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A NOTE ON THE C-NUMERICAL RADIUS AND THE λ-ALUTHGE TRANSFORM IN FINITE FACTORS

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Abstract. We prove that for any two elements A, B in a factor $\mathcal{M}$, if B commutes with all the unitary conjugates of A, then either A or B is in $\mathbb{C}I$. Then we obtain an equivalent condition for the situation that the C-numerical radius $\omega_C(\cdot)$ is a weakly unitarily invariant norm on finite factors, and we also prove some inequalities on the C-numerical radius on finite factors. As an application, we show that for an invertible operator T in a finite factor $\mathcal{M}$, $f(\triangle_\lambda(T))$ is in the weak operator closure of the set $\{\sum_{i=1}^n z_i U_i f(T) U_i^* \mid n \in \mathbb{N}, (U_i)_{1\leq i\leq n} \in \mathscr{U}(\mathcal{M}), \sum_{i=1}^n |z_i| \leq 1\}$, where f is a polynomial, $\triangle_\lambda(T)$ is the λ-Aluthge transform of T, and $0 \leq \lambda \leq 1$.

1. Introduction and preliminaries

Denote by $B(\mathscr{H})$ the set of bounded linear operators on a Hilbert space $\mathscr{H}$, and denote by $M_n(\mathbb{C})$ the self-adjoint algebra of the $n \times n$ matrices. A von Neumann algebra $\mathcal{M}$ on $\mathscr{H}$ is a unital weak operator closed $*$-algebra, and it is said to be a *factor* if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$, where I is the identity of $\mathcal{M}$. A von Neumann algebra $\mathcal{M}$ is finite if it has a faithful normal tracial state. If $\mathcal{M}$ is a finite factor with a faithful normal trace τ, denote by $\|\cdot\|_1$ the norm on $\mathcal{M}$ to be $\tau(|\cdot|)$. Then denote by $L^1(\mathcal{M}, \tau)$ the completion of $\mathcal{M}$ with respect to the $\|\cdot\|_1$-norm. Also to each normal linear functional f on $\mathcal{M}$ corresponds a unique element $X \in L^1(\mathcal{M}, \tau)$ such that $f(\cdot) = \tau(X\cdot)$. Denote by $\mathscr{U}(\mathcal{M})$ the set of all unitary operators in a von

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Neumann algebra $\mathcal{M}$. (For more background on finite von Neumann algebras, see [13].)

We next define the C-numerical radius on finite factors.

Definition 1.1. Let $\mathcal{M}$ be a finite factor with a faithful normal tracial state τ and for $A, C \in \mathcal{M}$, the *C-numerical radius* of A is defined as

$$\omega_C(A) = \sup_{U \in \mathscr{U}(\mathcal{M})} \left|\tau(CUAU^*)\right|.$$

Note that the C-numerical radius of A is a seminorm on $\mathcal{M}$. There are abundant results on the C-numerical radius on $M_n(\mathbb{C})$. We say that a norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is *weakly unitarily invariant* if $\|A\| = \|UAU^*\|$ for all $A \in M_n(\mathbb{C})$, $U \in \mathscr{U}(M_n(\mathbb{C}))$. Note that for every $C \in M_n(\mathbb{C})$, the C-numerical radius ω_C is a weakly unitarily invariant seminorm on $M_n(\mathbb{C})$. It is a norm on $M_n(\mathbb{C})$ if and only if C is not a scalar and has nonzero trace (see [3, Proposition IV.4.4]). The family ω_C of C-numerical radius, where C is not a scalar and has nonzero trace, plays a role analogous to that of Ky Fan norms in the family of unitarily invariant norms (see [3, Theorem IV.4.7]). A norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is called a *unitarily invariant norm* if $\|A\| = \|UAV^*\|$ for all $A \in M_n(\mathbb{C})$, $U, V \in \mathscr{U}(M_n(\mathbb{C}))$. The concept of unitarily invariant norms was introduced by von Neumann [14] for the purpose of metrizing matrix spaces. Von Neumann and his associates established that the class of unitarily invariant norms of $n \times n$ complex matrices coincides with the class of symmetric gauge functions of their s-numbers. These norms have now been variously generalized and utilized in many contexts. (For historical perspectives and surveys, we refer the reader to [3], [5], [7], [8] and the references therein.)

Let $T \in B(\mathscr{H})$, and let $T = U|T|$ be its polar decomposition. The Aluthge transform of T is the operator $\triangle(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This was first studied in [1] and has received much attention in recent years. One reason the Aluthge transform is interesting is in relation to the invariant subspace problem. Jung, Ko, and Pearcy [10, Theorem 1.15] proved that T has a nontrivial invariant subspace if and only if $\triangle(T)$ does. They also note that when T is quasiaffinity, then T has a nontrivial, hyperinvariant subspace if and only if $\triangle(T)$ does. A quasiaffinity is an operator with zero kernel and dense range. The invariant and hyperinvariant subspace problems are interesting only for quasiaffinities. As we know, for $A, B \in B(\mathscr{H})$, $\sigma(AB) = \sigma(BA)$ is not true in general since they may differ from zero, while the spectrum of $\triangle(T)$ equals that of T (see [9, Lemma 5]). Jung, Ko, and Percy further proved in [10, Theorems 1.3, 1.5] that other spectral data are also preserved by the Aluthge transform. Dykema and Schultz [4, Theorem 5.4] proved that Brown measures are unchanged by the Aluthge transform.

