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LIM'S CENTER AND FIXED-POINT THEOREMS FOR ISOMETRY MAPPINGS

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Abstract. In this article, we prove that if K is a nonempty weakly compact convex set in a Banach space such that K has the hereditary *fixed-point property* (FPP) and $\mathfrak{F}$ is a commuting family of isometry mappings on K, then there exists a point in $C(K)$ which is fixed by every member in $\mathfrak{F}$ whenever $C(K)$ is a compact set. Also, we give an example to show that $C(K)$, the Chebyshev center of K, need not be invariant under isometry maps. This example answers the question as to whether the Chebyshev center is invariant under isometry maps. Furthermore, we give a simple example to illustrate that Lim's center, as introduced by Lim, is different from the Chebyshev center.

1. Introduction and preliminaries

Let K be a nonempty bounded subset of a Banach space X. For $x \in X$, define $r(x,K) = \sup\{\|x-y\| : y \in K\}$, $r(K) = \inf\{r(x,K) : x \in K\}$, $\delta(K) = \sup\{r(x,K) : x \in K\}$, and $C(K) = \{x \in K : r(x,K) = r(K)\}$.

Definition 1.1 ([1, p. 837], [4, p. 38]). A nonempty bounded convex set K in a Banach space X is said to have *normal structure* if every nonempty convex set $C \subseteq K$ with more than one point has a point $x \in C$ such that $r(x,C) < \delta(C)$. Then the set $C(K)$ and the number $r(K)$ are called, respectively, the *Chebyshev center* of K and the *Chebyshev radius* of K.

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A mapping $T : K \to X$ is said to be *nonexpansive* (an isometry) if

$$\|Tx - Ty\| \le \|x - y\| \qquad \big(\|Tx - Ty\| = \|x - y\|\big) \quad \text{for } x, y \in K.$$

Brodskii and Milman [1] introduced the notion of normal structure and proved the following interesting result.

Theorem 1.2 ([1, p. 839]). *Let K be a nonempty weakly compact convex set in a Banach space X, and let $\mathfrak{F} = \{T : K \to K : T$ is a surjective isometry mapping$\}$. Furthermore, assume that K has normal structure. Then there exists an $x \in C(K)$ such that $Tx = x$, for every $T \in \mathfrak{F}$.*

By observing the results in [1], Lim [6] constructed a point, namely, the center of a convex set, which is defined as follows.

Definition 1.3 ([6, p. 345]). Let C_0 be a nonempty weakly compact convex subset of a Banach space. Define C_α for all ordinals α by transfinite induction as follows. Let $n \in \mathbb{N}$ be a finite ordinal number. Define $K_n = \{z \in C_{n-1} : z = \frac{x+y}{2}$ for some $x, y \in C_{n-1}$ with $\|x - y\| = \frac{\delta(C_{n-1})}{2}\}$ and $C_n = \overline{\text{co}}\{K_n\}$. Let ω be the first infinite ordinal number. Then define $C_\omega = \bigcap_{(n\in\mathbb{N};n<\omega)} C_{n-1}$. Let β be an infinite ordinal number.

If β is a limit ordinal (i.e., β does not have a predecessor), we set $C_\beta = \bigcap_{\alpha<\beta} C_\alpha$. Otherwise, let γ be the predecessor of β, and let $K_\beta = \{z \in C_\gamma : z = \frac{x+y}{2}$ for some $x, y \in C_\gamma$ with $\|x - y\| = \frac{\delta(C_\gamma)}{2}\}$. Then we set $C_\beta = \overline{\text{co}}(K_\beta)$.

Then it is known from [6] that the intersection of C_α over all ordinal numbers α (i.e., $\bigcap_{\alpha \text{ is ordinal}} C_\alpha$) contains exactly one point. This unique point is called the *center* of C_0.

Note: We call this center the *Lim's center* of the given convex set C_0.

Lim also established the next result.

Theorem 1.4 ([6, p. 345]). *Let K be a nonempty weakly compact convex set in a Banach space X. Then the Lim's center of K is a fixed point for every affine isometry mapping from K into K.*

Lim [5] introduced a notion of the asymptotic center of a decreasing net of bounded subsets of a Banach space. The notion of an asymptotic center is defined as follows.

