \ldots, n_2\} \times \{1, 2, \ldots, n_1\}$, we define a continuous map from $I_{j,i}$ to $X_{1,i}$ by

(1) $s_{j,i}(x, y, z, j) = (1, 0, 0, i)$, if $t_{j,i} = 0$;
(2) $s_{j,i}(x, y, z, j) = (g_{[\frac{2\pi(i-1)}{n_1}, \frac{2\pi i}{n_1}], t_{j,i}}(x, y, z), i)$, if $t_{j,i} \neq 0$.

Fix $j \in \{1, 2, \ldots, n_2\}$. Since $s_{j,i}$ always maps the boundary of $I_{j,i}$ to the point $(1, 0, 0, i) = \varnothing_{1,i}$, and recalling that $\pi_1(1, 0, 0, i) = \varnothing_1$ for each $i \in \{1, 2, \ldots, n_1\}$, by the gluing lemma in general topology, define a map $s^j : X_{2,j} = \bigcup_{1\leq i\leq n_1} I_{j,i} \to X_1$ by gluing together the $\pi_1 \circ s_{j,i}$, that is, $s^j(\xi) = \pi_1 \circ s_{j,i}(\xi)$ if $\xi \in I_{j,i}$. Note that each s^j maps $\varnothing_{2,j}$ to $\varnothing_1$, a repeated use of the gluing lemma. Define $s : X_2 \to X_1$ by $s(\zeta) = s^j(\xi)$, where $\zeta = \pi_2(\xi)$. Note that $s(\varnothing_2) = \varnothing_1$. Moreover, as β is injective, then for each $1 \leq i \leq n_1$, $\sum_{j=1}^{n_2} |t_{j,i}| > 0$, and it follows that s is surjective.

We now turn to check $s \circ \omega_2 = \omega_1 \circ s$. If $\zeta_2 = \varnothing_2$, it is evident that $s \circ \omega_2(\zeta_2) = \omega_1 \circ s(\zeta_2) = \varnothing_1$. If $\zeta_2 \neq \varnothing_2$, then there exists a unique $\xi_2 \in \bigsqcup_{1\leq j\leq n_2} X_{2,j}$ such that $\pi_2(\xi_2) = \zeta_2$. Let $j^* = P_4(\xi_2)$. Then $\xi_2 \in X_{2,j^*}$. Since $\pi_2(s_{j^*}(\partial I_{j,i})) = \varnothing_2$ for each $i \in \{1, 2, \ldots, n_1\}$ and $s_{j^*}(\xi_2) \neq \varnothing_2$, there exists a unique $i^* \in \{1, 2, \ldots, n_1\}$ such that $\xi_2 = (x, y, z, j^*) \in I^\circ_{j^*,i^*}$, where X° denotes the interior of a topological space X. We have the following.

(1) If $t_{j^*,i^*} \neq 0$, and $\eta_{1i^*} = \eta_{2j^*} = 1$, then

$$\begin{aligned}
\omega_1 \circ s(\zeta_2) &= \omega_1^{i^*}\big(s_{j^*,i^*}(\xi_2)\big)\\
&= \omega_1^{i^*}\big(g_{[\frac{2\pi(i^*-1)}{n_1}, \frac{2\pi i^*}{n_1}], t_{j^*,i^*}}(x, y, z), i^*\big)\\
&= \big([g_{[\frac{2\pi(i^*-1)}{n_1}, \frac{2\pi i^*}{n_1}], t_{j^*,i^*}}](x, y, z), i^*\big);\\
s \circ \omega_2(\zeta_2) &= s_{j^*,i^*}\big(\omega_2^{j^*}(\xi_2)\big)\\
&= s_{j^*,i^*}(x, y, z, j^*)\\
&= \big([g_{[\frac{2\pi(i^*-1)}{n_1}, \frac{2\pi i^*}{n_1}], t_{j^*,i^*}}(x, y, z), i^*\big).
\end{aligned}$$

