Ann. Funct. Anal. 9 (2018), no. 2, 220–232
https://doi.org/10.1215/20088752-2017-0049
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON GENERALIZED POINTWISE NONCYCLIC CONTRACTIONS WITHOUT PROXIMAL NORMAL STRUCTURE

MOOSA GABELEH

Communicated by M. A. Japón

Abstract. In this article, we introduce a new class of noncyclic mappings called *generalized pointwise noncyclic contractions*, and we prove a best proximity pair theorem for this class of noncyclic mappings in the setting of strictly convex Banach spaces. Our conclusions generalize a result due to Kirk and Royalty. We also study convergence of iterates of noncyclic contraction mappings in uniformly convex Banach spaces.

1. Introduction

Let (X, d) be a metric space. A self-mapping $T : X \to X$ is said to be *nonexpansive* provided that $d(Tx, Ty) \leq d(x, y)$. It is well known that if A is a nonempty, compact, and convex subset of a Banach space X, then every nonexpansive mapping of A into itself has a fixed point.

In 1965, Kirk proved that, if A is a nonempty, weakly compact, and convex subset of a Banach space with a geometric property, called *normal structure*, then every nonexpansive mapping $T : A \to A$ has a fixed point (see Kirk's fixed-point theorem [8]). Kirk and Royalty [9] replaced the geometric property of normal structure with another assumption on the nonexpansive mapping T and established the following interesting fixed-point theorem.

Theorem 1.1 ([9, Theorem 2.1], [10, Theorem 4.1]). *Let A be a nonempty, weakly compact, and convex subset of a Banach space X. If we suppose that $T : A \to A$*

Received Dec. 22, 2016; Accepted May 16, 2017.
First published online Dec. 7, 2017.
2010 *Mathematics Subject Classification.* 47H10, 47H09, 46B20.
Keywords. best proximity pair, generalized pointwise noncyclic contraction, strictly convex Banach space, iterate sequence.

is a nonexpansive mapping such that for each $x \in A$ there exist a positive integer $N(x)$ and an $\alpha(x) \in [0,1)$ such that

$$\|T^{N(x)}x - T^{N(x)}y\| \le \alpha(x)\|x-y\| \quad \text{for all } y \in A,$$

then T has a unique fixed point.

Now suppose that (A,B) is a nonempty pair of subsets of a metric space (X,d). A mapping $T : A \cup B \to A \cup B$ is said to be *noncyclic relatively nonexpansive* if T is noncyclic; that is, $T(A) \subseteq A, T(B) \subseteq B$, and $d(Tx,Ty) \le d(x,y)$ for all $(x,y) \in A \times B$. Under this weaker assumption over T, the existence of the so-called *best proximity pair*, that is, a point $(x^\star, y^\star) \in A \times B$ such that $x^\star = Tx^\star$, $y^\star = Ty^\star$ and $d(x^\star, y^\star) = \operatorname{dist}(A,B) := \inf\{d(x,y) : (x,y) \in A \times B\}$. The best proximity pair was first studied in [2]. The next theorem is a main result of [2] (see also [4] for a different approach to the same problem).

Theorem 1.2 ([2, Theorem 2.2])**.** *Let (A,B) be a nonempty, weakly compact, and convex pair of subsets of a strictly convex Banach space X, and suppose that (A,B) has proximal normal structure. If we assume that $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping, then T has a best proximity pair.*

This paper is organized as follows. In Section 2 we recall some definitions, notions, and previous results we will need. In Section 3 we introduce a new class of noncyclic mappings called *generalized pointwise noncyclic contractions*, and we prove a best proximity pair theorem in strictly convex Banach spaces. In this way, we extend a main result of [9]. In Section 4, we prove a convergence theorem of Picard iterates for noncyclic contractions in the setting of uniformly convex Banach spaces.

2. Preliminaries

To describe our results, we need some definitions and notation. We will say that a pair (A,B) of subsets of a Banach space X satisfies a property if both A and B satisfy that property. For example, (A,B) is convex if and only if both A and B are convex; $(A,B) \subseteq (C,D) \Leftrightarrow A \subseteq C$, and $B \subseteq D$. We also adopt the notation

$$\begin{aligned}
\delta_x(A) &= \sup\big\{d(x,y) \colon y \in A\big\} \quad \text{for all } x \in X,\\
\delta(A,B) &= \sup\big\{\delta_x(B) : x \in A\big\},\\
\operatorname{diam}(A) &= \delta(A,A).
\end{aligned}$$

The *closed and convex hull* of a set A will be denoted by $\overline{\operatorname{conv}}(A)$, and $\mathcal{B}(p,r)$ will denote the closed ball in the space X centered at $p \in X$ with radius $r > 0$.

If (A,B) is a pair of nonempty subsets of a Banach space, then its proximal pair is the pair (A_0,B_0) given by

$$\begin{aligned}
A_0 &= \big\{x \in A \colon \|x-y'\| = \operatorname{dist}(A,B) \text{ for some } y' \in B\big\},\\
B_0 &= \big\{y \in B \colon \|x'-y\| = \operatorname{dist}(A,B) \text{ for some } x' \in A\big\}.
\end{aligned}$$

Proximal pairs may be empty, but, if A and B are nonempty weakly compact and convex, then (A_0,B_0) is a nonempty, weakly compact, convex pair in X.