Another reason is related to the iterated Aluthge transform. Let $\triangle^0(T) = T$ and $\triangle^n(T) = \triangle(\triangle^{n-1}(T))$ for every $n \in \mathbb{N}$. It was conjectured in [10] that the sequence $\{\triangle^n(T)\}_{n\in\mathbb{N}}$ converges in the norm topology. (For more surveys, we refer the reader to [1], [2], [11], and [12]) The λ-Aluthge transform of T is defined in [11] by $\triangle_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$, $0 \le \lambda \le 1$. In particular, for $\lambda = \frac{1}{2}$, $\triangle_{\frac{1}{2}}(T)$ is just the Aluthge transform $\triangle(T)$. Okubo [11, Proposition 4] proved that for an invertible operator $T \in B(\mathscr{H})$, $\|f(\triangle_\lambda(T))\| \le \|f(T)\|$ for any polynomial f and

$\|\cdot\|$ a weakly unitarily invariant norm. (For more results on λ-Aluthge transforms, we refer the reader to [11] and [12].)

This article is organized as follows. The key motivation for studying the C-numerical radius ω_C on finite factors stems from the fact that for the finite-dimensional case—that is, $M_n(\mathbb{C})$—it has a relation with weakly unitarily invariant norms on $M_n(\mathbb{C})$. So in Section 2, we use some knowledge on dual norms to show that relation. In Section 3, we first prove that if $\mathcal{M}$ is a factor, then for any nontrivial projection P in $\mathcal{M}$, all the unitary conjugates of P generate the whole von Neumann algebra $\mathcal{M}$ (see Lemma 3.1). We then use this lemma to prove a technical result in this article.

Theorem 1.2 (see Theorem 3.2)**.** *Let $\mathcal{M}$ be a factor, and let $A, B \in \mathcal{M}$. If $UAU^*B = BUAU^*$ holds for every $U \in \mathscr{U}(\mathcal{M})$, then either A or B is in $\mathbb{C}I$.*

In Section 4, as one application of Theorem 1.2, we prove the following corollary.

Corollary 1.3 (see Corollary 4.1)**.** *Let $\mathcal{M}$ be a finite factor with a faithful normal trace τ. The C-numerical radius ω_C is a norm on $\mathcal{M}$ if and only if*

(1) *C is not a scalar multiple of I, and*
(2) *$\tau(C) \neq 0$.*

We also prove some inequalities for the C-numerical radius ω_C on finite factors (see Theorem 4.2). Then, in Section 5, we discuss some properties of the λ-Aluthge transform of an invertible operator in a finite factor. Using the *three lines theorem* and some results in Section 4, we obtain the following result.

Proposition 1.4 (see Proposition 5.3)**.** *Let M be a finite factor with a faithful normal trace τ. Assume that $T \in \mathcal{M}$ is an invertible operator with polar decomposition $T = U|T|$, and assume that f is a polynomial. Then for $0 \leq \lambda \leq 1$, $f(|T|^{\lambda}U|T|^{1-\lambda})$ is in the weak operator closure of the set $\{\sum_{i=1}^{n} z_i U_i f(T) U_i^* \mid n \in \mathbb{N}, (U_i)_{1\leq i\leq n} \in \mathscr{U}(\mathcal{M}), \sum_{i=1}^{n} |z_i| \leq 1\}$.*

Throughout this article, we assume that all the factors have separable preduals.

2. Relation between weakly unitarily invariant norms and the C-numerical radius ω_C on $M_n(\mathbb{C})$

In this section, a finite von Neumann algebra $(\mathcal{M}, \tau)$ means a finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ. Recall the definition and some properties of dual norms in [6]. Let $|||\cdot|||$ be a norm on a finite von Neumann algebra $(\mathcal{M}, \tau)$. For $T \in \mathcal{M}$, define

$$|||T|||_{\mathcal{M}}^{\sharp} = \sup\big\{\big|\tau(TX)\big| : X \in \mathcal{M}, |||X||| \leq 1\big\}.$$

When there is no chance for confusion, we write $|||\cdot|||^{\sharp}$ instead of $|||\cdot|||_{\mathcal{M}}^{\sharp}$.

Lemma 2.1 ([6, Lemma 6.1])**.** *We have that $|||\cdot|||^{\sharp}$ is a norm on $(\mathcal{M}, \tau)$.*

Definition 2.2 ([6, Definition 6.2]). A norm $|||\cdot|||^{\sharp}$ is called the *dual norm* of $|||\cdot|||$ on $\mathcal{M}$ with respect to τ.