(2) If $t_{j^*,i^*} \neq 0$, and $\eta_{1i^*} = \eta_{2j^*} = -1$, then

$$\begin{aligned}\omega_1 \circ s(\zeta_2) &= \omega_1^{i^*}\big(s_{j^*,i^*}(\xi_2)\big)\\ &= \omega_1^{i^*}\big(g_{[\frac{2\pi(i^*-1)}{n_1},\frac{2\pi i^*}{n_1}],t_{j^*,i^*}}(x,y,z),i^*\big)\\ &= \big([\lambda \circ g_{[\frac{2\pi(i^*-1)}{n_1},\frac{2\pi i^*}{n_1}],t_{j^*,i^*}}](x,y,z),i^*\big);\\ s\circ\omega_2(\zeta_2) &= s_{j^*,i^*}\big(\omega_2^{j^*}(\xi_2)\big)\\ &= s_{j^*,i^*}\big(\lambda(x,y,z),j^*\big)\\ &= \big([g_{[\frac{2\pi(i^*-1)}{n_1},\frac{2\pi i^*}{n_1}],t_{j^*,i^*}}\circ\lambda](x,y,z),i^*\big).\end{aligned}$$

(3) If η_{1i^*} and η_{2i^*} have different signs, then $t_{j^*,i^*} = 0$. Hence

$$\begin{aligned}\omega_1 \circ s(\zeta_2) &= \omega_1^{i^*}\big(s_{j^*,i^*}(\xi_2)\big)\\ &= \omega_1^{i^*}\big(\lambda(1,0,0),i^*\big) = (1,0,0,i^*);\\ s\circ\omega_2(\zeta_2) &= s_{j^*,i^*}\big(\omega_2^{j^*}(\xi_2)\big) = (1,0,0,i^*).\end{aligned}$$

By Section 3.2, $\lambda \circ g_{[\frac{2\pi(i^*-1)}{n_1},\frac{2\pi i^*}{n_1}],t_{j^*,i^*}} = g_{[\frac{2\pi(i^*-1)}{n_1},\frac{2\pi i^*}{n_1}],t_{j^*,i^*}} \circ \lambda$, and it follows that $\omega_1 \circ s(\zeta_2) = s \circ \omega_2(\zeta_2)$.

Lastly, by Section 2.1 and Section 2.2, it is standard to check that the induced map $s^* : \tilde{K}_0(X_1) \to \tilde{K}_0(X_2)$ is exactly β. □

4. The main result

We are now in a position to prove the main result.

Theorem 4.1. *Let A be the AF algebra whose scaled ordered group $K_0(A)$ is $(G \oplus H, (G_+ \setminus \{0\}) \oplus H \cup \{(0,0)\}, \tilde{g} \oplus \tilde{h})$, where $(G, G_+, \tilde{g})$ is the scaled ordered group $K_0(B)$ of a unital simple AF algebra B, H is a countable torsion-free Abelian group, and $\tilde{h} \in H$. Let σ be an order 2 scaled ordered automorphism of $K_0(A)$, defined by $\sigma(g,h) = (g, \eta(h))$, where $(g,h) \in G \oplus H$, and η is an order 2 automorphism of H which is of* type I. *Then there is an order* 2 *automorphism α of A such that $\alpha_* = \sigma$. Moreover, α could be constructed to have the tracial Rokhlin property. In this case, $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple AH algebra with no dimension growth.*

Proof. Since η is of type I, there is a sequence of positive integers $\{n_k : k \in \mathbb{N}\}$ and monomorphisms $\beta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_{k+1}}$ and of order 2 automorphisms $\eta_k : \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_k}$ such that the following diagram commutes

$$\begin{array}{ccccccc}\mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots H\\ \big\downarrow \eta_1 & & \big\downarrow \eta_2 & & \big\downarrow \eta_3 & & \\ \mathbb{Z}^{n_1} & \xrightarrow{\beta_1} & \mathbb{Z}^{n_2} & \xrightarrow{\beta_2} & \mathbb{Z}^{n_3} & \xrightarrow{\beta_3} & \cdots H\end{array}$$

and $H = \lim_{k\to\infty}(\mathbb{Z}^{n_k}, \beta_k)$, $\eta = \lim_{n\to\infty}\eta_k$. Moreover, under the canonical basis of $\mathbb{Z}^{n_k}$, η_k has the form

$$\eta_k = \mathrm{diag}\{\overbrace{1,\ldots,1}^{p_k};\overbrace{-1,\ldots,-1}^{q_k}\}$$

for suitable nonnegative integers p_k, q_k such that $p_k + q_k = n_k$, $k \in \mathbb{N}$.