Definition 2.1. A pair (A, B) in a Banach space is said to be *proximinal* if $A = A_0$, and $B = B_0$.

Definition 2.2. Let (A, B) be a nonempty pair of sets in a Banach space X. A point (p, q) in $A \times B$ is said to be a *diametral pair* if

$$\delta_p(B) = \delta_q(A) = \delta(A, B).$$

For a noncyclic mapping $T : A \cup B \to A \cup B$, we consider that a pair $(C, D) \subseteq (A, B)$ is T-invariant if T is noncyclic on $C \cup D$. We now state the following two lemmas, which will be used in our main results.

Lemma 2.3 ([2, proof of Theorem 2.1])**.** *Let (A, B) be a nonempty, weakly compact, and convex pair of subsets of a Banach space X, and let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. Then there exists $(K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)$ which is minimal with respect to being a nonempty, closed, convex, and T-invariant pair of subsets of (A, B) such that*

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B).$$

Moreover, the pair (K_1, K_2) is proximinal.

Lemma 2.4 ([5, Lemma 3.8])**.** *Let (A, B) be a nonempty, weakly compact, and convex pair in a strictly convex Banach space X. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping, and let $(K_1, K_2) \subseteq (A, B)$ be a minimal, weakly compact, and convex pair which is T-invariant such that*

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B).$$

Then each point $(p, q) \in K_1 \times K_2$ with $\|p - q\| = \operatorname{dist}(A, B)$ is a diametral pair.

We finish this section by recalling the following useful geometric concepts of Banach space.

Definition 2.5. A Banach space X is considered to be:

(i) uniformly convex if there exists a strictly increasing function $\delta : [0, 2] \to [0, 1]$ such that the following implication holds for all $x, y, p \in X, R > 0$, and $r \in [0, 2R]$:

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ \|x - y\| \geq r \end{cases} \Rightarrow \Big\|\frac{x + y}{2} - p\Big\| \leq \Big(1 - \delta\Big(\frac{r}{R}\Big)\Big)R;$$

(ii) strictly convex if the following implication holds for $x, y, p \in X$, and $R > 0$:

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ x \neq y \end{cases} \Rightarrow \Big\|\frac{x + y}{2} - p\Big\| < R.$$

It is well known that Hilbert spaces and l^p spaces ($1 < p < \infty$) are uniformly convex Banach spaces and that the Banach space l^1 with the norm

$$|x| = \sqrt{\|x\|_1 + \|x\|_2}, \quad \forall x \in l^1,$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the norms on l^1 and l^2, respectively, is strictly convex, which is not uniformly convex (see [12] for more details).

3. Generalized pointwise noncyclic contractions

A geometric notion of proximal normal structure on a nonempty and convex pair in a Banach space X was introduced in [2] as below.

Definition 3.1 ([2, Definition 1.2]). A convex pair (K_1, K_2) in a Banach space X is said to have *proximal normal structure* (PNS) if, for any bounded, closed, convex, and proximinal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $\operatorname{dist}(H_1, H_2) = \operatorname{dist}(K_1, K_2)$ and $\delta(H_1, H_2) > \operatorname{dist}(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that

$$\max\big\{\delta_{x_1}(H_2), \delta_{x_2}(H_1)\big\} < \delta(H_1, H_2).$$

It is worth noting that the pair (K, K) has PNS if and only if K has normal structure in the sense of Brodskii and Milman ([1]). Very recently, an extension version of Theorem 1.2 was proved for *generalized pointwise noncyclic relatively nonexpansive mappings*.

Theorem 3.2 ([7, Theorem 4.2])**.** *Let (A, B) be a nonempty, weakly compact, and convex pair of subsets of a strictly convex Banach space X, and suppose that (A, B) has* PNS*. Assume that $T : A \cup B \to A \cup B$ is a generalized pointwise noncyclic relatively nonexpansive mapping, that is, that T is noncyclic on $A \cup B$ and that, for any $(x, y) \in A \times B$, if $\|x - y\| = \operatorname{dist}(A, B)$, then $\|Tx - Ty\| = \operatorname{dist}(A, B)$ and that otherwise there exists a function $\alpha : A \times B \to [0, 1]$ such that*

$$\|Tx - Ty\| \le \alpha(x, y)\|x - y\| + \big(1 - \alpha(x, y)\big) \min\big\{\|x - Ty\|, \|Tx - y\|\big\}.$$

Hence T has a best proximity pair.

Motivated by Theorem 3.2, we introduce the following new class of noncyclic mappings and survey the existence of best proximity pairs for such mappings *without using the geometric notion of* PNS.

Definition 3.3. Let (A, B) be a nonempty pair of subsets of a Banach space X. A mapping $T : A \cup B \to A \cup B$ is said to be a generalized pointwise noncyclic contraction if T is noncyclic and if, for each $(x, y) \in A \times B$, there exist positive integers $N(x), N(y)$ and $\alpha(x), \alpha(y) \in [0, 1)$ such that

$$\|T^{N(x)}x - T^{N(x)}y\| \le \alpha(x)\|x - y\| + \big(1 - \alpha(x)\big) \operatorname{dist}(A, B) \quad \text{for all } y \in B,$$
$$\|T^{N(y)}x - T^{N(y)}y\| \le \alpha(y)\|x - y\| + \big(1 - \alpha(y)\big) \operatorname{dist}(A, B) \quad \text{for all } x \in A.$$

The next theorem is the main result of this section.