Definition 2.3. A norm $\|\cdot\|$ on $(\mathcal{M},\tau)$ is weakly unitarily invariant if $\|UTU^*\| = \|T\|$ for all $T \in \mathcal{M}$ and $U \in \mathscr{U}(\mathcal{M})$.

Using the same trick as in [6, Lemma 6.18], we can obtain the following lemma and state it without proof.

Lemma 2.4. *If $\|\cdot\|$ is a norm on $(M_n(\mathbb{C}),\mathrm{tr})$ and $\|\cdot\|^\sharp$ is the dual norm with respect to* tr, *then $\|\cdot\| = \|\cdot\|^{\sharp\sharp}$.*

Lemma 2.5. *If $\|\cdot\|$ is a weakly unitarily invariant norm on a finite von Neumann algebra $(\mathcal{M},\tau)$, then $\|\cdot\|^\sharp$ is also a weakly unitarily invariant norm on $(\mathcal{M},\tau)$.*

Proof. Let $U \in \mathscr{U}(\mathcal{M})$. Then $\|UTU^*\|^\sharp = \sup\{|\tau(UTU^*X)| : X \in \mathcal{M}, \|X\| \leq 1\} = \sup\{|\tau(TU^*XU)| : X \in \mathcal{M}, \|U^*XU\| \leq 1\} = \|T\|^\sharp$. □

We now proceed to the relation between weakly unitarily invariant norms and the C-numerical radius on $(M_n(\mathbb{C}),\mathrm{tr})$.

Proposition 2.6. *If $\|\cdot\|$ is a weakly unitarily invariant norm on $(M_n(\mathbb{C}),\mathrm{tr})$, then $\|T\| = \sup_{\|X\|^\sharp\leq 1}\omega_X(T)$.*

Proof. For $T \in (M_n(\mathbb{C}),\mathrm{tr})$, by Lemmas 2.4 and 2.5 and the definition of the dual norm, we have

$$\begin{aligned}
\|T\| = \|T\|^{\sharp\sharp} &= \sup_{U\in\mathscr{U}(\mathcal{M})} \|UTU^*\|^{\sharp\sharp} \\
&= \sup_{U\in\mathscr{U}(\mathcal{M})} \sup_{\|X\|^\sharp\leq 1} \left\{\left|\tau(TUXU^*)\right|, X \in M_n(\mathbb{C})\right\} \\
&= \sup_{\|X\|^\sharp\leq 1} \sup_{U\in\mathscr{U}(\mathcal{M})} \left\{\left|\tau(TUXU^*)\right|, X \in M_n(\mathbb{C})\right\} \\
&= \sup_{\|X\|^\sharp\leq 1} \omega_X(T).
\end{aligned}$$

□

Note that when proving Proposition 2.6, we use Lemma 2.4, so we may ask whether this result can be generalized to finite factors.

3. A result on factors

In this section, we show a technical result (Theorem 3.2), which is the most difficult part of this article. To prove that result, we first need the following lemma.

Lemma 3.1. *Let $\mathcal{M}$ be a factor, and let P be a nontrivial projection in $\mathcal{M}$. Then the von Neumann algebra generated by $\{UPU^* : U \in \mathscr{U}(\mathcal{M})\}$ is $\mathcal{M}$.*

Proof. We divide the proof into four cases according to the type of $\mathcal{M}$.

(i) *The case $\mathcal{M} = B(\mathscr{H})$, where* $\dim(\mathscr{H}) \leq \infty$*:* Take two projections $P_0 \leq P$ and $P_1 \leq 1 - P$ with $\dim(P_i(H)) = 1$ for $i = 0, 1$, and write $Q = P - P_0 + P_1$. Then $P_0 = P(1-Q)$, and we can find some unitary operator $V \in \mathscr{U}(\mathcal{M})$ such that $VPV^* = Q$, since P and Q are equivalent. Then we have $\{UP_0U^* : U \in \mathscr{U}(\mathcal{M})\}'' \subseteq \{UPU^* : U \in \mathscr{U}(\mathcal{M})\}''$. Note that the von Neumann algebra generated by $\{UP_0U^* : U \in \mathscr{U}(\mathcal{M})\}$ is $\mathcal{M}$. Hence we have proved our result.

(ii) *The case where $\mathcal{M}$ is a II_1 factor with a faithful normal tracial state τ:* Write $\tau(P) = \lambda \in (0,1)$, and we may assume that $\lambda \leq \frac{1}{2}$. Then for any $0 < t \leq \lambda$, we can find two projections $P_t \leq P$ and $F_t \leq 1-P$ with $\tau(P_t) = \tau(F_t) = t$. Write $Q_t = P - P_t + F_t$. Then $P_t = P(1-Q_t)$. Again, we can find some unitary operator $V \in \mathscr{U}(\mathcal{M})$ such that $VPV^* = Q_t$. Hence $\{UP_tU^* : \tau(P_t) = t \in (0,\lambda], P_t \leq P, U \in \mathscr{U}(\mathcal{M})\}'' \subseteq \{UPU^* : U \in \mathscr{U}(\mathcal{M})\}''$. Note that the von Neumann algebra generated by $\{UP_tU^* : \tau(P_t) = t \in (0,\lambda], P_t \leq P, U \in \mathscr{U}(\mathcal{M})\}$ is the whole $\mathcal{M}$. Then we have our result.