Set $X_k = S^2 \vee S^2 \vee \cdots \vee S^2$, where S^2 repeats n_k times, $k \in \mathbb{N}$. Then by Section 3.3 and Lemma 3.1, there exist an order 2 homeomorphism ω_k of X_k such that ${\omega_k}^* = \eta_k$, and a surjective continuous map s_k from X_{k+1} to X_k such that the induced map $s_k^* : \tilde{K}^0(X_k) \to \tilde{K}^0(X_{k+1})$ is exactly β_k, $k \in \mathbb{N}$; and the following diagram commutes:

$$\begin{array}{ccccccc}
X_1 & \xleftarrow{s_1} & X_2 & \xleftarrow{s_2} & X_3 & \xleftarrow{s_3} & \cdots \\
\uparrow{\scriptstyle\omega_1} & & \uparrow{\scriptstyle\omega_2} & & \uparrow{\scriptstyle\omega_3} & & \\
X_1 & \xleftarrow{s_1} & X_2 & \xleftarrow{s_2} & X_3 & \xleftarrow{s_3} & \cdots
\end{array}$$

Without loss of generality, we may assume that there exists $h_1 \in \mathbb{Z}^{n_1}$ such that $\beta_{1,\infty}(h_1) = \tilde{h}$. It is easy to see that $\eta_1(h_1) = h_1$. Denote

$$h_1 = (\lambda_1, \ldots, \lambda_{p_1}; \zeta_1, \ldots, \zeta_{q_1}) \in \mathbb{Z}^{n_1}.$$

Since $\eta_1(h_1) = h_1$, we have

$$\zeta_1 = \cdots = \zeta_{q_1} = 0.$$

By an elementary fact of K-theory (see, e.g., [19, Exercise 11.2]), there exists a projection $q_1 \in M_\bullet(C(X_1))$, where $\bullet$ is a large positive integer, such that

$$[q_1] = \big(\mathrm{rank}(q_1), h_1\big) \in \mathbb{Z} \oplus \mathbb{Z}^{n_1} = K_0\big(C(X_1)\big) \qquad \text{and} \qquad w_1^*(q_1) = q_1,$$

where the first coordinate of $\mathbb{Z} \oplus \mathbb{Z}^{n_1}$ denotes the rank part (in fact, we could choose $\bullet = 2$ and $\mathrm{rank}(q_1) = 1$).

For fixed $j < k \in \mathbb{N}$, since X_j is a compact metric space, there exist $\gamma(j,k)$ open balls of radius $\frac{1}{k}$, $D_1^{(j,k)}, \ldots, D_{\gamma(j,k)}^{(j,k)}$, in X_j such that

$$\bigcup_{1\le i\le\gamma(j,k)} D_i^{(j,k)} = X_j.$$

Let $\Gamma_{k,j} = s_j \circ \cdots \circ s_{k-1}$ be the surjective map from X_k to X_j. Define $\Omega_i^{(j,k)} \doteq \Gamma_{k,j}^{-1}(D_i^{(j,k)})$. Then $\Omega_i^{(j,k)}$ is an open set in X_k, $1 \le i \le \gamma(j,k)$; moreover,

$$\bigcup_{1\le i\le\gamma(j,k)} \Omega_i^{(j,k)} = X_k.$$

Let $\xi_{j,k,1}, \xi_{j,k,2}, \ldots, \xi_{j,k,\gamma(j,k)}$ be a sequence of points of X_k such that $\xi_{j,k,i} \in \Omega_i^{(j,k)}$, $1 \le i \le \gamma(j,k)$, and denote

