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NONLINEAR HARMONIC ANALYSIS OF INTEGRAL OPERATORS IN WEIGHTED GRAND LEBESGUE SPACES AND APPLICATIONS

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ABSTRACT. In this article, we give a boundedness criterion for Cauchy singular integral operators in generalized weighted grand Lebesgue spaces. We establish a necessary and sufficient condition for the couple of weights and curves ensuring boundedness of integral operators generated by the Cauchy singular integral defined on a rectifiable curve. We characterize both weak and strong type weighted inequalities. Similar problems for Calderón–Zygmund singular integrals defined on measured quasimetric space and for maximal functions defined on curves are treated. Finally, as an application, we establish existence and uniqueness, and we exhibit the explicit solution to a boundary value problem for analytic functions in the class of Cauchy-type integrals with densities in weighted grand Lebesgue spaces.

1. Introduction

The theory of grand Lebesgue spaces $L^{p)}$ introduced by Iwaniec and Sbordone in [11] is nowadays the focus of one of the most intensively developing directions in modern analysis. The necessity of introducing and studying these spaces grew out of their rather essential role in various fields. For example, besides the original question about minimal hypotheses to ensure the integrability of the Jacobian determinant, we recall applications in partial differential equations (PDEs)

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(see, e.g., [9]), geometric function theory (see, e.g., [4]), Sobolev spaces theory (see, e.g., [8], [6]), and Banach function spaces theory (see, e.g., [3], [6], [7]). In order to study the existence and uniqueness of solutions for the nonhomogeneous n-harmonic equation $\operatorname{div} a(x, \nabla u) = \mu$, Greco, Iwaniec, and Sbordone [9] defined the more general grand $L^{p),\theta}$ spaces. Afterwards, it turned out that, in the theory of PDEs, generalized $L^{p),\theta}$ spaces were also appropriate for treating some regularity problems (see, e.g., [6]).

An intensive study of the boundedness of singular integrals in weighted grand Lebesgue spaces was carried out in [13, Chapter 14]. In the present article, we explore the same problem in the more general grand Lebesgue spaces introduced, in the unweighted case, by Capone, Formica, and Giova in [3]. We treat both cases of weighted spaces differing by the position of the weight function in the norms.

Let $\Gamma = \{t \in \mathbb{C} : t = t(s), 0 \le s \le l < \infty\}$ be a simple rectifiable curve with arc-length measure. In the remainder of this article, we use the notation

$$D(t,r) = \Gamma \cap B(t,r), \quad 0 < r < \operatorname{diam} \Gamma,$$

where $B(t,r) = \{z \in \mathbb{C} : |z-t| < r\}$. A rectifiable curve Γ is called a *Carleson (regular) curve* if

$$\sup_{t\in\Gamma, 0<r<\operatorname{diam}\Gamma} \frac{|D(t,r)|}{r} < \infty.$$

Here $|D(t,r)|$ denotes the arc-length measure of the portion $D(t,r)$.

Now let w be a weight, that is, an almost everywhere positive integrable function on a given rectifiable simple curve Γ; for arbitrary Borel sets e on Γ, we denote

$$we = \int_e w(t)\, ds.$$

Let $1 < p < \infty$, and let δ be a positive, nondecreasing, bounded function on $(0, p-1)$, $\delta(0+) = 0$. By $L_w^{p),\delta}(\Gamma)$, we denote the set of all measurable functions on Γ for which

$$\|f\|_{L_w^{p),\delta}(\Gamma)} = \sup_{0<\varepsilon<p-1} \Big(\delta(\varepsilon) \int_\Gamma \big|f(t)\big|^{p-\varepsilon} w(t)\, ds\Big)^{\frac{1}{p-\varepsilon}} < \infty.$$

In the following, we will consider also the generalized weighted grand Lebesgue space $\mathcal{L}_w^{p),\delta}(\Gamma)$ defined by the norm

$$\|f\|_{\mathcal{L}_w^{p),\delta}(\Gamma)} = \sup_{0<\varepsilon<p-1} \Big(\delta(\varepsilon) \int_\Gamma \big|f(t)w(t)\big|^{p-\varepsilon}\, ds\Big)^{\frac{1}{p-\varepsilon}}.$$

When $w(t) \equiv 1$, we put $L^{p),\delta}(\Gamma) = L_1^{p),\delta}(\Gamma)$, $\mathcal{L}^{p),\delta}(\Gamma) = \mathcal{L}_1^{p),\delta}(\Gamma)$. It is clear that in such a case, when $\Gamma = (0,1)$, both these spaces coincide with the space $L^{p),\delta}$ introduced in [3].