(iii) *The case where $\mathcal{M}$ is a II_∞ factor with a faithful normal tracial weight* Tr*:* Write $\mathrm{Tr}(P) = \lambda \in (0,\infty]$, and we may assume that $\mathrm{Tr}(1-P) \geq \mathrm{Tr}(P)$. Then using the same trick as in case (ii), we prove our result.

(iv) *The case where $\mathcal{M}$ is a type III factor:* This case is trivial, since all the nontrivial projections in a type *III* factor are equivalent. □

Our main theorem is the following.

Theorem 3.2. *Let $\mathcal{M}$ be a factor, and let $A, B \in \mathcal{M}$. If $UAU^*B = BUAU^*$ holds for any $U \in \mathscr{U}(\mathcal{M})$, then either A or B is in $\mathbb{C}I$.*

Proof. Let P be a projection in $\mathcal{M}$. Then we can write A and B in the matrix form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where $A_{11}, B_{11} \in P\mathcal{M}P$, $A_{12}, B_{12} \in P\mathcal{M}P^\perp$, $A_{21}, B_{21} \in P^\perp\mathcal{M}P$, $A_{22}, B_{22} \in P^\perp\mathcal{M}P^\perp$. Let $\theta \in [0,2\pi]$ and $U = \begin{pmatrix} e^{i\theta}P_n & 0 \\ 0 & P_n^\perp \end{pmatrix}$. It is then clear, for this case, that U is a unitary operator. Then we have

$$UAU^* = \begin{pmatrix} A_{11} & e^{i\theta}A_{12} \\ e^{-i\theta}A_{21} & A_{22} \end{pmatrix},$$

$$UAU^*B = \begin{pmatrix} A_{11} & e^{i\theta}A_{12} \\ e^{-i\theta}A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + e^{i\theta}A_{12}B_{21} & * \\ * & * \end{pmatrix},$$

and

$$BUAU^* = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} A_{11} & e^{i\theta}A_{12} \\ e^{-i\theta}A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} B_{11}A_{11} + e^{-i\theta}B_{12}A_{21} & * \\ * & * \end{pmatrix}.$$

It follows that

$$A_{11}B_{11} - B_{11}A_{11} + e^{i\theta}A_{12}B_{21} - e^{-i\theta}B_{12}A_{21} = 0 \tag{3.1}$$

since $UAU^*B = BUAU^*$. Note that (3.1) holds for any $\theta \in [0,2\pi]$; an easy calculation implies that

$$A_{11}B_{11} = B_{11}A_{11}, \qquad A_{12}B_{21} = B_{12}A_{21} = 0. \tag{3.2}$$

Note that for any $U, V \in \mathscr{U}(\mathcal{M})$, $UVAV^*U^*B = BUVAV^*U^*$ still holds; in particular, we can choose $V = \begin{pmatrix} V_1 & 0 \\ 0 & P^\perp \end{pmatrix}$, where $V_1 \in \mathscr{U}(P\mathcal{M}P)$. Then

$$V_1A_{11}V_1^*B_{11} = B_{11}V_1A_{11}V_1^*. \tag{3.3}$$

(i) *The case $\mathcal{M} = B(\mathscr{H})$, where* $\dim(\mathscr{H}) = \infty$*:* For $n \in \mathbb{N}$, let P_n be a projection of dimension n, and let $P_n \leq P_{n+1}$. By a result of the finite-dimensional case — that is, if $A, B \in M_n(\mathbb{C})$ and $UAU^*B = BUAU^*$ holds for any $U \in \mathscr{U}(M_n(\mathbb{C}))$ — then either A or B is in $\mathbb{C}I_n$, where I_n is the identity of $M_n(\mathbb{C})$

(see the proof of [3, Proposition IV.4.4]). Then by (3.3), we have that either A_{11} or B_{11} is in $\mathbb{C}I_n$; that is, P_nAP_n or P_nBP_n is in $\mathbb{C}I_n$, for any $n \in \mathbb{N}$. Assume that P_nAP_n is in $\mathbb{C}I_n$, while P_nBP_n is not. For $m > n$, if P_mAP_m is not in $\mathbb{C}I_m$, while P_mBP_m is in $\mathbb{C}I_m$, then this would contradict the assumption that P_nBP_n is not in $\mathbb{C}I_n$. Hence we have that for all $n \in \mathbb{N}$, P_nAP_n is in $\mathbb{C}I_n$, which implies that A is in $\mathbb{C}I$.