$$E_{j,k} = \{\xi_{j,k,1}, \xi_{j,k,2}, \ldots, \xi_{j,k,\gamma(j,k)}\}.$$

For $k \ge 2$, rewrite the finite set $\bigcup_{1\le j<k} E_{j,k}$ as

$$\{\xi_{k,1}, \ldots, \xi_{k,2}, \xi_{k,\gamma(k)}\} \doteq E_k.$$

It follows from Lemma 2.4 that $(G,G_+)=\lim_{k\to\infty}(G_k,(G_k)_+,\varphi_k)$, where G_k is $\mathbb{Z}^{m_k}$ and $m_k\geq 2, k\in\mathbb{N}$. Denote by $\varphi_k^{(i,j)}$ the partial map of φ_k from the ith summand of $\mathbb{Z}^{m_k}$ to the jth summand of $\mathbb{Z}^{m_{k+1}}$. By Lemma 2.4, we may assume that $d(k,i,j)\geq 2\gamma(k)+4$, where $d(k,i,j)$ is the multiplicity of the partial map of $\varphi_k^{(i,j)}$. Also, we may assume that

$$\lim_{k\to\infty}\min_{1\leq i\leq m_k,1\leq j\leq m_{k+1}} d(k,i,j)=+\infty.$$

When $d(k,i,j)$ is odd, define $r'(k,i,j)=2$; when $d(k,i,j)$ is even, define $r'(k,i,j)=1$, $1\leq i\leq m_k, 1\leq j\leq m_{k+1}, k\in\mathbb{N}$. Set

$$r(k,i,j)=d(k,i,j)-2\gamma(k)-1,$$

and define

$$r''(k,i,j)=\frac{r(k,i,j)-r'(k,i,j)}{2}.$$

Then $r''(k,i,j)$ is a positive integer, $1\leq i\leq m_k, 1\leq j\leq m_{k+1}, k\in\mathbb{N}$. By the Effros–Handelman–Shen theorem (see [9]; see also [13, Proposition 3.4.9]), there is a unital AF algebra $B=\lim_{k\to\infty}(B_k,\psi_k)$, where

$$B_k=M_{l(k,1)}\oplus\cdots\oplus M_{l(k,m_k)},$$

such that

$$\big(K_0(B_k),K_0(B_k)_+,[1_{B_k}]\big)=\big(G_k,(G_k)_+,g_k\big)$$

and

$$(\psi_k)_*=\varphi_k,\quad k=1,2,\dots,\qquad \lim_{k\to\infty}\varphi_{k,+\infty}(g_k)=\tilde{g}.$$

Since B is simple, we may assume without loss of generality that $l(1,1)$ is larger than $\bullet$.

Define

$$C_k=\big(M_{2l(k,1)}\big(C(X_k)\big)\big)\oplus\Big(\bigoplus_{i=2}^{m_k}M_{2l(k,i)}\Big).$$

Therefore,

$$\begin{aligned}&\big(K_0(C_k),K_0(C_k)_+,[1_{C_k}]\big)\\&\quad=\big(\mathbb{Z}^{m_k}\oplus\mathbb{Z}^{n_k},\big[(\mathbb{N}\oplus\mathbb{Z}^{n_k})\cup\big\{(0,0)\big\}\big]\oplus\mathbb{Z}_+^{m_k-1},2g_k\oplus 0\big),\\&K_1(C_k)=0.\end{aligned}$$

Define

$$\pi_1^{(k)}:C_k\to M_{2l(k,1)}\big(C(X_k)\big)$$

and

$$\pi_i^{(k)}:C_k\to M_{2l(k,i)}\quad\text{for } 2\leq i\leq m_k$$

as the quotient maps. Let

$$Z_k=\big\{y:y\in X_k,\omega_k(y)=y\big\}.$$

Choose $z_k^*\in Z_k$.