In the following, we will also need the definition of Muckenhoupt-type weights suited to the curves. We set

$$A_p(\Gamma) = \Big\{w : \sup_{t\in\Gamma, 0<r<\operatorname{diam}\Gamma} \Big(\frac{1}{r}\int_{D(t,r)} w(\tau)\, ds\Big)\Big(\frac{1}{r}\int_{D(t,r)} w^{1-p'}(\tau)\, ds\Big)^{p-1} < \infty\Big\}.$$

Together with $L_w^{p),\delta}(\Gamma)$ spaces, we are interested in the weak grand Lebesgue spaces $WL_w^{p),\delta}(\Gamma)$, which we define by the quasinorm

$$\|f\|_{WL_w^{p),\delta}(\Gamma)} = \sup_{\lambda>0} \lambda \sup_{0<\varepsilon<p-1} \big(\delta(\varepsilon) w\big\{t \in \Gamma : \big|f(t)\big| > \lambda\big\}\big)^{\frac{1}{p-\varepsilon}}.$$

It is clear that $L_w^{p),\delta}(\Gamma) \hookrightarrow WL_w^{p),\delta}(\Gamma)$.

2. Boundedness of the Cauchy singular integral operator in $L_w^{p),\delta}(\Gamma)$

In this section, we prove the following statement.

Theorem 2.1. *Let $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *the Cauchy singular integral operator*

$$S_\Gamma f(t) = \int_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau$$

is bounded in $L_w^{p),\delta}(\Gamma)$;

(ii) *S_Γ is bounded from $L_w^{p),\delta}(\Gamma)$ to $WL_w^{p),\delta}(\Gamma)$;*

(iii) *Γ is a Carleson curve and $w \in A_p(\Gamma)$.*

Corollary 2.2. *Let $1 < p < \infty$. Then the operator S_Γ is bounded in $L^{p),\delta}(\Gamma)$ if and only if Γ is a Carleson curve.*

It is evident that, after formally setting $\delta(\varepsilon) \equiv 1$, Corollary 2.2 is G. David's well-known result.

Proof. First, we show that (iii) $\Longrightarrow$ (i). It is well known that Carleson curves with arc-length measure and Euclidean distance are spaces of homogeneous type; moreover, the Muckenhoupt class of weights defined on spaces of homogeneous type is always open (see, e.g., [16]).

From the openness of $A_p(\Gamma)$, there exists a small positive number σ such that $w \in A_{p-\sigma}(\Gamma)$. Furthermore, by the definition of $L_w^{p),\delta}(\Gamma)$, we have that $f \in L_w^{p-\sigma}(\Gamma)$ for $0 < \sigma < p - 1$. Using Hölder's inequality, we get

$$\int_\Gamma \big|f(t)\big|\, ds \le \|f\|_{L_w^{p-\sigma}(\Gamma)} \cdot \Big(\int_\Gamma w^{1-(p-\sigma)'}(t)\, ds\Big)^{\frac{p-\sigma-1}{p-\sigma}} < \infty;$$

hence, if $f \in L_w^{p),\delta}(\Gamma)$, then from $f \in L_w^{p-\sigma}(\Gamma)$ and $w \in A_{p-\sigma}(\Gamma)$ we get $f \in L^1(\Gamma)$, and the Cauchy singular integral $S_\Gamma f(t)$ exists almost everywhere on Γ. This latter existence of $S_\Gamma f(t)$ almost everywhere when Γ is an arbitrary rectifiable curve is in fact a consequence of Calderón's well-known result in [2, Theorem 1] on the boundedness of S_Γ in the case of Lipschitz curves when Lipschitz constants are sufficiently small (for details, we refer the reader to [5, Theorem 2.23, p. 215]). It is well known that for the operator S_Γ to be bounded in $L_w^p(\Gamma)$ $(1 < p < \infty)$ it is necessary and sufficient that Γ be a Carleson curve and that $w \in A_p(\Gamma)$. This result was established independently in [1, Theorem 4.8 and Chapter 5] and [12, Theorem 4.2], using different approaches.

Fix some σ, $0<\sigma<p-1$ such that $w \in A_{p-\sigma}(\Gamma)$. Due to the aforementioned results, we have

$$\|S_\Gamma\|_{L_w^{p-\sigma}(\Gamma)} \le M_1 \|f\|_{L_w^{p-\sigma}(\Gamma)}$$

and

$$\|S_\Gamma\|_{L_w^{p}(\Gamma)} \le M_2 \|f\|_{L_w^{p}(\Gamma)}.$$

By the Riesz–Thorin interpolation theorem, there exists a constant $M>0$ such that the inequality

$$\|S_\Gamma f\|_{L_w^{p-\varepsilon}(\Gamma)} \le M \|f\|_{L_w^{p-\varepsilon}(\Gamma)}$$

holds uniformly for all ε, $0<\varepsilon\le\sigma$ and for all $f \in L_w^{p-\varepsilon}(\Gamma)$. From the latter inequality, it follows that

$$\sup_{0<\varepsilon\le\sigma} \big(\delta(\varepsilon)\big)^{\frac{1}{p-\varepsilon}} \|S_\Gamma f\|_{L_w^{p-\varepsilon}(\Gamma)} \le M \sup_{0<\varepsilon\le\sigma} \big(\delta(\varepsilon)\big)^{\frac{1}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon}(\Gamma)}. \tag{2.1}$$