Definition 1.5 ([5, p. 421]). Let A be a nonempty subset of a Banach space X. Let $\{B_n : n \in \mathbb{N}\}$ be a decreasing sequence of bounded subsets of X. For each $x \in X$ and each $n \in \mathbb{N}$, define

$$r_n(x) = \sup\big\{\|x - y\| : y \in B_n\big\} \qquad \text{and} \qquad r(x) = \lim_n r_n(x) = \inf_n r_n(x).$$

Then the nonnegative real number $\text{ar}(\{B_n\}, A) := \inf\{r(x) : x \in A\} = r$ and the set $\text{AC}(\{B_n\}, A) := \{x \in A : r(x) = r\}$ are called, respectively, the *asymptotic radius* and *asymptotic center* of $\{B_n\}$ with respect to A.

Remark 1.6. Note that $r_n(x) = r(x, B_n)$ for $x \in X$.

Lim also proved the following.

Lemma 1.7 ([5, p. 426])**.** *Let K be a nonempty weakly compact convex set in a Banach space, and let $T : K \to K$ be a nonexpansive map. Then the asymptotic center of $\{T^n(K) : n = 0, 1, 2, \dots\}$ is invariant under T.*

Motivated by Theorem 1.2 of Brodskii and Milman [1] and the fact that $T(C(K)) = C(K)$ whenever T is a surjective isometry on K, Lim et al. [7] raised the following questions.

Question 1. Let T be an isometry on K which is not surjective. Does one still have $T(C(K)) \subseteq C(K)$?

Question 2. Let K be a nonempty weakly compact convex subset of a Banach space, and assume that K has normal structure. Does there exist a point in $C(K)$ which is fixed by every isometry from K into K?

In the case of uniformly convex Banach spaces, Lim et al. [7] affirmatively answered the above questions. Moreover, Lim et al. [7] established the next result (Theorem 1.8) by using Lemma 1.7 and the notion of the hereditary fixed-point property (FPP). A nonempty weakly compact convex set K in a Banach space is considered to have the *fixed-point property* (FPP) if every nonexpansive map from K into K has a fixed point. The set K is said to have the *hereditary FPP* if every closed convex nonempty subset of K has the FPP.

Theorem 1.8 ([7, p. 5])**.** *Let K be a nonempty weakly compact convex set in a Banach space, and let T be an isometry from K into K. Furthermore, assume that K has the hereditary FPP. Then T has a fixed point in $C(K)$.*

We proved in [8], in the setting of strictly convex Banach spaces, that there exists a common fixed point in $C(K)$ for a commuting family of isometry mappings whenever K is a nonempty weakly compact convex set having normal structure.

Next, in connection with common fixed points of a commuting family of nonexpansive maps, we state the following theorem.

Theorem 1.9 ([2, p. 261])**.** *Let K be a nonempty weakly compact convex set in a Banach space, and let $\mathfrak{F}$ be a finite family of commuting nonexpansive mappings on K. Furthermore, assume that K has the hereditary FPP. Then there exists an $x_0 \in K$ such that $Tx_0 = x_0$, for all $T \in \mathfrak{F}$.*

In this article, we prove that every finite family of isometry mappings has a common fixed point in $C(K)$ (see Theorem 3.2). In the case of an arbitrary family of commuting isometry mappings, we prove the existence of a common fixed point in $C(K)$ (see Theorem 3.4) whenever K is a nonempty weakly compact convex set in a Banach space such that K has the hereditary FPP and $C(K)$ is a compact set. Also, we show that $C(K)$ need not be invariant under isometry maps (see Example 3.8). That is, $T(C(K)) \nsubseteq C(K)$ for some K and an isometry map $T : K \to K$, where K is a nonempty weakly compact convex set in a Banach space X and K has normal structure. This example (Example 3.8) provides a negative answer to the question (Question 1) raised by Lim et al. in [7].

2. Lim's center and the Chebyshev center

In this section, we discuss the problem of whether the Lim's center of a set K, where K is a nonempty weakly compact convex set in a Banach space X, is a Chebyshev center of K. The notion modulus of convexity is defined as follows.

Definition 2.1 ([4, p. 52]). The *modulus of convexity* of a Banach space X is the function $\delta_X : [0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x-y\| \geq \epsilon\Big\}.$$

A Banach space X is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for $\epsilon \in (0,2]$.