Theorem 3.4. *Let (A, B) be a nonempty, weakly compact, and convex pair of subsets of a strictly convex Banach space X, and suppose that $T : A \cup B \to A \cup B$ is a noncyclic, relatively nonexpansive mapping. If T is a generalized pointwise noncyclic contraction, then T has a best proximity pair.*

Proof. It follows from Lemma 2.3 that there exists a proximinal pair (K_1, K_2) in (A, B) which is minimal with respect to being nonempty, closed, convex, and T-invariant such that $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$. Furthermore, by Lemma 2.4, for each $(p, q) \in K_1 \times K_2$ with $\|p - q\| = \operatorname{dist}(A, B)$ we have

$$\delta_p(K_2) = \delta_q(K_1) = \delta(K_1, K_2).$$

Suppose that $\delta(K_1, K_2) = \operatorname{dist}(A, B)$. If $(x, y) \in K_1 \times K_2$, then

$$\operatorname{dist}(A, B) = \operatorname{dist}(K_1, K_2) \leq \|x - y\| \leq \delta(K_1, K_2) = \operatorname{dist}(A, B).$$

If x' is another element of K_1, then $\|x - y\| = \|x' - y\| = \operatorname{dist}(A, B)$, and, by the strict convexity of X and the fact that (A, B) is a convex pair, we obtain

$$\operatorname{dist}(A, B) \leq \left\|\frac{x + x'}{2} - y\right\| < \frac{1}{2}\big(\|x - y\| + \|x' - y\|\big) = \operatorname{dist}(A, B),$$

which is a contradiction. This implies that K_1 is singleton. Similarly, K_2 is also singleton, and the result follows. We therefore assume that $\delta(K_1, K_2) > \operatorname{dist}(A, B)$. Let $(x, y) \in K_1 \times K_2$, let $N_1 = N_1(x), N_2 = N_2(y)$, and let $N = \max\{N_1, N_2\}$. Next put

$$\begin{aligned} r_1 &:= \alpha(x)\delta(K_1, K_2) + \big(1 - \alpha(x)\big) \operatorname{dist}(A, B), \\ r_2 &:= \alpha(y)\delta(K_1, K_2) + \big(1 - \alpha(y)\big) \operatorname{dist}(A, B). \end{aligned}$$

We may assume that $r_2 \leq r_1$. Define

$$\begin{aligned} F_1 &:= \big\{v \in K_2 : \|v - T^{iN}x\| \leq r_1 \text{ for almost all } i \geq 1\big\}, \\ E_1 &:= \big\{u \in K_1 : \|u - T^{iN}y\| \leq r_1 \text{ for almost all } i \geq 1\big\}, \\ F_2 &:= \big\{v \in K_2 : \|v - T^{[iN+1]}x\| \leq r_1 \text{ for almost all } i \geq 1\big\}, \\ E_2 &:= \big\{u \in K_1 : \|u - T^{[iN+1]}y\| \leq r_1 \text{ for almost all } i \geq 1\big\}, \\ &\ \vdots \\ F_N &:= \big\{v \in K_2 : \|v - T^{[(i+1)N-1]}x\| \leq r_1 \text{ for almost all } i \geq 1\big\}, \\ E_N &:= \big\{u \in K_1 : \|u - T^{[(i+1)N-1]}y\| \leq r_1 \text{ for almost all } i \geq 1\big\}. \end{aligned}$$

Note that $T^N y \in F_1$. Indeed, since T is a noncyclic, relatively nonexpansive, and generalized pointwise noncyclic contraction, we have

$$\begin{aligned} \|T^N y - T^{iN}x\| \leq \|T^{N_2}y - T^{iN_2}x\| &= \big\|T^{N_2}y - T^{N_2}(T^{(i-1)N_2}x)\big\| \\ &\leq \alpha(y)\|y - T^{(i-1)N_2}x\| + \big(1 - \alpha(y)\big) \operatorname{dist}(A, B) \\ &\leq \alpha(y)\delta(K_1, K_2) + \big(1 - \alpha(y)\big) \operatorname{dist}(A, B) \leq r_1. \end{aligned}$$

Moreover, if $v \in F_1$, then

$$\|Tv - T^{(iN+1)}x\| \leq \|v - T^{iN}x\| \leq r_1;$$

that is, $Tv \in F_2$; hence $T(F_1) \subseteq F_2$. Similarly, we can see that $T(F_i) \subseteq F_{i+1}$ for all $i = 1, 2, \ldots, N-1$. If we now let $v \in F_N$, then $\|v - T^{[(i+1)N-1]}x\| \leq r_1$ for almost all $i \geq 1$. This implies that $\|Tv - T^{iN}x\| \leq r_1$ for almost all $i \geq 1$; that is, $Tv \in F_1$; thus $T(F_N) \subseteq F_1$. We also observe that $\{F_j\}_{j=1}^N$ is a finite family of nonempty and convex subsets of K_2. Equivalently, $\{E_j\}_{j=1}^N$ is a finite family of nonempty and convex subsets of K_1, and we have $T(E_i) \subseteq E_{i+1}$ for all $i = 1, 2, \dots, N-1$, and $T(E_N) \subseteq E_1$. Suppose that $i \geq j \geq N$. If we say that $j = N + k$ and that $i = N + k + s$, then

$$\begin{aligned}\|T^i y - T^j x\| &= \|T^{N+k+s} y - T^{N+k} x\| \leq \|T^{N+s} y - T^N x\| \\ &= \big\|T^N(T^s y) - T^N x\big\| \leq \alpha(x)\|x - T^s y\| + \big(1 - \alpha(x)\big)\operatorname{dist}(A, B) \\ &\leq \alpha(x)\delta(K_1, K_2) + \big(1 - \alpha(x)\big)\operatorname{dist}(A, B) \leq r_1.\end{aligned}$$