Now let $\varepsilon>\sigma$. Using Hölder's inequality with the exponent $\frac{p-\sigma}{p-\varepsilon}$, we get

$$\|S_\Gamma f\|_{L_w^{p-\varepsilon}(\Gamma)} \le \|S_\Gamma f\|_{L_w^{p-\sigma}(\Gamma)} \cdot (wI)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}}; \tag{2.2}$$

therefore,

$$\begin{aligned}
\|S_\Gamma f\|_{L_w^{p),\delta}(\Gamma)} &= \max\Big\{ \sup_{0<\varepsilon\le\sigma} \big(\delta(\varepsilon)\big)^{\frac{1}{p-\varepsilon}} \|S_\Gamma f\|_{L_w^{p-\varepsilon}(\Gamma)}, \sup_{\sigma<\varepsilon<p-1} \big(\delta(\varepsilon)\big)^{\frac{1}{p-\varepsilon}} \|S_\Gamma f\|_{L_w^{p-\varepsilon}(\Gamma)} \Big\} \\
&\le \max\Big\{ \sup_{0<\varepsilon\le\sigma} \big(\delta(\varepsilon)\big)^{\frac{1}{p-\varepsilon}} \|S_\Gamma f\|_{L_w^{p-\varepsilon}(\Gamma)}, \\
&\qquad \sup_{\sigma<\varepsilon<p-1} \big(\delta(\varepsilon)\big)^{\frac{1}{p-\varepsilon}} \|S_\Gamma f\|_{L_w^{p-\sigma}(\Gamma)} (wI)^{\frac{\varepsilon-\sigma}{(p-\varepsilon)(p-\sigma)}} \Big\} \\
&= \max\Big\{ \sup_{0<\varepsilon\le\sigma} \big(\delta(\varepsilon)\big)^{\frac{1}{p-\varepsilon}} \|S_\Gamma f\|_{L_w^{p-\varepsilon}(\Gamma)}, \\
&\qquad \sup_{\sigma<\varepsilon<p-1} \delta(\sigma)^{-\frac{1}{p-\sigma}} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \delta(\sigma)^{\frac{1}{p-\sigma}} \|S_\Gamma f\|_{L_w^{p-\sigma}(\Gamma)} (wI)^{\frac{\varepsilon-\sigma}{(p-\varepsilon)(p-\sigma)}} \Big\} \\
&\le I \cdot \max\Big\{ 1, \delta(\sigma)^{-\frac{1}{p-\sigma}} \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} (1+w\Gamma)^{\frac{p-1-\sigma}{p-\sigma}} \Big\},
\end{aligned}$$

where

$$I = \sup_{0<\varepsilon\le\sigma} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|S_\Gamma f\|_{L_w^{p-\varepsilon}(\Gamma)}.$$

From estimate (2.1), we conclude that

$$\|S_\Gamma f\|_{L_w^{p),\delta}(\Gamma)} \le M \|f\|_{L_w^{p),\delta}} \cdot \delta(\sigma)^{-\frac{1}{p-\sigma}} \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} (1+w\Gamma)^{\frac{p-1-\sigma}{p-\sigma}}.$$

Thus we have proved the implication (iii) $\Longrightarrow$ (i) and, consequently, since again by [5, Theorem 2.23, p. 215] (i) $\Longrightarrow$ (ii) holds, we have that (iii) $\Longrightarrow$ (ii). It remains to prove that (ii) $\Longrightarrow$ (iii). The proof of this implication is divided into two steps.

Step 1. We prove that, from condition (ii), the inequality

$$\int_{D(t,r)} w(\tau)^{-\frac{1}{p-1}} \, ds < \infty$$

holds for arbitrary $t \in \Gamma$ and r, $0 < r < \operatorname{diam}\Gamma$. Let us suppose the contrary, namely, that

$$\int_{D(t,r)} w(\tau)^{-\frac{1}{p-1}}\, ds = \infty$$

holds for some $t \in \Gamma$ and r, $0 < r < \operatorname{diam}\Gamma$, which we may assume to be sufficiently small. We have $w^{-\frac{1}{p}} \notin L^{\frac{p}{p-1}}(D(t,r))$; therefore, there exists a nonnegative $h \in L^p(D(t,r))$ such that $h = 0$ outside of $D(t,r)$ and

$$\int_{D(t,r)} w(\tau)^{-\frac{1}{p}} h(\tau)\, ds = \infty. \tag{2.3}$$

Set $f = w^{-\frac{1}{p}} h$. It is obvious that $f \in L^p_w(\Gamma)$; consequently, $f \in L^{p),\delta}_w(\Gamma)$. Recalling that the point $t \in \Gamma$ and the radius r are fixed, let $x \in \Gamma$ be such that $|x-t| = 3r$. Introduce the function