(ii) *The case where $\mathcal{M}$ is a II_1 factor with trace τ or a type III factor:* If $\mathcal{M}$ is a II_1 factor, then assume that $\tau(P) = \frac{1}{2}$. Otherwise, if $\mathcal{M}$ is a type III factor, then assume that $P \neq 0$ and $P \neq 1$. Then we have $\mathcal{M} \cong M_2(\mathbb{C}) \otimes P\mathcal{M}P$, and we can write A, B in the matrix form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad A_{ij}, B_{ij} \in P\mathcal{M}P \text{ for } 1 \leq i,j \leq 2.$$

Let $V_1, V_2 \in \mathscr{U}(P\mathcal{M}P)$, and put $V = \left(\begin{smallmatrix} V_1 & 0 \\ 0 & V_2 \end{smallmatrix}\right)$. Then we have

$$VAV^* = \begin{pmatrix} V_1A_{11}V_1^* & V_1A_{12}V_2^* \\ V_2A_{21}V_1^* & V_2A_{22}V_2^* \end{pmatrix}.$$

It follows that $V_1A_{12}V_2^*B_{21} = 0$, since $UVAV^*U^*B = BUVAV^*U^*$ for any $U, V \in \mathscr{U}(\mathcal{M})$ and (3.2). If $A_{12} \neq 0$, then $A_{12}V_2^*B_{21} = B_{21}^*V_2A_{12}^* = 0$ for all unitary operators $V_2 \in \mathscr{U}(P\mathcal{M}P)$, which implies that $B_{21} = 0$. Moreover, put $V' = \left(\begin{smallmatrix} 0 & V_1 \\ V_2 & 0 \end{smallmatrix}\right)$. Then

$$V'AV'^* = \begin{pmatrix} V_1A_{22}V_1^* & V_1A_{21}V_2^* \\ V_2A_{12}V_1^* & V_2A_{11}V_2^* \end{pmatrix}.$$

Using the same trick as above, we obtain that if $A_{12} \neq 0$, then $B_{12} = 0$. Thus we have that if $A_{12} \neq 0$, then $B_{21} = B_{12} = 0$. Similarly, we have that if $A_{21} \neq 0$, then $B_{21} = B_{12} = 0$. Note that if we replace A with UAU^* for every $U \in \mathscr{U}(\mathcal{M})$ and if we replace B with VBV^* for every $V \in \mathscr{U}(\mathcal{M})$, then the above fact still holds, and we can argue as follows.

Assume that $A \notin \mathbb{C}I$. We try to show that $B \in \mathbb{C}I$.

Case 1: If there exists $U \in \mathscr{U}(\mathcal{M})$ such that $(UAU^*)_{12}$ or $(UAU^*)_{21}$ is nonzero, then from the above we know that $(VBV^*)_{12} = (VBV^*)_{21} = 0$ for every $V \in \mathscr{U}(\mathcal{M})$. Hence $VBV^*P = PVBV^*$ for every $V \in \mathscr{U}(\mathcal{M})$. Then apply Lemma 3.1 to get $B \in \mathbb{C}I$.

Case 2: If for every $U \in \mathscr{U}(\mathcal{M})$, $(UAU^*)_{12} = (UAU^*)_{21} = 0$, then $UAU^*P = PUAU^*$ for every $U \in \mathscr{U}(\mathcal{M})$. Again using Lemma 3.1, we have $A \in \mathbb{C}I$, which is a contradiction. Hence this case actually does not appear under the assumption that $A \notin \mathbb{C}I$.

(iii) *The case where $\mathcal{M}$ is a II_∞ factor:* Note that $\mathcal{M} = B(\mathscr{H}) \otimes \mathcal{N}$, where $\mathcal{N}$ is a II_1 factor. For any $n \in \mathbb{N}$, let P_n' be a projection of dimension n in $B(\mathscr{H})$, let I' be the identity of $\mathcal{N}$, and let $P_n = P_n' \otimes I'$. Then $P_n\mathcal{M}P_n$ is a type II_1 factor. Hence using the same trick in case (i) and the result in case (ii), our result follows. □

4. The C-numerical radius ω_C on finite factors

In this section, we show some applications of Theorem 3.2 and discuss some properties of the C-numerical radius ω_C on finite factors. We use Theorem 3.2 and the same technique as in [3, Proposition IV.4.4] to prove our next corollary. We include the proof below for the reader's convenience.

Corollary 4.1. *Let $\mathcal{M}$ be a finite factor with trace τ. The C-numerical radius ω_C is a weakly unitarily invariant norm on $\mathcal{M}$ if and only if*

(1) *C is not a scalar multiple of I, and*
(2) *$\tau(C) \neq 0$.*

Proof. If $C = \lambda I$ for some $\lambda \in \mathbb{C}$, then $\omega_C(A) = |\lambda||\tau(A)|$, and this is zero if $\tau(A) = 0$, which means that ω_C cannot be a norm on $\mathcal{M}$. If $\tau(C) = 0$, then $\omega_C(I) = 0$. Again, ω_C is not a norm.