Thus $T^i(y) \in [\mathcal{B}(T^j x; r_1) \cap K_2]$ for each $i \geq j \geq N$; hence the family of $\{\mathcal{B}(T^j x; r_1) \cap K_2\}_{j \geq N}$ has the finite intersection property. It now follows from the weak compactness of K_2 that $\bigcap_{j=N}^\infty [\mathcal{B}(T^j x; r_1) \cap K_2]$ is nonempty. Using a similar approach, we can see that $\bigcap_{j=N}^\infty [\mathcal{B}(T^j y; r_1) \cap K_1]$ is also nonempty; hence the pair $(\bigcap_{j=1}^N E_j, \bigcap_{j=1}^N F_j)$ is nonempty. If we set

$$E := \Big[\bigcap_{j=1}^N E_j\Big] \cap K_1, \qquad F := \Big[\bigcap_{j=1}^N F_j\Big] \cap K_2,$$

then $(\overline{E}, \overline{F}) \subseteq (K_1, K_2)$ is a nonempty, closed, convex, and T-invariant pair. Minimality of (K_1, K_2) implies that $\overline{E} = K_1, \overline{F} = K_2$. In particular, $\overline{E_1} = K_1$ and $\overline{F_1} = K_2$; thus, if $v \in K_2$, and if $\varepsilon > 0$ is chosen such that $r_1 + \varepsilon < \delta(K_1, K_2)$, then we have

$$\|v - T^{iN} x\| \leq r_1 + \varepsilon < \delta(K_1, K_2),$$

for i is sufficiently large. Specifically, if $\{v_1, v_2, \dots, v_j\}$ is a subset of K_2, then by the recent relation for each $l = 1, 2, \dots, j$ we have $\|v_l - T^{iN} x\| \leq r_1 + \varepsilon$ for almost all $i \geq 1$. Consequently, $T^{iN} x \in [\bigcap_{l=1}^j \mathcal{B}(v_l; r_1)] \cap K_1$ for i is sufficiently large; that is, $[\bigcap_{l=1}^j \mathcal{B}(v_l; r_1)] \cap K_1 \neq \emptyset$. Hence the family $\{\mathcal{B}(v; r_1 + \varepsilon) \cap K_1 : v \in K_2\}$ consisting of weakly compact sets has the finite intersection property, and, accordingly, $[\bigcap_{v \in K_2} \mathcal{B}(v; r_1 + \varepsilon)] \cap K_1$ is nonempty. Assume that

$$x^\star \in \Big[\bigcap_{v \in K_2} \mathcal{B}(v; r_1 + \varepsilon)\Big] \cap K_1.$$

Since (K_1, K_2) is a proximinal pair, there exists an element $y^\star \in K_2$ such that $\|x^\star - y^\star\| = \operatorname{dist}(A, B)$. On the other hand, for each $v \in K_2$ we have

$$\|x^\star - v\| \leq r_1 + \varepsilon < \delta(K_1, K_2);$$

hence $\delta_{x^\star}(K_2) < \delta(K_1, K_2)$. That is, the point $(x^\star, y^\star)$ is not a diametral pair, and this is a contradiction, by Lemma 2.4. □

The following fixed-point theorem is an extension of a theorem by Kirk and Royalty in the setting of strictly convex Banach spaces.

Theorem 3.5 ([9, Theorem 2.1])**.** *Let (A, B) be a nonempty, weakly compact, and convex pair of subsets of a strictly convex Banach space X, and let $T : A \cup B \to A \cup B$ be a noncyclic, relatively nonexpansive mapping. If we suppose that, for each*

$(x,y) \in A \times B$, there exist positive integers $N(x)$, $N(y)$ and $\alpha(x), \alpha(y) \in [0,1)$ such that

$$\|T^{N(x)}x - T^{N(x)}y\| \le \alpha(x)\|x-y\| \quad \textit{for all } y \in B,$$
$$\|T^{N(y)}x - T^{N(y)}y\| \le \alpha(y)\|x-y\| \quad \textit{for all } x \in A,$$

then $A \cap B$ is nonempty, and T has a unique fixed point in $A \cap B$.

Note that in the special case of $A = B$, we do not need to require the strict convexity of the Banach space X.

It is possible to reformulate Theorem 3.5 as a common fixed-point theorem for two mappings as follows.