$$g(\tau) = f(\tau)\, \frac{d\tau}{|d\tau|} e^{i \arg(t-x)}. \tag{2.4}$$

Then for arbitrary $z \in D(x,r)$, we have

$$\big|S_\Gamma g(z)\big| \geq \frac{c}{r} \int_{D(t,r)} w^{-\frac{1}{p}}(\tau) h(\tau)\, ds,$$

and, taking (2.3) into account, we conclude that

$$\big|S_\Gamma g(z)\big| = \infty \tag{2.5}$$

for $z \in D(x,r)$. Therefore, according to (ii) there exists $c > 0$ such that

$$\big(\delta(\varepsilon) w\big\{t \in \Gamma : \big|(S_\Gamma \varphi)(t)\big| > \lambda\big\}\big)^{\frac{1}{p-\varepsilon}} \leq \frac{c}{\lambda} \|\varphi\|_{L^{p),\delta}_w}$$

for arbitrary $\varphi \in L^{p),\delta}_w(\Gamma)$, $\lambda > 0$ and ε, $0 < \varepsilon < p-1$. Then the last inequality for $\varphi \equiv g$ and arbitrary $\lambda > 0$, by virtue of (2.5), yields

$$wD(x,r) \leq \frac{c}{\lambda} \|g\|_{L^{p),\delta}_w}.$$

Consequently, $wD(x,r) = 0$. The latter expression contradicts the assumption that $w(t)$ is positive almost everywhere.

Now, substituting in (2.4) the function f by $w^{-\frac{1}{p-1}} \chi_{D(t,r)}$, for $z \in D(x,r)$, we have

$$\big|S_\Gamma g(z)\big| \geq \frac{2}{r} \int_{D(t,r)} w^{-\frac{1}{p-1}}(\tau)\, ds. \tag{2.6}$$

For $\lambda = \frac{2}{r} \int_{D(t,r)} w^{-\frac{1}{p-1}}(\tau)\, ds$, condition (ii) turns into the estimate

$$\frac{1}{r} \int_{D(t,r)} w^{-\frac{1}{p-1}}(\tau)\, ds \big(\delta(\varepsilon) wD(x,r)\big)^{\frac{1}{p-\varepsilon}} \leq c \|w^{-\frac{1}{p-1}}\|_{L^{p),\delta}(D(t,r))},$$

which in turn implies that

$$\frac{1}{r} \int_{D(t,r)} w^{-\frac{1}{p-1}}(\tau)\, ds \|\chi_{D(x,r)}\|_{L^{p),\delta}_w} \leq c \|w^{-\frac{1}{p-1}}\|_{L^{p),\delta}_w(D(t,r))} \tag{2.7}$$

with a constant c that does not depend on t, x, and r.

Step 2. We now assert that for arbitrary $D(t,r)$ and $f \in L_w^{p),\delta}(\Gamma)$, the inequality

$$\|f\|_{L_w^{p),\delta}(D(t,r))} \le \big(wD(t,r)\big)^{-\frac{1}{p}} \|f\|_{L_w^p(D(t,r))} \|\chi_{D(t,r)}\|_{L_w^{p),\delta}} \tag{2.8}$$

holds; in fact, by applying Hölder's inequality with the exponent $\frac{p}{p-\varepsilon}$, we get the chain of inequalities

$$\begin{aligned}\|f\|_{L_w^{p),\delta}(D(t,r))} &\le \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \Big(\int_{D(t,r)} w(\tau)\,ds\Big)^{\frac{\varepsilon}{(p-\varepsilon)p}} \cdot \|f\|_{L_w^p(D(t,r))} \\ &= \sup_{0<\varepsilon<p-1} \delta(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L_w^p(D(t,r))} \big(wD(t,r)\big)^{-\frac{1}{p}} \big(wD(t,r)\big)^{\frac{1}{p-\varepsilon}} \\ &= \big(wD(t,r)\big)^{-\frac{1}{p}} \|f\|_{L_w^p(D(t,r))} \|\chi_{D(t,r)}\|_{L_w^{p),\delta}}.\end{aligned}$$

Setting $f = w^{-\frac{1}{p-1}}$ in (2.8) and using (2.7), we deduce that

$$\begin{aligned}&\frac{1}{r} \int_{D(t,r)} w^{-\frac{1}{p-1}}(\tau)\,ds \|\chi_{D(x,r)}\|_{L_w^{p),\delta}} \\ &\quad \le c\Big(\int_{D(t,r)} w^{-\frac{1}{p-1}}(\tau)\,ds\Big)^{\frac{1}{p}} \big(wD(t,r)\big)^{-\frac{1}{p}} \|\chi_{D(t,r)}\|_{L_w^{p),\delta}},\end{aligned}$$

and hence

$$\big(wD(t,r)\big)^{\frac{1}{p}} \Big(\int_{D(t,r)} w^{-\frac{1}{p-1}}(\tau)\,ds\Big)^{\frac{1}{p'}} \|\chi_{D(x,r)}\|_{L_w^{p),\delta}} \le cr \|\chi_{D(t,r)}\|_{L_w^{p),\delta}}. \tag{2.9}$$