The next result claims that, in the case of uniformly convex Banach spaces, the center of a convex set C_0 can be defined using the finite induction method.

Proposition 2.2. *Let C_0 be a nonempty bounded closed convex set in a uniformly convex Banach space X, and let $K_n := \{z \in C_{n-1} : z = \frac{x+y}{2}$ for some $x, y \in C_{n-1}$ with $\|x-y\| = \frac{\delta(C_{n-1})}{2}\}$, for $n \in \mathbb{N}$. Then $\delta(C_n) \leq \alpha_0^2 \delta(C_{n-1})$, where $C_n = \overline{\text{co}}(K_n)$ and $\alpha_0 = (1 - \delta_X(\frac{1}{2})) < 1$.*

Proof. Let $z_1, z_2 \in K_1$. Then for $i = 1, 2$, $z_i = \frac{x_i+y_i}{2}$ for some $x_i, y_i \in C_0$ with $\|x_i - y_i\| = \frac{d_0}{2}$, where $d_0 = \delta(C_0)$. Note that $\|z_1 - x_2\| \leq r(z_1, C_0)$ and $\|z_1 - y_2\| \leq r(z_1, C_0)$. Hence $\|z_1 - z_2\| \leq (1 - \delta_X(\frac{d_0}{2r(z_1,C_0)}))r(z_1, C_0)$, where $\delta_X(\cdot)$ is the modulus of convexity function.

Since for any $u \in C_0$, $\|x_1 - u\| \leq d_0$, $\|y_1 - u\| \leq d_0$, and $\|x_1 - y_1\| = \frac{d_0}{2}$, $\|z_1 - u\| \leq (1 - \delta_X(\frac{d_0}{2d_0}))d_0$. Therefore, $r(z_1, C_0) \leq (1 - \delta_X(\frac{1}{2}))d_0$. Also, as $\frac{d_0}{2d_0} \leq \frac{d_0}{2r(z_1,C_0)}$ and $\delta_X(\cdot)$ is an increasing function, we have $1 - \delta_X(\frac{d_0}{2r(z_1,C_0)}) \leq 1 - \delta_X(\frac{1}{2})$. Hence $\|z_1 - z_2\| \leq \alpha_0^2 d_0$, where $\alpha_0 = 1 - \delta_X(\frac{1}{2})$. Therefore, $\delta(C_1) \leq \alpha_0^2 \delta(C_0)$, as $C_1 = \overline{\text{co}}(K_1)$.

Again, since $C_2 = \overline{\text{co}}(K_2)$, where $K_2 = \{z \in C_1 : z = \frac{x+y}{2}$ for some $x, y \in C_1$ with $\|x - y\| = \frac{\delta(C_1)}{2}\}$, by repeating the above arguments we can prove that $\delta(C_2) \leq \alpha_0^2 \delta(C_1)$. Hence by induction, we can see that $\delta(C_n) \leq \alpha_0^2 \delta(C_{n-1})$, for all $n \in \mathbb{N}$. □

The following example illustrates that the Lim's center of a weakly compact convex set C_0 need not be a Chebyshev center of C_0.

Example 2.3. Consider the Banach space $X = \mathbb{R}^2$ with the norm

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x \in Q_1 \cup Q_3, \\ \|x\|_1 & \text{if } x \in Q_2 \cup Q_4, \end{cases}$$

where Q_i is the ith quadrant, which also contains the boundary in $\mathbb{R}^2$ for $i = 1, 2, 3, 4$, and $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ and $\|x\|_1 = |x_1| + |x_2|$.

Let C_0 be the convex hull of $\{(-1,0), (1,0), (0,1)\}$. Note that $\delta(C_0) = 2$ and that for any $(x,y) \in C_0$ with $(x,y) \neq (0,0)$, either $\|(x,y) - (-1,0)\| > 1$ or $\|(1,0) - (x,y)\| > 1$. Hence, $(0,0)$ is the unique Chebyshev center of C_0.

We claim that $(0,0)$ is not the Lim's center of C_0. Note that it is enough to show that $(0,0) \notin C_\alpha$, for some ordinal number α. We claim that $(0,0) \notin C_3$.