Corollary 3.6. *Let (A,B) be a nonempty, weakly compact, and convex pair of subsets of a strictly convex Banach space X, and let $f : A \to A$ and $g : B \to B$ be two self-mappings. If, for each $(x,y) \in A \times B$, there exist positive integers $N(x), N(y)$ and $\alpha(x), \alpha(y) \in [0,1)$ such that*

$$\|f^{N(x)}x - g^{N(x)}y\| \le \alpha(x)\|x-y\| \quad \textit{for all } y \in B,$$
$$\|f^{N(y)}x - g^{N(y)}y\| \le \alpha(y)\|x-y\| \quad \textit{for all } x \in A,$$

then there exists a unique $x^\star \in A \cap B$ such that

$$fx^\star = gx^\star = x^\star.$$

The next theorem follows immediately from Theorem 3.4.

Theorem 3.7. *Let (A,B) be a nonempty, weakly compact, and convex pair of subsets of a strictly convex Banach space X, and suppose that $T : A \cup B \to A \cup B$ is a pointwise noncyclic contraction; that is, for each $(x,y) \in A \times B$ there exist $\alpha(x), \alpha(y) \in [0,1)$ such that*

$$\|Tx - Ty\| \le \alpha(x)\|x-y\| + \big(1-\alpha(x)\big)\operatorname{dist}(A,B) \quad \textit{for all } y \in B,$$
$$\|Tx - Ty\| \le \alpha(y)\|x-y\| + \big(1-\alpha(y)\big)\operatorname{dist}(A,B) \quad \textit{for all } x \in A.$$

Consequently, T has a best proximity pair.

As a result of Theorem 3.7 we obtain the following corollary.

Corollary 3.8 ([5, Theorem 3.10])**.** *Let (A,B) be a nonempty, weakly compact, and convex pair of subsets of a strictly convex Banach space X, and suppose that $T : A \cup B \to A \cup B$ is a noncyclic contraction mapping, that is, that T is noncyclic on $A \cup B$ and that*

$$\|Tx - Ty\| \le r\|x-y\| + (1-r)\operatorname{dist}(A,B),$$

for some $r \in [0,1)$ and for all $(x,y) \in A \times B$. In that case, T has a best proximity pair.

Let us illustrate Theorem 3.4 with the following example.

Example 3.9. Let $A=[0,1]$ and $B=[2,3]$ be subsets of $\mathbb{R}$ endowed with the Euclidean metric. Define a noncyclic mapping $T:A\cup B\to A\cup B$ as follows:

$$T(x)=\begin{cases}\sqrt{x} & \text{if } x\in\mathbb{Q}\cap(A-\{0\}),\\ 1 & \text{if } x\in(\mathbb{Q}^c\cap A)\cup\{0\},\\ 2 & \text{if } x\in B.\end{cases}$$

Then

- T is a noncyclic, relatively nonexpansive mapping.
  Case 1. If $x\in\mathbb{Q}\cap A$, and $y\in B$, then
  $$|Tx-Ty|\le 2-\sqrt{x}\le|x-y|.$$
  Case 2. If $x\in\mathbb{Q}^c\cap A$, and $y\in B$, then
  $$|Tx-Ty|=1\le|x-y|.$$
- T is a generalized pointwise noncyclic contraction.
  Case 1. If $y\in B$, then for each $x\in\mathbb{Q}\cap A$, by the definition of T, there exists $N(x)\in\mathbb{N}$ so that $T^{N(x)}x=1$. Now for all $\alpha\in[0,1)$, we have
  $$|T^{N(x)}x-T^{N(x)}y|=|1-2|\le\alpha|x-y|+(1-\alpha)\operatorname{dist}(A,B).$$
  It is clear that, if $x\in\mathbb{Q}^c\cap A$, then the above relation holds for $N(x)=1$.
  Case 2. If $x\in A$, then there exists $N(x)\in\mathbb{N}$ such that $T^{N(x)}x=1$. Now for any $y\in B$ and $\alpha\in[0,1)$, if we set $N(y):=N(x)$, then we obtain
  $$|T^{N(y)}x-T^{N(y)}y|=|1-2|\le\alpha|x-y|+(1-\alpha)\operatorname{dist}(A,B).$$
- T is not a noncyclic contraction. If we suppose the contrary, then there exists $\alpha\in[0,1)$ such that
  $$|Tx-Ty|\le\alpha|x-y|+(1-\alpha),\quad\forall(x,y)\in A\times B.$$
  Now, if $x=\frac{1}{n^2}$ for $n\in\mathbb{N}$, and $y=2$, then we have
  $$2-\frac{1}{n}\le\alpha\Big(2-\frac{1}{n^2}\Big)+(1-\alpha)=1+\alpha\Big(1-\frac{1}{n^2}\Big).$$
  Thus we conclude that $\frac{n}{n+1}\le\alpha$ for any $n\in\mathbb{N}$, which is a contradiction.
  We note that all of the conditions of Theorem 3.4 hold; therefore T has a best proximity pair.

The next example shows that the strict convexity of the underlying space is just a sufficient condition in Theorem 3.4.