For $i=1$ and $j=1$, define $\Phi_{k,1,1}: M_{2l(k,1)}(C(X_k)) \to M_{2l(k+1,1)}(C(X_{k+1}))$ by

$$\begin{aligned}\Phi_{k,1,1}(f) = \operatorname{diag}\big(f\circ s_k; \overbrace{f(z_k^*),\ldots,f(z_k^*)}^{r'(k,i,j)+2r''(k,i,j)}; f(\xi_{k,1}), f\big(\omega_k(\xi_{k,1})\big);\ldots;\\ f(\xi_{k,\gamma(k)}), f\big(\omega_k(\xi_{k,\gamma(k)})\big)\big).\end{aligned}$$

For $i=1$, and $j>1$, define $\Phi_{k,1,j}: M_{2l(k,1)}(C(X_k)) \to M_{2l(k+1,j)}$ by

$$\begin{aligned}\Phi_{k,1,j}(f) = \operatorname{diag}\big(f(z_k^*); \overbrace{f(z_k^*),\ldots,f(z_k^*)}^{r'(k,i,j)+2r''(k,i,j)}; f(\xi_{k,1}), f\big(\omega_k(\xi_{k,1})\big);\ldots;\\ f(\xi_{k,\gamma(k)}), f\big(\omega_k(\xi_{k,\gamma(k)})\big)\big).\end{aligned}$$

For $i>1$, and $j=1$, define $\Phi_{k,i,j}: M_{2l(k,i)} \to M_{2l(k+1,1)}(C(X_{k+1}))$ by

$$\Phi_{k,i,j}(a) = \iota_{k+1,1}\circ \operatorname{diag}(\overbrace{a,\ldots,a}^{r'(k,i,j)+1}; \quad \overbrace{a,\ldots,a}^{2r''(k,i,j)+2\gamma(k)} \quad),$$

where $\iota_{k+1,1}$ is the embedding of $M_{2l(k+1,1)}$ into $M_{2l(k+1,1)}C(X_{k+1})$.

For $i>1$ and $j>1$, define $\Phi_{k,i,j}: M_{2l(k,i)} \to M_{2l(k+1,j)}$ by

$$\Phi_{k,i,j}(a) = \operatorname{diag}(\overbrace{a,\ldots,a}^{r'(k,i,j)+1}; \quad \overbrace{a,\ldots,a}^{2r''(k,i,j)+2\gamma(k)} \quad).$$

Set $\Phi_k = \bigoplus_{1\le j\le m_{k+1}}(\bigoplus_{1\le i\le m_k}\Phi_{k,i,j})$. Then $(\Phi_k)_* = \varphi_k\oplus\beta_k$. Let

$$\mathfrak{C} = \lim_{k\to\infty}(C_k,\Phi_k).$$

Therefore, $\mathfrak{C}$ is an AH algebra and

$$\begin{aligned}\big(K_0(\mathfrak{C}),K_0(\mathfrak{C})_+\big) &= \lim_{k\to\infty}\big(\mathbb{Z}^{m_k}\oplus\mathbb{Z}^{n_k}, \big[(\mathbb{N}\oplus\mathbb{Z}^{n_k})\cup\{(0,0)\}\big]\oplus\mathbb{Z}_+^{m_k-1}, \varphi_k\oplus\beta_k\big),\\ K_1(\mathfrak{C}) &= 0.\end{aligned}$$

Note that

$$\begin{aligned}&\big(G\oplus H, \big(G_+\setminus\{0\}\big)\oplus H\cup\{(0,0)\}\big)\\ &\quad= \lim_{k\to\infty}\big(\mathbb{Z}^{m_k}\oplus\mathbb{Z}^{n_k}, \big(\mathbb{Z}_+^{m_k}\setminus\{0\}\big)\oplus\mathbb{Z}^{n_k}\big)\cup\{(0,0)\}, \varphi_k\oplus\beta_k\big).\end{aligned}$$

Moreover, since $d(k,i,j)>0$ for $1\le i\le m_k, 1\le j\le m_{k+1}, k\in\mathbb{N}$, it is straightforward to check that

$$\big(K_0(\mathfrak{C}),K_0(\mathfrak{C})_+,[1_{\mathfrak{C}}]\big) \cong \big(G\oplus H, \big(G_+\setminus\{0\}\big)\oplus H\cup\{(0,0)\}, 2\tilde{g}\oplus 0\big).$$

We next show that $\mathfrak{C}$ is simple. Fix $j\in\mathbb{N}$, for a nonzero element $f\in C_j$.