Note that for $n \in \mathbb{N}$, $K_n := \{z \in C_{n-1}\colon z = \frac{x+y}{2}$ for some $x, y \in C_{n-1}$ with $\|x-y\| = \frac{\delta(C_{n-1})}{2}\}$, where $C_{n-1} = \overline{\text{co}}\{K_{n-1}\}$ and $K_0 = \{(-1,0),(1,0),(0,1)\}$.

Hence if $z \in K_n$, then there exist x and y in C_{n-1} such that

P1: $\|x-z\| = \frac{\delta(C_{n-1})}{4} = \|y-z\|$;

P2: $[x,y] := \{(1-t)x+ty : t \in [0,1]\} \subseteq C_{n-1}$;

P3: consider any straight line $L(z)$, different from $L[x,y] := \{(1-t)x+ty : t \in \mathbb{R}\}$, passing through z in $\mathbb{R}^2$; then x and y are in different open half-spaces determined by the complement of $L(z)$ in $\mathbb{R}^2$.

Construction of C_1: We claim that $C_1 = \overline{\text{co}}\{(\frac{-1}{2},0),(\frac{1}{2},0),(\frac{1}{4},\frac{3}{4}),(\frac{3}{4},\frac{1}{4}),(\frac{-1}{2},\frac{1}{2})\}$. From the definition of K_1, it is easy to see that $(\frac{\pm 1}{2},0),(\frac{1}{4},\frac{3}{4})$, $(\frac{3}{4},\frac{1}{4})$, and $(\frac{-1}{2},\frac{1}{2})$ belong to K_1, as $\frac{\delta(C_1)}{2} = 1 = \|(0,0)-(\pm 1,0)\| = \|(-1,0)-(0,1)\|$ and $\|(1,0)-(\frac{1}{2},\frac{1}{2})\| = 1 = \|(0,1)-(\frac{1}{2},\frac{1}{2})\|$.

Let $S_1 = \{(1-\lambda)(\frac{1}{2},0)+\lambda(\frac{3}{4},\frac{1}{4}) : \lambda \in \mathbb{R}\}$, $S_2 = \{(1-\lambda)(\frac{-1}{2},0)+\lambda(\frac{-1}{2},\frac{1}{2}) : \lambda \in \mathbb{R}\}$ and $S_3 = \{(1-\lambda)(\frac{-1}{4},\frac{3}{4})+\lambda(\frac{1}{4},\frac{3}{4}) : \lambda \in \mathbb{R}\}$. Note that if S is a straight line in $\mathbb{R}^2$, then S^c (the complement of S in $\mathbb{R}^2$) contains two disjoint open half-spaces in $\mathbb{R}^2$. Since S_1, S_2, and S_3 are straight lines in $\mathbb{R}^2$, they also determine open half-spaces in $\mathbb{R}^2$.

Suppose that H_1 is the open half-space, which contains $(1,0)$, determined by S_1; that H_2 is the open half-space, which contains $(-1,0)$, determined by S_2; and that H_3 is the open half-space, which contains $(0,1)$, determined by S_3.

Now, note that $C_0 \cap H_i \neq \emptyset$ and $\delta(C_0 \cap H_i) \leq \frac{1}{2} = \frac{\delta(C_0)}{4}$ for $i = 1,2,3$.

Let $x = (x_1,x_2), y = (y_1,y_2) \in C_0 \cap H_1$. Then $\frac{1}{2} < x_1, y_1 \leq 1$, $0 \leq x_2, y_2 < \frac{1}{4}$.

Note that either $\|x-y\| = \|x-y\|_1$ or $\|x-y\| = \|x-y\|_\infty$.

Suppose that $\|x-y\| = \|x-y\|_\infty$. Then it is easy to see that $\|x-y\| < \frac{1}{2}$, as $\frac{1}{2} < x_1, y_1 \leq 1$, $0 \leq x_2, y_2 < \frac{1}{4}$.

Now, assume that $\|x-y\| = \|x-y\|_1$. Then $\|x-y\| = |x_1-y_1| + |x_2-y_2|$.

Suppose that either $\frac{1}{2} < x_1, y_1 \leq \frac{3}{4}$ or $\frac{3}{4} < x_1, y_1 \leq 1$. Then it is apparent that $\|x-y\|_1 < \frac{1}{2}$. Assume that $\frac{1}{2} < x_1 \leq \frac{3}{4}$ and $\frac{3}{4} < y_1 \leq 1$. In this case, $\|x-y\|_1 = y_1 - x_1 + |x_2-y_2|$.