Example 3.10. Let X be the Banach space $\mathbb{R}^3$ endowed with the supremum norm, let $\{e_1,e_2,e_3\}$ be the canonical basis of $\mathbb{R}^3$, and let e_0 be the zero of $\mathbb{R}^3$. We note that the Banach space X is not strictly convex. Let

$$A:=\overline{\operatorname{conv}}\big(\{e_0,e_1,e_3\}\big)\quad\text{and}\quad B:=\{te_2:0\le t\le1\}.$$

It is clear that (A, B) is a bounded, closed, and convex pair in X. If we define a noncyclic mapping $T : A \cup B \to A \cup B$ as follows:

$$Tx = \begin{cases} e_0 & \text{if } x \neq e_1, \\ e_3 & \text{if } x = e_1, \end{cases}$$

then T is noncyclic, relatively nonexpansive on $A \cup B$. Moreover, for each $(x, y) \in A \times B$ and $\alpha \in [0, 1)$ we have

$$\|T^2x - T^2y\|_\infty = 0 \leq \alpha\|x - y\|_\infty.$$

We note that $A \cap B$ is nonempty and that T has a unique fixed point in $A \cap B$.

4. Convergence of iterate sequences for noncyclic contractions in uniformly convex Banach spaces

In this section, we establish a best proximity pair theorem for noncyclic contractions defined on *unbounded* pairs of subsets of a uniformly convex Banach space X. To this end, we recall the following inequality which is a characterization of uniformly convex Banach spaces.

Proposition 4.1 (see [11]). *A Banach space X is uniformly convex if and only if, for each fixed number $r > 0$, there exits a continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) = 0 \Leftrightarrow t = 0$ such that*

$$\big\|\lambda x + (1 - \lambda)y\big\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\varphi\big(\|x - y\|\big),$$

for all $\lambda \in [0, 1]$ and all $x, y \in X$ so that $\|x\| \leq r$, and $\|y\| \leq r$.

We also need the following auxiliary lemmas.

Lemma 4.2 ([3, Lemma 3.7]). *Let (A, B) be a nonempty, closed, and convex pair in a uniformly convex Banach space X. If we assume that $\{x_n\}$ and $\{z_n\}$ are sequences in A and that $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$, and we assume that, for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $m > n \geq N_0$, $\|x_m - y_n\| \leq \operatorname{dist}(A, B) + \varepsilon$, then there exists $N_1 \in \mathbb{N}$ such that, for all $m > n \geq N_1$, $\|x_m - z_n\| < \varepsilon$.*

Lemma 4.3 ([3, Lemma 3.8]). *Let (A, B) be a nonempty, closed, and convex pair in a uniformly convex Banach space X. If we assume that $\{x_n\}$ and $\{z_n\}$ are sequences in A and that $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(A, B)$, and $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$, then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Lemma 4.4 ([6, Remark 4.2]). *Let (A, B) be a nonempty pair of subsets of a metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is a noncyclic contraction. If, for an arbitrary element $(x_0, y_0) \in A \times B$, we define $x_n = T^n x_0$ and $y_n = T^n y_0$ for every $n \in \mathbb{N}$, then $\lim_n d(x_n, y_n) = \operatorname{dist}(A, B)$. Moreover, $\{x_n\}$ and $\{y_n\}$ are bounded.*

Next we state the main result of this section.

Theorem 4.5. *Let (A, B) be a nonempty, closed, and convex pair of subsets of a uniformly convex Banach space X such that B is bounded. If we let $T : A \cup B \to A \cup B$ be a noncyclic contraction mapping, then T has a best proximity pair. Moreover, for $x_0 \in A$, if we define $x_n = T^n x_0$, then $\{x_n\}$ converges to a fixed point of T in A.*

Proof. Let $x_0 \in A$ be a fixed element, and define a function $f : B \to [0, \infty)$ by

$$f(y) := \limsup_{n\to\infty} \|T^n x_0 - y\|^2, \quad \forall y \in B.$$

In view of the fact that X is uniformly convex and that B is bounded, closed, and convex, f attains its minimum in exactly one point in B, namely $v \in B$. For all $m, n, l \in \mathbb{N} \cup \{0\}$, by Proposition 4.1 we have

$$\begin{aligned}
&\Big\|T^{(l+m+n)}x_0 - \frac{1}{2}(T^l v + T^m v)\Big\|^2 \\
&= \Big\|\frac{T^{(l+m+n)}x_0 - T^l v}{2} + \frac{T^{(l+m+n)}x_0 - T^m v}{2}\Big\|^2 \\
&\le \frac{1}{2}\|T^{(l+m+n)}x_0 - T^l v\|^2 + \frac{1}{2}\|T^{(l+m+n)}x_0 - T^m v\|^2 - \frac{1}{4}\varphi\big(\|T^m z - T^l v\|\big) \\
&\le \frac{1}{2}\|T^{(m+n)}x_0 - v\|^2 + \frac{1}{2}\|T^{(l+n)}x_0 - v\|^2 - \frac{1}{4}\varphi\big(\|T^m v - T^l v\|\big).
\end{aligned}$$