Changing the roles of $D(t,r)$ and $D(x,r)$, we get

$$\big(wD(x,r)\big)^{\frac{1}{p}} \Big(\int_{D(x,r)} w^{-\frac{1}{p-1}}(\tau)\,ds\Big)^{\frac{1}{p'}} \|\chi_{D(t,r)}\|_{L_w^{p),\delta}} \le cr \|\chi_{D(x,r)}\|_{L_w^{p),\delta}}. \tag{2.10}$$

Multiplying the inequalities (2.9) and (2.10) term by term and taking into account the rectifiablity of Γ and the estimate

$$r \le \big|D(x,r)\big| \le \big(wD(x,r)\big)^{\frac{1}{p}} \Big(\int_{D(x,r)} w^{-\frac{1}{p-1}}(\tau)\,ds\Big)^{\frac{1}{p'}},$$

we come to the condition

$$\frac{1}{r} \int_{D(t,r)} w(\tau)\,ds \Big(\frac{1}{r} \int_{D(t,r)} w^{1-p'}(\tau)\,ds\Big)^{p-1} \le c,$$

where c does not depend on $t \in \Gamma$ and $0 < r < \operatorname{diam}\Gamma$. Theorem 1 is proved. □

By analyzing the proof of Theorem 2.1, we can conclude that, in fact, for Calderón–Zygmund singular integrals defined on measured quasimetric space, a more general statement is true.

Definition 2.3. Let (X,d,μ) be a quasimetric measured space. The conditions

$$\mu B(x,r) \le c_1 r^{\alpha} \tag{2.11}$$

and

$$\mu B(x,r) \ge c_2 r^{\beta}$$

imposed on measure μ are known as the upper and lower *Ahlfors conditions* of orders α and β, respectively. The first one is also referred to as the *growth condition.* Here the constants c_1 and c_2 do not depend on $x \in X$ and $0 < r < \operatorname{diam} X$, and $B(x,r)$ denotes the ball in X with center $x \in X$ and radius r.

Theorem 2.4. *Let (X,d,μ) be a quasimetric measured space. Let $1 < p < \infty$, and let δ be a positive, nondecreasing, bounded function on $(0,p-1)$, $\delta(0+) = 0$. Then the following assertions are true.*

(i) *If $w \in A_p(X)$ and μ satisfies the upper Ahlfors condition of order 1, then the Calderón–Zygmund operator*

$$Kf(x) = \int_X k(x,y) f(y)\, d\mu$$

is bounded in $L_w^{p),\delta}(X)$.

(ii) *Let μ satisfy the lower Ahlfors condition of order 1. Suppose that the kernel $k : X \times X \longrightarrow \mathbb{R}$ is such that for arbitrary $\varphi \in L^1(X,\mu)$, there exist a measurable function g and balls B, B_1 in X such that $\operatorname{supp} g \subset B_1$ and*

$$\big|Kg(z)\big| \ge c_0 \int_{B_1} \frac{|\varphi(y)|}{d(z,y)}\, d\mu(y) \quad \forall z \in B,$$

where c_0 does not depend on φ and z. Then from the boundedness of K from $L^{p),\delta}(X)$ to $WL_w^{p),\delta}(X)$, it follows that the measure μ satisfies the upper Ahlfors condition of order 1 and that $w \in A_p(X)$.

3. Boundedness of the Cauchy singular integral operator in $\mathcal{L}_w^{p),\delta}(\Gamma)$

In this section, we establish the boundedness of S_Γ in the other weighted generalized grand Lebesgue spaces, namely, the generalization where weights are interpreted as multipliers.

Theorem 3.1. *If $1 < p < \infty$, Γ is a Carleson curve, $w^p \in A_p(\Gamma)$, and δ is as in Theorem 2.1, then S_Γ is bounded in $\mathcal{L}_w^{p),\delta}(\Gamma)$.*

Proof. By the virtue of the openness of $A_p(\Gamma)$ and the assumption that $w^p \in A_p(\Gamma)$, there exists a small positive number σ such that $w^p \in A_{p-\sigma}(\Gamma)$. Note that it is also $w^{p-\sigma} \in A_{p-\sigma}$; in fact, by applying Jensen's inequality, we derive the estimate

$$\begin{aligned}
&\Big(\frac{1}{r}\int_{D(t,r)} w^{p-\sigma}(t)\, ds\Big)^{\frac{1}{p-\sigma}} \Big(\frac{1}{r}\int_{D(t,r)} w^{(p-\sigma)[1-(p-\sigma)']}(t)\, ds\Big)^{\frac{1}{(p-\sigma)'}} \\
&\quad \le \Big(\frac{1}{r}\int_{D(t,r)} w^{p}(t)\, ds\Big)^{\frac{1}{p}} \Big(\frac{1}{r}\int_{D(t,r)} w^{p[1-(p-\sigma)']}(t)\, ds\Big)^{\frac{p-\sigma}{(p-\sigma)'}\cdot\frac{1}{p}}.
\end{aligned}$$

Since $\frac{p-\sigma}{(p-\sigma)'} = p - \sigma - 1$, we conclude that $w^{p-\sigma} \in A_{p-\sigma}$.