Now note that if $\frac{1}{2} < x_1 \leq \frac{3}{4}$, then $x_2 < x_1 - \frac{1}{2}$. Similarly, it can be seen that if $\frac{3}{4} < y_1 \leq 1$, then $y_2 \leq 1 - y_1$. This implies that

$$\|x-y\|_1 = \begin{cases} y_1 - x_1 + x_2 - y_2 < y_1 - \frac{1}{2} - y_2 < \frac{1}{2} & \text{if } x_2 \geq y_2, \\ y_1 - x_1 + y_2 - x_2 < 1 - (x_1+x_2) < \frac{1}{2} & \text{if } y_2 \geq x_2. \end{cases}$$

Therefore, if $z \in C_0 \cap H_1$, then there is no $x \in C_0 \cap H_1$ such that $\|x-z\| = \frac{\delta(C_0)}{4}$. Hence, by the properties P1, P2, and P3 of K_1, we have $z \notin K_1$ for any $z \in C_0 \cap H_1$ and consequently $K_1 \subseteq C_0 \cap H_1^c$, where $H_1^c = \{(x,y) \in \mathbb{R}^2 : (x,y) \notin H_1\}$. In a similar manner, it can been seen that $K_1 \subseteq C_0 \cap H_i^c$ for $i = 2,3$, since $\delta(C_0 \cap H_2) \leq \frac{\delta(C_0)}{4}$ and $\delta(C_0 \cap H_3) \leq \frac{\delta(C_0)}{4}$.

Therefore, $K_1 \subseteq \bigcap_{i=1}^3 (C_0 \cap H_i^c)$. Further, as each $C_0 \cap H_i^c$ is a closed convex set, $C_1 = \overline{\text{co}}(K_1) \subseteq \bigcap_{i=1}^3 (C_0 \cap H_i^c)$. Hence, $\delta(C_1) \leq \delta(\bigcap_{i=1}^3 (C_0 \cap H_i^c))$. Also, since $\delta(\bigcap_{i=1}^3 (C_0 \cap H_i^c)) = \frac{3}{2}$ and the points $(\frac{-1}{2},\frac{1}{2})$ and $(\frac{3}{4},\frac{1}{4})$ belong to K_1, we have the diameter $\delta(C_1) \leq \frac{3}{2}$ and $\delta(C_1) \geq \|(\frac{-1}{2},\frac{1}{2})-(\frac{3}{4},\frac{1}{4})\| = \|(\frac{-5}{4},\frac{1}{4})\| = \frac{3}{2}$.

Construction of C_2: We claim that $(x,0) \notin C_2 = \overline{\text{co}}(K_2)$ for all $x \in (\frac{1}{8}, \frac{1}{2}]$. Note that the points $(\frac{1}{8}, 0), (\frac{-1}{8}, 0), (\frac{-3}{8}, \frac{1}{4}), (\frac{-1}{8}, \frac{5}{8}), (\frac{7}{16}, \frac{9}{16}), (\frac{9}{16}, \frac{7}{16})$, and $(\frac{3}{8}, \frac{1}{8})$ belong to K_2. Consider $C_1 = \overline{\text{co}}\{(\frac{-1}{2}, 0), (\frac{1}{2}, 0), (\frac{1}{4}, \frac{3}{4}), (\frac{3}{4}, \frac{1}{4}), (\frac{-1}{2}, \frac{1}{2})\}$. Then $\delta(C_1) = \frac{3}{2}$.

Now, since $\|x-y\| = \|(\frac{-1}{4}, 0) - (\frac{1}{2}, 0)\| = \frac{3}{4} = \frac{\delta(C_1)}{2}$, we have $\frac{x+y}{2} = (\frac{1}{8}, 0) \in K_2$. In a similar way, it can be seen that $(\frac{-1}{8}, 0)$ belongs to K_2. Also note that since $\|x-y\| = \|(\frac{-1}{2}, \frac{1}{2}) - (\frac{1}{4}, \frac{3}{4})\| = \frac{3}{4} = \frac{\delta(C_1)}{2} = \|(0,0) - (\frac{3}{4}, \frac{1}{4})\|$, we have $\frac{x+y}{2} = (\frac{-1}{8}, \frac{5}{8})$ and $\frac{x+y}{2} = (\frac{3}{8}, \frac{1}{8}) \in K_2$.