Case 1: $\pi_i^{(j)}(f)\neq 0$ for some $i>1$. As each partial map of $\Phi_j = \Phi_{j,j+1}$ has positive multiplicity, $\Phi_{j,j+1}(f)$ is full in C_{n+1}.

Case 2: $\pi_1^{(j)}(f)\neq 0$. Choose $x^*\in X_j$ such that $\pi_1^{(j)}(f)(x^*)\neq 0$. Since $\pi_1^{(j)}(f)$ is a continuous map on the compact metric space X_j, there exists a $\delta>0$ such that

$$\big\|\pi_1^{(j)}(f)(x) - \pi_1^{(j)}(f)(x^*)\big\| < \frac{\|\pi_1^{(j)}(f)(x^*)\|}{2},$$

provided that $\operatorname{dist}(x, x^*) < \delta$, where $x \in X_j$. Choose $k \in \mathbb{N}$, $k > j$, such that $\frac{1}{k} < \frac{\delta}{2}$. Then we could find $\xi \in E_k$ such that $\operatorname{dist}(\Gamma_{k,j}(\xi), x^*) < \frac{1}{k}$. Now $\Phi_{j,k}(f)$ is full in C_k.

By [7, Proposition 2.1(iii)], it follows that $\mathfrak{C}$ is simple.

Next we will show that $\mathrm{RR}(\mathfrak{C}) = 0$. Fix $j \in \mathbb{N}$. For any $f = f^* \in C_j$, note that X_j is a connected 2-dimensional compact metric space and that $\pi_1^{(j)}(f)$ is a continuous map on X_j. Recall the fact that a continuous map from a compact metric space to another metric space is automatically uniformly continuous. Hence, for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left\|\pi_1^{(j)}(f)(x) - \pi_1^{(j)}(f)(y)\right\| < \frac{\varepsilon}{10},$$

provided that $d(x, y) < \delta$. Choose $k \in \mathbb{N}$, $k > j$, such that $\frac{1}{k} < \frac{\delta}{2}$. According to the choice of E_k, we have

$$\max\Big\{\sup\Big\{\big|\lambda_i\big(\Phi_{j,k}(f)(x)\big) - \lambda_i\big(\Phi_{j,k}(f)(y)\big)\big|; x, y \in X_k\Big\} : 1 \le i \le 2l(k,1)\Big\} \le \frac{\varepsilon}{5},$$

where $\lambda_i(T) =$ the ith lowest eigenvalue of a matrix T, counted with multiplicity. Therefore, by [5, Theorem 1.3, Note added in proof], $\mathrm{RR}(\mathfrak{C}) = 0$. Hence, a priori, $\mathfrak{C}$ is an AH algebra, but by Theorem 2.5, $\mathfrak{C}$ turns out to be an AF algebra.

For $k \in \mathbb{N}$, define $\rho_k = w_k^* \oplus \mathrm{id}_{\bigoplus_{i=2}^{m_k} M_{2l(k,i)}}$. Set $u_1 = \mathrm{id}_{C_1}$, and for $1 \le i \le m_k, 1 \le j \le m_{k+1}, k \in \mathbb{N}$, define

$$u_{k+1,i,j} = \operatorname{diag}\Big\{\overbrace{1, \dots, 1}^{r'(k,i,j)+1}; \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{r''(k,i,j)+\gamma(k)}\Big\} \otimes \mathrm{id}_{M_{2l(k,i)}}.$$

Next define

$$u_k = \Big(\bigoplus_{i=1}^{m_k} u_{k+1,i,1}\Big) \otimes 1_{C(X_{k+1})} \oplus \bigoplus_{j=2}^{m_{k+1}} \Big(\bigoplus_{i=1}^{m_k} u_{k+1,i,j}\Big), \quad k \in \mathbb{N}.$$

Then it is straightforward to check that

$$\Phi_k \circ (\rho_k) = \operatorname{Ad} u_{k+1} \circ (\rho_{k+1}) \circ \Phi_k \qquad \text{and} \qquad u_{k+1}^2 = \mathrm{id}_{C_{k+1}}, \quad k \in \mathbb{N}.$$