Now let us consider the operator

$$f \longrightarrow K_w f := w S_\Gamma\Big(\frac{f}{w}\Big).$$

It is obvious that the boundedness of S_Γ in $\mathcal{L}^p_w(\Gamma)$ is equivalent to the boundedness of K_w in $L^p(\Gamma)$. Since $w^p \in A_p$ and $w^{p-\sigma} \in A_{p-\sigma}$, the inequalities

$$\|K_w f\|_{L^p} \le c_1 \|f\|_{L^p}$$

and

$$\|K_w f\|_{L^{p-\sigma}} \le c_2 \|f\|_{L^{p-\sigma}}$$

hold, and by repeating verbatim the arguments used to prove Theorem 2.1, we get the boundedness of K_w in $L^{p),\delta}(\Gamma)$, which in turn is equivalent to the boundedness of S_Γ in $\mathcal{L}^{p),\delta}_w(\Gamma)$. Theorem 3.1 is proved. □

4. Hardy–Littlewood maximal function defined on curve

Let Γ be a rectifiable curve of finite length. Let us consider the Hardy–Littlewood maximal function defined as

$$(M_\Gamma f)(t) = \sup \frac{1}{r} \int_{D(t,r)} \left| f(\tau) \right| ds,$$

where the supremum is taken over all r, $0 < r < \operatorname{diam} \Gamma$. Using the arguments in Section 2, we are able to prove the following statement.

Theorem 4.1. *Let $1 < p < \infty$, and let δ be a positive, nondecreasing, bounded function on $(0, p-1)$, $\delta(0+) = 0$. The following conditions are equivalent:*

(i) *M_Γ is bounded in $L^{p),\delta}_w(\Gamma)$;*
(ii) *M_Γ is bounded from $L^{p),\delta}_w(\Gamma)$ to $WL^{p),\delta}_w(\Gamma)$;*
(iii) *Γ is a Carleson curve and $w \in A_p(\Gamma)$.*

We provide a sketch of the proof.

Sufficient part: The operator M_Γ is bounded in $L^p(\Gamma)$ $(1 < p < \infty)$ when Γ is Carleson curve and $w \in A_p(\Gamma)$. This follows from the fact that Γ with arc-length measure and Euclidean distance is a space of homogeneous type. On the other hand, the Hardy–Littlewood maximal function defined on a space of homogeneous type is bounded in L^p_w, if w is a Muckenhoupt weight (see [16]).

Necessary part: It is sufficient to note that from the definition of the maximal function, we have that

$$\frac{1}{r} \int_{D(t,r)} \left| f(\tau) \right| ds \le c M_\Gamma (f \cdot \chi_{D(t,r)})(z), \quad z \in D(t,r).$$

The remaining part of the proof is once more analogous to that of Theorem 2.1, and so we omit the details. Finally, we state the following result, whose proof follows the same lines as that of Theorem 3.1.

Theorem 4.2. *Let $1 < p < \infty$ and $w^p \in A_p(\Gamma)$. Then the operator M_Γ is bounded in $\mathcal{L}^{p),\delta}_w(\Gamma)$.*

5. The Riemann problem for analytic functions in the class of Cauchy-type integrals with densities in $L^{p),\delta}(\Gamma)$

In this section, we show an application of Theorem 2.1 to boundary value problems for analytic functions, a topic which was formulated for the first time by Riemann and which arose from several important results established by Hilbert, Sokhotski, Plemely, Poincaré, Bertrand, Noether, Carleman, Gakhov, Muskhelishvili, and Vekua.

We begin by introducing the necessary notation and definitions. Let

$$\mathcal{C}_\Gamma^{p),\delta} = \Big\{\Phi : \Phi(z) = \mathcal{C}_\Gamma(\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)\,d\tau}{\tau - z}, \varphi \in L^{p),\delta}, z \notin \Gamma\Big\}$$

and

$$\mathcal{C}_\Gamma = \Big\{\Phi : \Phi(z) = \mathcal{C}_\Gamma(\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)\,d\tau}{\tau - z}, \varphi \in L^1, z \notin \Gamma\Big\}.$$

The function φ is known as the *density* of the Cauchy-type integral $\mathcal{C}_\Gamma(\varphi)$.

Definition 5.1 ([15, p. 203]). Let D be a simply connected domain in the complex plane with rectifiable Jordan boundary Γ. The Smirnov class $E^s(D)$ $(s > 0)$ is the set of analytic functions on D enjoying the following property. For every sequence $(\Gamma_n)_{n=1,2,\ldots}$, $\Gamma_n \subset D$ being rectifiable Jordan curves approaching Γ as $n \longrightarrow \infty$, in the sense that if D_n is the bounded domain with boundary Γ_n, we have

$$D_1 \subset D_2 \subset \cdots \subset D \quad \text{and} \quad \bigcup_{n=1}^{\infty} D_n = D,$$

the condition

$$\sup_n \int_{\Gamma_n} \big|f(z)\big|^s\,|dz| < \infty$$

is fulfilled.