Conversely, suppose that ω_C is not a norm on $\mathcal{M}$ and that $\omega_C(A) = 0$ for some $A \neq 0$. If $A = \lambda I$ for some $\lambda \in \mathbb{C}$, this would mean that $\tau(C) = 0$. So, if $\tau(C) \neq 0$, then $A \notin \mathbb{C}I$. We claim that $C \in \mathbb{C}I$. Since e^{itK} is in $\mathscr{U}(\mathcal{M})$ for all $t \in \mathbb{R}$ and $K = K^* \in \mathcal{M}$, the condition $\omega_C(A) = 0$ implies in particular that $\tau(Ce^{itK}Ae^{-itK}) = 0$ if $t \in \mathbb{R}$ and $K = K^* \in \mathcal{M}$. Differentiating this relation at $t = 0$, one gets $\tau((AC - CA)K) = 0$ for all $K = K^* \in \mathcal{M}$. Hence we obtain that $\tau((AC - CA)T) = 0$ for all $T \in \mathcal{M}$. Hence $AC = CA$. Note that $\omega_C(A) = \omega_C(UAU^*)$ for all $U \in \mathscr{U}(\mathcal{M})$, so that $UAU^*C = CUAU^*$ for all $U \in \mathscr{U}(\mathcal{M})$. Hence the result that C is in $\mathbb{C}I$ follows from Theorem 3.2. □

Note that for $A, C \in \mathcal{M}$, by the definition of the C-numerical radius ω_C, we have that $\omega_C(A) = \omega_A(C)$ and that $\omega_C(\cdot)$ is continuous in the strong operator topology on the unit ball of $\mathcal{M}$.

Theorem 4.2. *Let $\mathcal{M}$ be a finite factor with a faithful normal trace τ. For $A, B \in \mathcal{M}$, the following conditions are equivalent:*

(1) *$\omega_C(A) \leq \omega_C(B)$ for all operators $C \in \mathcal{M}$ that are not scalars and have nonzero trace;*
(2) *$\omega_C(A) \leq \omega_C(B)$ for all operators $C \in \mathcal{M}$;*
(3) *let $K = \{\sum_{i=1}^n z_i U_i B U_i^* \mid n \in \mathbb{N}, (U_i)_{1\leq i\leq n} \in \mathscr{U}(\mathcal{M}), \sum_{i=1}^n |z_i| \leq 1\}$, and let Γ be the weak operator closure of K; then $A \in \Gamma$.*

Proof. (1) ⇒ (2). Assume that $C \in \mathcal{M}$ and $\tau(C) = 0$. Put $C_n = C + \frac{1}{n}$. Then $\tau(C_n) = \frac{1}{n}$ and $\|C_n - C\| \to 0$. Moreover, we have

$$\begin{aligned}\big|\omega_A(C_n) - \omega_A(C)\big| &\leq \sup_{U\in\mathscr{U}(\mathcal{M})} \big|\tau\big(AU(C_n - C)U^*\big)\big| \\ &= \sup_{U\in\mathscr{U}(\mathcal{M})} \frac{1}{n}\big|\tau(A)\big| \\ &\to 0.\end{aligned}$$

Similarly, we would have $\omega_B(C_n) \to \omega_B(C)$. Note that $\omega_A(C_n) \leq \omega_B(C_n)$. Then we have $\omega_A(C) \leq \omega_B(C)$.

Let $P \in \mathcal{M}$ be a projection with trace not equal to 0 or 1. Let $C_n = P + (1 - \frac{1}{n})(1-P)$. Then C_n is not a scalar, $\tau(C_n) \neq 0$, and $\|C_n - 1\| \to 0$. Hence we have $\omega_A(C_n) \le \omega_B(C_n)$ and for any operator $T \in \mathcal{M}$,

$$\begin{aligned}\big|\omega_T(C_n) - \omega_T(I)\big| &\le \big|\omega_T(C_n - I)\big| \\ &= \sup_{U \in \mathscr{U}(\mathcal{M})} \big|\tau\big(TU(C_n - I)U^*\big)\big| \\ &\le \|C_n - 1\| \|T\|_1 \\ &\to 0.\end{aligned}$$

It follows that $\omega_A(I) \le \omega_B(I)$.

(2) $\Rightarrow$ (3). Assume that $A \notin \Gamma$. Then there exists a linear normal functional f on $\mathcal{M}$ and $a > b$ such that $\operatorname{Re} f(A) \ge a > b \ge \operatorname{Re} f(D)$, $\forall D \in \Gamma$. Since f is a normal linear functional on $\mathcal{M}$, there exists a $C \in L^1(\mathcal{M}, \tau)$ such that $f(T) = \tau(CT)$ for all $T \in \mathcal{M}$.

Note that $\omega_C(A) = \sup_{U \in \mathscr{U}(\mathcal{M})} |\tau(CUAU^*)| \ge |\tau(CA)| = |f(A)|$ and

$$\operatorname{Re} f(A) > \sup_{D \in \Gamma} \operatorname{Re} f(D) \ge \sup_{\theta, U} \operatorname{Re} f(e^{i\theta} UBU^*) = \sup_{U \in \mathscr{U}(\mathcal{M})} \big|f(UBU^*)\big| = \omega_C(B).$$

Let $C = V|C|$ be the polar decomposition of C in $L^1(\mathcal{M}, \tau)$, and let $H_n = \chi_{[0,n]}(|C|)|C|$. Then $\|H_n - |C|\|_1 \to 0$. Put $C_n = VH_n$. Then we have

$$\begin{aligned}\big|\omega_{C_n}(A) - \omega_C(A)\big| &= \big|\omega_A(C_n) - \omega_A(C)\big| \\ &\le \sup_{U \in \mathscr{U}(\mathcal{M})} \big|\tau\big((C_n - C)UAU^*\big)\big| \\ &\le \|C_n - C\|_1 \|A\| \\ &\to 0.\end{aligned}$$

Similarly, $|\omega_{C_n}(B) - \omega_C(B)| \to 0$. Hence there exists $m \in \mathbb{N}$ such that $\omega_{C_m}(A) > \omega_{C_m}(B)$, which contradicts condition (2) since $C_m \in \mathcal{M}$.