Taking $\limsup$ with respect to n and $l = 1$, $m = 0$, we obtain

$$f(v) \le f\Big(\frac{Tv + v}{2}\Big) \le f(v) - \frac{1}{4}\varphi\big(\|v - Tv\|\big),$$

which implies that $v = Tv$. Due to the fact that T is a noncyclic contraction,

$$\|x_n - v\| = \|T^n x_0 - T^n v\| \le \alpha^n \|x_0 - v\| + (1 - \alpha^n)\operatorname{dist}(A, B), \quad \forall n \in \mathbb{N};$$

hence $\lim_{n\to\infty} \|x_n - v\| = \operatorname{dist}(A, B)$. Besides, by Lemma 4.4 the sequence $\{T^n x_0\}$ is bounded. If we assume that $\{x_n\}$ does not converge, then there exists a subsequence $\{x_{n_k}\}$ such that $\inf_{i\ne j} \|x_{n_i} - x_{n_j}\| > 0$. Passing to a next subsequence, we can assume that $\{x_{n_k}\}$ converges weakly to some $u_0 \in A$. Since X is uniformly convex, X has the Kadec–Klee property; thus

$$\|u_0 - v\| < \liminf_{k\to\infty} \|x_{n_k} - v\| = \operatorname{dist}(A, B),$$

which is a contradiction. Hence, $\{x_n\}$ converges in norm to some $u \in A$. Note that

$$\|u - v\| = \lim_{n\to\infty} \|x_n - v\| = \operatorname{dist}(A, B).$$

Moreover, $\|Tu - Tv\| \le \|u - v\| = \operatorname{dist}(A, B)$. Now, if $u \ne Tu$, then by the strict convexity of X,

$$\operatorname{dist}(A, B) \le \Big\|Tu - \frac{v + Tv}{2}\Big\| < \frac{1}{2}\big(\|Tu - v\| + \|Tu - Tv\|\big) = \operatorname{dist}(A, B),$$

which is impossible, and the result follows. ∎

Remark 4.6. It is worth noting that the existence of a best proximity pair for the noncyclic contraction mapping T in Theorem 4.5 cannot be concluded from Corollary 3.8 because of the fact that the considered pair (A, B) in Theorem 4.5 may not necessarily be bounded. Another observation about Theorem 4.5 is that the convergence of the iterate sequence $x_{n+1} = T^n x_0$ to the fixed point of T is concluded without the continuity of the mapping T.

Proposition 4.7. *Under the assumptions of Theorem 4.5 the proximal pair (A_0, B_0) is a nonempty, bounded, closed, and convex pair, and T is noncyclic on $A_0 \cup B_0$.*

Proof. From Theorem 4.5 the pair (A_0, B_0) is nonempty, and it is easy to verify that (A_0, B_0) is also closed and convex and that T is noncyclic on $A_0 \cup B_0$. We prove that A_0 is bounded. If we suppose the contrary, then there exists a sequence $\{x_n\}$ in A_0 such that $\|x_n\| \geq n$ for all $n \in \mathbb{N}$. Since X is strictly convex, for all $n \in \mathbb{N}$, there exists a unique element $y_n \in B_0$ such that $\|x_n - y_n\| = \operatorname{dist}(A, B)$. We now have

$$n \leq \|x_n\| \leq \|y_n\| + \operatorname{dist}(A, B), \quad \forall n \in \mathbb{N},$$

which is a contradiction by the fact that B_0 is bounded. □

The next convergence result is a straightforward consequence of Theorem 4.5 and Proposition 4.7.

Corollary 4.8. *Under the assumptions of Theorem 4.5, if $x_{n+1} = T^n x_0$, and $y_{n+1} = T^n y_0$ for an arbitrary element $(x_0, y_0) \in A_0 \times B_0$, then the iterate sequence $\{(x_n, y_n)\}$ in $A_0 \times B_0$ converges to a best proximity pair for the mapping T.*

Let us illustrate Theorem 4.5 with the following examples.

Example 4.9. Consider the Hilbert space $X = l_2$ with the canonical basis $\{e_n\}$. If we let

$$A = \{te_1 : t \geq 0\}, \qquad B = \{se_2 + e_3 : 0 \leq s \leq 1\},$$

then (A, B) is a closed, convex, and unbounded pair in X, and it is clear that $\operatorname{dist}(A, B) = 1$. Define the mapping $T : A \cup B \to A \cup B$ with

$$T(te_1) = \begin{cases} \frac{t}{2} e_1 & \text{if } t \in \mathbb{Q} \cap [0, \infty), \\ \frac{t}{4} e_1 & \text{if } t \in \mathbb{Q}^c \cap [0, \infty), \end{cases} \qquad T(se_2 + e_3) = \frac{s}{2} e_2 + e_3.$$

Obviously, T is noncyclic on $A \cup B$, and, for $\mathbf{x} = te_1$, $\mathbf{y} = se_2 + e_3$, and $\alpha = \frac{1}{2}$ if $t \in \mathbb{Q}^c \cap [0, \infty)$, then

$$\begin{aligned} \|T\mathbf{x} - T\mathbf{y}\|_2 &= \big\|T(te_1) - T(se_1 + e_2)\big\|_2 = \left\|\frac{t}{4} e_1 - \frac{s}{2} e_2 - e_3\right\|_2 \\ &= \sqrt{\frac{t^2}{16} + \frac{s^2}{4} + 1} \leq \frac{1}{2}\sqrt{t^2 + s^2 + 1} + \frac{1}{2} \\ &= \alpha\|\mathbf{x} - \mathbf{y}\|_2 + (1 - \alpha) \operatorname{dist}(A, B). \end{aligned}$$

Moreover, if $t \in \mathbb{Q} \cap [0, \infty)$, then

$$\begin{aligned}\|T\mathbf{x} - T\mathbf{y}\|_2 = \big\|T(te_1) - T(se_1 + e_2)\big\|_2 &= \Big\|\frac{t}{2}e_1 - \frac{s}{2}e_2 - e_3\Big\|_2 \\ &= \sqrt{\frac{t^2}{4} + \frac{s^2}{4} + 1} \le \frac{1}{2}\sqrt{t^2 + s^2 + 1} + \frac{1}{2} \\ &= \alpha\|\mathbf{x} - \mathbf{y}\|_2 + (1 - \alpha)\operatorname{dist}(A, B).\end{aligned}$$

It follows from Theorem 4.5 that T has a best proximity pair which is the point $(x^*, y^*) = (0, e_3)$. Moreover, if $\mathbf{x} = te_1$ with $t \ge 0$, then the iterate sequence $T^n(\mathbf{x})$ converges to the fixed point of $T|_A$. It is interesting to note that $T|_A$ is not continuous.