The classes $E^p(D)$ generalize the well-known Hardy classes.

Proposition A ([10, p. 501])**.** *Let D be a simply connected domain in the complex plane with rectifiable Jordan regular boundary Γ. Then $\Phi \in E^s(D)$ $(1 < s < \infty)$ if and only if it is represented by a Cauchy-type integral with density from $L^s(\Gamma)$.*

Proposition B ([15, p. 205])**.** *The class $E^1(D)$ coincides with the class of functions representable in D by a Cauchy-type integral.*

By using Theorem 2.1 and following the approach developed in [14, Section 4.18], we can establish the following result. Here we just recall that for Φ defined in $\mathbb{C}$, the symbol Φ^+ denotes the restriction of Φ in $D^+ := D$, and the symbol Φ^- denotes the restriction of Φ in $D^- := \mathbb{C} \setminus \overline{D}$.

Theorem 5.2. *Let Γ be a simple closed rectifiable Carleson curve, and let $G(t)$ be continuous on Γ, with $G(t) \neq 0$, $t \in \Gamma$, $p > 1$, and $g(t) \in L^{p),\delta}(\Gamma)$. If $\varkappa = \operatorname{ind} G(t) = \frac{1}{2\pi}[\arg G(t)]_\Gamma$, then*

(i) *for $\varkappa = 0$, the boundary value problem*

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \quad \textit{a.e. on } \Gamma \tag{5.1}$$

is uniquely solvable in $\mathcal{C}_\Gamma^{p),\delta}$ and the solution is given by

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_\Gamma \frac{g(t)}{X^+(t)(t-z)}, \tag{5.2}$$

where

$$X(z) = \exp \mathcal{C}_\Gamma(\ln G)(z), \quad z \notin \Gamma;$$

(ii) *for $\varkappa > 0$, problem (5.1) is unconditionally solved in $\mathcal{C}_\Gamma^{p),\delta}$ and its general solution is given by the equality*

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_\Gamma \frac{g(t)}{X^+(t)(t-z)} + X(z)Q_{\varkappa-1}(z), \tag{5.3}$$

where

$$X(z) = \begin{cases} \exp h(z), & z \in D^+, \\ (t-z_0)^{-\varkappa}\exp h(z), & z \in D^-, z_0 \in D^+, \end{cases}$$
$$h(z) = \mathcal{C}_\Gamma\big(\ln G(t)(t-z_0)^{-\varkappa}\big)(z),$$

and $Q_{\varkappa-1}(z)$ is an arbitrary polynomial of order $\varkappa - 1$ ($Q_{-1}(z) \equiv 0$).

(iii) *for $\varkappa < 0$, problem (5.1) is solvable in $\mathcal{C}_\Gamma^{p),\delta}$ if and only if*

$$\int_\Gamma \frac{g(t)}{X^+(t)} t^k\, dt = 0, \quad k = \overline{0, \varkappa - 1}.$$

If the latter conditions are fulfilled, then problem (5.1) has a unique solution given by equality (5.3) for $Q_{\varkappa-1}(z) \equiv 0$.

Proof. Select a rational function $\widetilde{G}(t)$ such that

$$\sup_{t\in\Gamma}\left|\frac{G(t)}{\widetilde{G}(t)} - 1\right| < \frac{1}{2}\big(1 + \|S_\Gamma\|_{L^{p(\cdot),\theta}\to L^{p(\cdot),\theta}}\big)^{-1}. \tag{5.4}$$

It is clear that $\operatorname{ind}\widetilde{G}(t) = 0$; indeed,

$$\operatorname{ind} G = \operatorname{ind}\frac{G}{\widetilde{G}} + \operatorname{ind}\widetilde{G},$$

and by (5.4), we have $\operatorname{ind}\frac{G}{\widetilde{G}} = 0$, so that $\operatorname{ind}\widetilde{G}(t) = 0$.

Consider now the function

$$\widetilde{X}(z) = \exp \mathcal{C}_\Gamma(\ln \widetilde{G})(z), \quad z \notin \Gamma, \tag{5.5}$$

so that the functions $[\widetilde{X}(z)]^{\pm 1}$ are bounded and

$$\widetilde{G}(t) = \frac{\widetilde{X}^+(t)}{\widetilde{X}^-(t)}.$$

Let us rewrite condition (5.1) in the form

$$\Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{+} = \frac{G}{\widetilde{\widetilde{G}}}\Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{-} + \frac{g}{\widetilde{\widetilde{X}}^{+}}. \tag{5.6}$$