(3) $\Rightarrow$ (1). For all operators $C \in \mathcal{M}$ that are not scalars and have nonzero trace, by Corollary 4.1, we obtain that ω_C is a norm, and hence $\omega_C(T) \le \omega_C(B)$ for all $T \in K$. Hence our result follows since ω_C is normal. □

Remark 4.3. If $|\!|\!| \cdot |\!|\!|$ is a weakly unitarily invariant norm on $(M_n(\mathbb{C}), \mathrm{tr})$, then by Theorem 4.2 and Proposition 2.6, we have [3, Theorem IV.4.7].

5. λ-Aluthge transform of an invertible operator in a finite factor

Let $T \in B(\mathscr{H})$, and let $T = U|T|$ be its polar decomposition. The Aluthge transform of T is the operator $\triangle(T) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. The λ-Aluthge transform of T is defined by $\triangle_\lambda(T) = |T|^\lambda U |T|^{1-\lambda}$, $0 \le \lambda \le 1$. In this section, we show some results on the λ-Aluthge transform of an invertible operator in a finite factor.

For the infinite factor $B(\mathscr{H})$, Okubo [11, Proposition 4] proved that if $T \in B(\mathscr{H})$ is an invertible operator, then for any polynomial f, $0 \le \lambda \le 1$ and $\|\cdot\|$ a weakly unitarily invariant norm, we have $\|f(\triangle_\lambda(T))\| \le \|f(T)\|$. Note that the C-numerical radius is a weakly unitarily invariant seminorm on a finite factor $\mathcal{M}$

and that we have already given an equivalent condition for the situation when this seminorm is a norm in Section 4.

The idea of proving the following theorem comes from [11, Theorem 3].

Theorem 5.1. *Let $\mathcal{M}$ be a finite factor with a faithful normal trace τ, let $T \in \mathcal{M}$ be an invertible operator with polar decomposition $T = U|T|$, and let $B \in \mathcal{M}$ commute with T. Let $\omega_C(\cdot)$ be the C-numerical radius on $\mathcal{M}$. Then*

$$\omega_C\big(|T|^{\lambda} BU|T|^{1-\lambda}\big) \leq \omega_C(BT) \quad \text{for } 0 \leq \lambda \leq 1. \tag{5.1}$$

Proof. On the strip $\{z : -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}\}$, consider the operator-valued function $\phi(z)$ defined by $\phi(z) = |T|^{\frac{1}{2}-z} BU|T|^{\frac{1}{2}+z}$. It is clear that $\phi(z)$ is analytic in the interior of the strip.

For any $U \in \mathscr{U}(\mathcal{M})$, define $f_U(z) = \tau(CU\phi(z)U^*)$. Then $f_U(z)$ is uniformly bounded on the strip and analytic since τ is linear and $\phi(z)$ is analytic. Applying the *three lines theorem* (see [7, pp. 136–137]) to $f_U(z)$, we obtain that the function $x \mapsto \mathrm{Log}\sup_{y\in\mathbb{R}} |f_U(x+iy)|$ is a convex function on $[-\frac{1}{2}, \frac{1}{2}]$.

Put $F_U(x) = \mathrm{Log}\sup_{y\in\mathbb{R}} |f_U(x+iy)|$. Then for $-\frac{1}{2} \leq x \leq \frac{1}{2}$,

$$F_U(x) \leq F_U\Big(\frac{1}{2}\Big)\Big(x + \frac{1}{2}\Big) + F_U\Big(-\frac{1}{2}\Big)\Big(\frac{1}{2} - x\Big),$$

so that

$$\sup_{U\in\mathscr{U}(\mathcal{M})} F_U(x) \leq \sup_{U\in\mathscr{U}(\mathcal{M})} F_U\Big(\frac{1}{2}\Big)\Big(x + \frac{1}{2}\Big) + \sup_{U\in\mathscr{U}(\mathcal{M})} F_U\Big(-\frac{1}{2}\Big)\Big(\frac{1}{2} - x\Big). \tag{5.2}$$

For $-\infty < y < \infty$, since $|T|^{\pm iy}$ is a unitary operator and $\phi(\frac{1}{2}+iy) = |T|^{-iy}BU \times |T||T|^{iy}$ and $\omega_C(\cdot)$ is a weakly unitarily invariant seminorm on M, we have $\omega_C(\phi(\frac{1}{2}+iy)) = \omega_C(BU|T|)$. Note that

$$\phi\Big(-\frac{1}{2} + iy\Big) = |T|^{-iy}|T|BU|T|^{iy} = |T|^{-iy}U^*U|T|BU|T|^{iy}.$$

By using the commutativity of T and B, we have $\omega_C(\phi(-\frac{1}{2}+iy)) = \omega_C(BU|T|)$.