Set $v_1 = \mathrm{id}_{C_1}$, and define $v_{k+1} = \Phi_k(v_k)u_{k+1}$ inductively. Then $v_k = v_k^*$ and $v_k^2 = \mathrm{id}_{C_k}$, $k = 1, 2, \dots$. Define $\alpha_k = \operatorname{Ad} v_k \circ \rho_k$, $k \in \mathbb{N}$. So one can easily construct the following commutative diagram:

$$\begin{array}{ccccccc}
C_1 & \xrightarrow{\Phi_1} & C_2 & \xrightarrow{\Phi_2} & C_3 & \xrightarrow{\Phi_3} & \cdots \mathfrak{C} \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
C_1 & \xrightarrow{\Phi_1} & C_2 & \xrightarrow{\Phi_2} & C_3 & \xrightarrow{\Phi_3} & \cdots \mathfrak{C}
\end{array}$$

Hence the automorphisms α_k define

$$\alpha : \mathfrak{C} = \lim_{k\to\infty} (C_k, \Phi_k) \to \mathfrak{C} = \lim_{k\to\infty} (C_k, \Phi_k),$$

which is a symmetry with

$$\alpha_* : G \oplus H \to G \oplus H \qquad \text{and} \qquad \alpha_*(g,h) = \big(g, \eta(h)\big).$$

We are now in a position to show that the action of $\mathbb{Z}_2$ generated by α has the tracial Rokhlin property. To this end, we fix a finite set $F \subset \mathfrak{C}_1 \doteq \{a : a \in \mathfrak{C}, \|a\| \leq 1\}$ and an $\varepsilon > 0$. Then there exists a positive integer k such that $F \subset_{\frac{\varepsilon}{2}} C_k$ and

$$\max_{1\leq i\leq m_k, 1\leq j\leq m_{k+1}} \frac{3}{d(k,i,j)} < \frac{\varepsilon}{2}.$$

Define

$$e_{k+1,i,j} = \operatorname{diag}\Big\{\overbrace{0,\dots,0}^{r'(k,i,j)+1}; \overbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix},\dots,\begin{bmatrix}1&0\\0&0\end{bmatrix}}^{r''(k,i,j)+\gamma(k)}\Big\} \otimes \mathrm{id}_{M_{2l(k,i)}}, \quad \text{for } 1 \leq i \leq m_k, 1 \leq j \leq m_{k+1},$$

and take

$$e = \Big(\bigoplus_{i=1}^{m_k} e_{k+1,i,1}\Big) \otimes 1_{C(X_{k+1})} \oplus \bigoplus_{j=2}^{m_{k+1}} \Big(\bigoplus_{i=1}^{m_k} e_{k+1,i,j}\Big).$$

Then $\alpha_{k+1}(e)e = 0$, and e commutates with $\Phi_k(C_k)$. Moreover, for any tracial state τ on A_{k+1},

$$\tau\big(1 - e - \alpha_{k+1}(e)\big) \leq \max_{1\leq i\leq m_k, 1\leq j\leq m_{k+1}} \frac{3}{d(k,i,j)} < \frac{\varepsilon}{2}.$$

Then α has the tracial Rokhlin property.

Set $p_1 = (q_1 \oplus 1_{M_{l(1,1)-\mathrm{rank}(q_1)}}) \oplus 1_{M_{l(1,2)}} \oplus \cdots \oplus 1_{M_{l(1,m_1)}}$. Inductively, we define p_{k+1} as $p_{k+1} = \Phi_k(p_k)$, $k \in \mathbb{N}$. Noting that $p_1 \in C_1^{\alpha_1}$ and $\alpha_2 \circ \Phi = \Phi \circ \alpha_1$, it follows from Section 2.3 that $p_2 \in C_2^{\alpha_2}$; similarly, we have $p_k \in C_k^{\alpha_k}$, $k \in \mathbb{N}$. For $k \in \mathbb{N}$, let $A_k = p_k C_k p_k$. Since $p_k \in C_k^{\alpha_k}$, it is routine to check that Φ_k maps A_k into A_{k+1}, α_k maps A_k onto A_k and is also an order 2 automorphism of A_k, $k \in \mathbb{N}$. Let

$$\mathfrak{A} = \lim_{k\to\infty} (A_k, \Phi_k).$$

Then it is straightforward to check that

$$\big(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, [1_{\mathfrak{A}}]\big) = \big(G \oplus H, \big(G_+ \setminus \{0\}\big) \oplus H \cup \big\{(0,0)\big\}, \tilde{g} \oplus \tilde{h}\big).$$