Since the sought function $\Phi \in \mathcal{C}_\Gamma^{p),\delta}$, we have that $\Phi(z) \in E^{p-\varepsilon}$ for $0 < \varepsilon < p-1$ and that, according to Theorem 2.1, $\Phi^{\pm}(t) \in L^{p),\delta}(\Gamma)$. The fact that $\frac{1}{\widetilde{X}(z)}$ is bounded implies that

$$\frac{\Phi(z)}{\widetilde{\widetilde{X}}(z)} \in E^{\underline{p}-\varepsilon}(D^{\pm}) \subset E^{1}(D^{\pm}).$$

Due to the latter inclusion, the function $\frac{\Phi}{\widetilde{\widetilde{X}}}$ belongs to $\mathcal{C}_\Gamma^{p(\cdot),\theta}$; indeed,

$$\mathcal{C}_\Gamma\Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{+}(z) = \begin{cases} \frac{\Phi(z)}{\widetilde{\widetilde{X}}(z)}, & z \in D^{+},\\ 0, & z \in D^{-} \end{cases}$$

and

$$\mathcal{C}_\Gamma\Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{-}(z) = \begin{cases} 0, & z \in D^{+},\\ -\frac{\Phi(z)}{\widetilde{\widetilde{X}}(z)}, & z \in D^{-}. \end{cases}$$

Thus,

$$\frac{\Phi(z)}{\widetilde{\widetilde{X}}(z)} = \mathcal{C}_\Gamma\Big[\Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{+} - \Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{-}\Big](z)$$

and

$$\Big[\Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{+} - \Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{-}\Big] \in L^{p),\delta}(\Gamma).$$

Therefore,

$$\frac{\Phi}{\widetilde{\widetilde{X}}} \in \mathcal{C}_\Gamma^{p),\delta};$$

that is,

$$\frac{\Phi(z)}{\widetilde{\widetilde{X}}(z)} = \mathcal{C}_\Gamma(\varphi)(z), \quad \varphi \in L^{p),\delta}(\Gamma).$$

The equality

$$\Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{+} - \Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{-} = \Big(\frac{G}{\widetilde{\widetilde{G}}} - 1\Big)\Big(\frac{\Phi}{\widetilde{\widetilde{X}}}\Big)^{-} + \frac{g}{\widetilde{\widetilde{X}}^{+}}$$

follows from (5.6). Consequently, by the Sokhotski–Plemely formula we derive that

$$\varphi(t) = \Big(\frac{G}{\widetilde{\widetilde{G}}} - 1\Big)\Big(-\frac{1}{2}\varphi(t) + \frac{1}{2}(S_\Gamma\varphi)(t)\Big) + \frac{g(t)}{\widetilde{\widetilde{X}}^{+}(t)}. \tag{5.7}$$

The latter is the equation

$$\varphi = M\varphi, \tag{5.8}$$

where, due to Theorem 2.1, the operator M is bounded in $L^{p),\delta}(\Gamma)$. Then, by virtue of condition (5.4), the norm of the operator M is less than 1 and it is a contractive operator. Therefore, equation (5.8) is uniquely solvable in $L^{p),\delta}(\Gamma)$. Hence, $\Phi(z) = \widetilde{X}(z)\mathcal{C}_\Gamma(\varphi)(z) \in \mathcal{C}_\Gamma^{p),\delta} \subset \mathcal{C}_\Gamma$.

Now if we consider the function

$$X(z) = \exp\big(\mathcal{C}_\Gamma(\ln G)\big)(z), \tag{5.9}$$

then

$$G(t) = \frac{X^+(t)}{X^-(t)}, \quad t \in \Gamma,$$

and the boundary condition (5.1) can be rewritten in the form

$$\left(\frac{\Phi}{X}\right)^+ - \left(\frac{\Phi}{X}\right)^- = \frac{g}{X^+}. \tag{5.10}$$

Since the boundary Γ is a Carleson curve, we have that (see [12])

$$\Phi \in E^{\underline{p}-\varepsilon}(D^\pm), \quad \frac{1}{X(z)} \in \bigcap_{\mu=1}^{\infty} E^\mu(D^+), \quad \left(\frac{1}{X(z)} - 1\right) \in \bigcap_{\mu=1}^{\infty} E^\mu(D^-).$$

Therefore, $\frac{\Phi(z)}{X(z)} \in E^1(D^\pm)$ and, consequently, $\frac{\Phi}{X} \in \mathcal{C}_\Gamma$. But equation (5.10) has the unique solution

$$\Phi(z) = X(z)\mathcal{C}_\Gamma\left(\frac{g}{X^+}\right)(z) \tag{5.11}$$

in $\mathcal{C}_\Gamma$, and then $\Phi \in \mathcal{C}_\Gamma^{p),\delta}$.

If we take into account that the function $\Phi^+(t) - (t-z_0)^{\varkappa}\Phi^-(t)$ belongs to $L^{p),\delta}(\Gamma)$, then the remaining part of the proof of Theorem 5.2 for the case $\varkappa \neq 0$ is valid. Theorem 5.2 is proved. □

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