Note that

$$\begin{aligned}
\sup_{U\in\mathscr{U}(\mathcal{M})} F_U\Big(-\frac{1}{2}\Big) &= \sup_{U\in\mathscr{U}(\mathcal{M})} \mathrm{Log}\sup_{y\in\mathbb{R}}\Big|f_U\Big(-\frac{1}{2}+iy\Big)\Big| \\
&= \mathrm{Log}\sup_{y\in\mathbb{R}}\sup_{U\in\mathscr{U}(\mathcal{M})}\Big|f_U\Big(-\frac{1}{2}+iy\Big)\Big| \\
&= \mathrm{Log}\sup_{y\in\mathbb{R}}\sup_{U\in\mathscr{U}(\mathcal{M})}\Big|\tau\Big(CU\phi\Big(-\frac{1}{2}+iy\Big)U^*\Big)\Big| \\
&= \mathrm{Log}\sup_{y\in\mathbb{R}}\omega_C\Big(\phi\Big(-\frac{1}{2}+iy\Big)\Big) \\
&= \mathrm{Log}\,\omega_C\big(BU|T|\big).
\end{aligned}$$

Similarly,

$$\sup_{U\in\mathscr{U}(\mathcal{M})} F_U\Big(\frac{1}{2}\Big) = \sup_{U\in\mathscr{U}(\mathcal{M})} \mathrm{Log}\sup_{y\in\mathbb{R}}\Big|f_U\Big(\frac{1}{2}+iy\Big)\Big|$$

$$\begin{aligned}
&= \operatorname{Log} \sup_{y\in\mathbb{R}} \sup_{U\in\mathscr{U}(\mathcal{M})} \Big| f_U\Big(\frac{1}{2}+iy\Big)\Big| \\
&= \operatorname{Log} \sup_{y\in\mathbb{R}} \sup_{U\in\mathscr{U}(\mathcal{M})} \Big| \tau\Big(CU\phi\Big(\frac{1}{2}+iy\Big)U^*\Big)\Big| \\
&= \operatorname{Log} \sup_{y\in\mathbb{R}} \omega_C\Big(\phi\Big(\frac{1}{2}+iy\Big)\Big) \\
&= \operatorname{Log} \omega_C\big(BU|T|\big).
\end{aligned}$$

Then inequality (5.2) implies that for $-\frac{1}{2} \le x \le \frac{1}{2}$,

$$\begin{aligned}
\sup_{U\in\mathscr{U}(\mathcal{M})} F_U(x) &= \sup_{U\in\mathscr{U}(\mathcal{M})} \operatorname{Log} \sup_{y\in\mathbb{R}} \big| f_U(x+iy)\big| \\
&= \operatorname{Log} \sup_{y\in\mathbb{R}} \omega_C\big(\phi(x+iy)\big) \\
&\le \operatorname{Log} \omega_C(BT),
\end{aligned}$$

which means that

$$\omega_C\big(\phi(x+iy)\big) \le \omega_C(BT), \quad -\frac{1}{2} \le x \le \frac{1}{2}, -\infty < y < \infty,$$

and hence that

$$\omega_C\big(|T|^\lambda BU|T|^{1-\lambda}\big) \le \omega_C(BT) \quad \text{for } 0 \le \lambda \le 1. \qquad \square$$

The proof of the following proposition is exactly the same as [11, Proposition 4], so we state it as follows without a proof.

Proposition 5.2. *Let $\mathcal{M}$ be a finite factor with a faithful normal trace τ, and let $T \in \mathcal{M}$ be an invertible operator with polar decomposition $T = U|T|$. Let $\omega_C(\cdot)$ be the C-numerical radius on $\mathcal{M}$, and let $f(x)$ be a polynomial. Then*

$$\omega_C\big(f\big(|T|^\lambda U|T|^{1-\lambda}\big)\big) \le \omega_C\big(f(T)\big) \quad \textit{for } 0 \le \lambda \le 1.$$

Applying Theorem 4.2 and Proposition 5.2, we obtain the following.

Proposition 5.3. *Let $\mathcal{M}$ be a finite factor with a faithful normal trace τ. Assume that $T \in \mathcal{M}$ is an invertible operator with polar decomposition $T = U|T|$, and assume that f is a polynomial. Then for $0 \le \lambda \le 1$, $f(|T|^\lambda U|T|^{1-\lambda})$ is in the weak operator closure of the set $\{\sum_{i=1}^n z_i U_i f(T) U_i^* \mid n \in \mathbb{N}, (U_i)_{1\le i\le n} \in \mathscr{U}(\mathcal{M}), \sum_{i=1}^n |z_i| \le 1\}$.*

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