Next we give an example to show that the existence of best proximity pairs for noncyclic contractions in Theorem 4.5 cannot be established when the Banach space X is not strictly convex.

Example 4.10. Consider the Banach space $X = l_\infty$ with the supremum norm. We know that X is not a strictly convex Banach space. If we let

$$A = \{te_1 : 0 \le t \le 1\}, \qquad B = \{se_2 : 1 \le s \le 2\},$$

then (A, B) is a compact and convex pair in X, and $\operatorname{dist}(A, B) = 1$. Clearly, $A_0 = A$, and $B_0 = \{e_2\}$. If we assume that $T : A \cup B \to A \cup B$ defined as

$$T(te_1) = \begin{cases} \frac{\sqrt{2}}{2}e_1 & \text{if } t \in \mathbb{Q} \cap [0, 1], \\ 0 & \text{if } t \in \mathbb{Q}^c \cap [0, 1], \end{cases} \qquad T(se_2) = \frac{s+1}{2}e_2,$$

where $t \in [0, 1]$, and $s \in [1, 2]$, then, for any $(\mathbf{x}, \mathbf{y}) = (te_1, se_2) \in A \times B$ and for $\alpha = \frac{1}{2}$, we have

$$\|T\mathbf{x} - T\mathbf{y}\|_\infty = \frac{s+1}{2} = \frac{1}{2}s + \frac{1}{2} = \alpha\|\mathbf{x} - \mathbf{y}\|_\infty + (1 - \alpha)\operatorname{dist}(A, B);$$

that is, T is a noncyclic contraction. Notice that T does not have any best proximity pair because the fixed-point set of $T|_A$ is empty.

Acknowledgments. The author would like to thank the reviewers for their valuable suggestions. The author's work was partially supported by IPM grant 96470046.

References

1. M. S. Brodskii and D. P. Milman, *On the center of a convex set*, Dokl. Akad. Nauk **59** (1948), 837–840. Zbl 0030.39603. MR0024073. 223
2. A. A. Eldred, W. A. Kirk, and P. Veeramani, *Proximal normal structure and relatively nonexpansive mappings*, Studia Math. **171** (2005), no. 3, 283–293. Zbl 1078.47013. MR2188054. DOI 10.4064/sm171-3-5. 221, 222, 223
3. A. A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323** (2006), no. 2, 1001–1006. Zbl 1105.54021. MR2260159. DOI 10.1016/j.jmaa.2005.10.081. 228
4. R. Espínola, *A new approach to relatively nonexpansive mappings*, Proc. Amer. Math. Soc. **136** (2008), no. 6, 1987–1996. Zbl 1141.47035. MR2383505. DOI 10.1090/S0002-9939-08-09323-4. 221

5. R. Espínola and M. Gabeleh, *On the structure of minimal sets of relatively nonexpansive mappings*, Numer. Funct. Anal. Optim. **34** (2013), no. 8, 845–860. Zbl 1285.47062. MR3175599. DOI 10.1080/01630563.2013.763824. 222, 226
6. M. Gabeleh, *Semi-normal structure and best proximity pair results in convex metric spaces*, Banach J. Math. Anal. **8** (2014), no. 2, 214–228. Zbl 1286.54041. MR3189552. DOI 10.15352/bjma/1396640065. 228
7. M. Gabeleh and O. O. Otafudu, *Generalized pointwise noncyclic relatively nonexpansive mappings in strictly convex Banach spaces*, J. Nonlinear Convex Anal. **17** (2016), no. 6, 1117–1128. Zbl 06629834. MR3540231. 223
8. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), no. 9, 1004–1006. Zbl 0141.32402. MR0189009. DOI 10.2307/2313345. 220
9. W. A. Kirk and W. D. Royalty, *Fixed point theorems for certain nonlinear nonexpansive mappings*, Illinois J. Math. **15** (1971), no. 4, 533–693. Zbl 0221.47039. MR0291919. 220, 221, 225
10. W. A. Kirk, P. S. Srinivasan, and P. Veeramani, *Fixed points for mappings satisfying cyclic contractive conditions*, Fixed Point Theory **4** (2003), no. 1, 79–86. Zbl 1052.54032. MR2031823. 220
11. H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), no. 12, 1127–1138. Zbl 0757.46033. MR1111623. DOI 10.1016/0362-546X(91)90200-K. 228
12. V. Zizler, *On some rotundity and smoothness properties of Banach spaces*, Dissertationes Math. **87** (1971), 1–33. Zbl 0231.46036. MR0300060.

Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran.

E-mail address: gab.moo@gmail.com; Gabeleh@abru.ac.ir