Also, as the following diagram commutes:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & \cdots\mathfrak{A} \\
\big\downarrow{\scriptstyle\alpha_1} & & \big\downarrow{\scriptstyle\alpha_2} & & \big\downarrow{\scriptstyle\alpha_3} & & \\
A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & \cdots\mathfrak{A}
\end{array}$$

the automorphisms α_k define $\alpha : \mathfrak{A} \to \mathfrak{A}$, which is a symmetry with $\alpha_* : G \oplus H \to G \oplus H$, such that $\alpha_*(g,h) = (g, \eta(h)) = \sigma(g,h)$.

Define $p = \Phi_{1,\infty}(p_1)$. Then it is obvious that $\mathfrak{A} = p\mathfrak{C}p$; hence $\mathfrak{A}$ is an AF algebra. Therefore, by Elliott's classification theorem of AF algebras (see, e.g., [13, Theorem 3.4.8]), $\mathfrak{A} \cong A$. Finally, noting that $p \in \mathfrak{A}^\alpha$, by [16, Lemma 3.7], the $\mathbb{Z}_2$-action α of A has the tracial Rokhlin property.

Hence, by [16, Corollary 1.6, Theorem 2.6], $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple, separable C^*-algebra with tracial rank zero. Since C_k is nuclear, $A_k = p_k C_k p_k$ is a hereditary subalgebra of A_k, hence A_k is nuclear. Note that $\mathbb{Z}_2$ is amenable and compact. It follows from [20, Corollary 7.18] and [8, Proposition 6.1] that $C^*(\mathbb{Z}_2, A_k, \alpha_k)$ is nuclear and satisfies the universal coefficient theorem. Hence, by [18, Proposition 2.4.7(ii)],

$$C^*(\mathbb{Z}_2, A, \alpha) = \lim_{k\to\infty} \big(C^*(\mathbb{Z}_2, A_k, \alpha_k), \Phi_k\big)$$

is also nuclear and satisfies the universal coefficient theorem. Therefore, by [14, Theorem 5.2] and its proof, $C^*(\mathbb{Z}_2, A, \alpha)$ is a unital simple AH algebra with no dimension growth. □

Remark 4.2. Let A be the unital simple AF algebra with

$$\big(K_0(A), K_0(A)_+, [1_A]\big) = \Big(\mathbb{Z}\Big[\frac{1}{2}\Big] \oplus \mathbb{Z}, \mathbb{Z}\Big[\frac{1}{2}\Big]_{>0} \oplus \mathbb{Z} \cup \big\{(0,0)\big\}, (1,0)\Big).$$

Let σ be the order 2 scaled ordered automorphism of $K_0(\mathfrak{A})$ defined by $\sigma(a,b) = (a,-b)$, where $(a,b) \in \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$. Let γ be any order 2 automorphism of A such that $\gamma_* = \sigma$. As the proof is outlined in [3, Section 10.11.3], $C^*(\mathbb{Z}_2, A, \gamma)$ is not an AF algebra. Hence, by [16, Theorem 2.2], γ does not have the Rokhlin property.

Remark 4.3. Let (G, G_+) be the ordered group of a unital simple AF algebra B, and let H be a countable torsion-free Abelian group. From the proof of the above theorem, $\mathfrak{C}$ is a unital, simple AH algebra of no dimension growth and of real rank zero. Hence, by [11, p. 571], $(K_0(\mathfrak{C}), K_0(\mathfrak{C})_+) = (G \oplus H, (G_+ \setminus \{0\}) \oplus H \cup \{(0,0)\})$ has the Riesz interpolation property.

Remark 4.4. If we further assume that η is the minus one map in the assumption of Theorem 4.1, the corresponding proof becomes relatively easy because we do not need to cut down the algebra $\mathfrak{C}$ with projection p.

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