Similarly, as $\|(\frac{-1}{4}, 0) - (\frac{-1}{2}, \frac{1}{2})\| = \|(\frac{1}{4}, \frac{-1}{2})\|_1 = \frac{3}{4}$, we have $(\frac{-3}{8}, \frac{1}{4}) \in K_2$.

Moreover, as the points $(\frac{3}{4}, \frac{1}{4})$ and $(\frac{1}{4}, \frac{3}{4})$ are in C_1 and $\|(\frac{3}{4}, \frac{1}{4}) - (\frac{1}{4}, \frac{3}{4})\| = 1 > \frac{\delta(C_1)}{2}$, it can be seen that $(\frac{7}{16}, \frac{9}{16})$ and $(\frac{9}{16}, \frac{7}{16})$ belong to K_2. This implies that $\delta(K_2) \geq \|(\frac{9}{16}, \frac{7}{16}) - (\frac{-1}{8}, 0)\| = \frac{11}{16}$.

Now we claim that $(x,0) \notin C_2 = \overline{\text{co}}(K_2)$, for every $x \in (\frac{1}{8}, \frac{1}{2}]$. Fix $x \in (\frac{1}{8}, \frac{1}{2}]$, and let $\alpha = \frac{x+\frac{1}{8}}{2}$ and $S_\alpha = \{(1-\lambda)(\alpha, 0) + \lambda(\frac{3}{8} + \alpha, \alpha - \frac{1}{8}) : \lambda \in \mathbb{R}\}$. Note that $(\frac{3}{8} + \alpha, \alpha - \frac{1}{8})$ is in $F = \{(1-t)(\frac{1}{2}, 0) + t(\frac{3}{4}, \frac{1}{4}) : t \in [0,1]\} \subseteq C_1$.

Let H_α be the open half-space in $\mathbb{R}^2$, which contains $(\frac{1}{2}, 0)$, determined by S_α. Now, consider the set $C_1 \cap H_\alpha$. Note that $(x,0) \in C_1 \cap H_\alpha$. Also, it is apparent that $\overline{C_1 \cap H_\alpha} = \overline{\text{co}}\{(\alpha, 0), (\frac{1}{2}, 0), (\frac{3}{8} + \alpha, \alpha - \frac{1}{8})\}$. This implies that the diameter $\delta(C_1 \cap H_\alpha) = \|(\frac{3}{8} + \alpha, \alpha - \frac{1}{8}) - (\alpha, 0)\| = \frac{3}{8} = \frac{\delta(C_1)}{4}$.

Now, it follows from the properties P1, P2, and P3 of K_2 that $K_2 \subseteq C_1 \cap H_\alpha^c$ for all $\alpha = \frac{x+\frac{1}{8}}{2}$, where $x \in (\frac{1}{8}, \frac{1}{2}]$ and $H_\alpha^c = \{(x,y) \in \mathbb{R}^2 : (x,y) \notin H_\alpha\}$. Hence for $x \in (\frac{1}{8}, \frac{1}{2}]$, $(x,0) \notin C_2 = \overline{\text{co}}(K_2) \subseteq C_1 \cap H_\alpha^c$, as $C_1 \cap H_\alpha^c$ is a closed convex set and $(x,0) \in C_1 \cap H_\alpha$.

Therefore, for every $x \in (\frac{1}{8}, \frac{1}{2}]$ there exists a unique $y_x \in (0,1)$ such that $(x, y_x) \in C_2$ and $(x,y) \notin C_2$ for all $y \in [0, y_x)$, since $(\frac{1}{8}, 0)$ and $(\frac{9}{16}, \frac{7}{16})$ are in the convex set C_2. Furthermore, note that for every $x \in (\frac{1}{8}, \frac{3}{8}]$ we have $y_x \leq \frac{1}{8}$, since the line segment joining $(\frac{1}{8}, 0)$ and $(\frac{3}{8}, \frac{1}{8})$ is contained in C_2.

Construction of C_3: We claim that $(0,0) \notin C_3$. Since $\frac{9}{64} \in (\frac{1}{8}, \frac{3}{8})$, there exists $y_0 \in (0, \frac{1}{8})$ such that $(\frac{9}{64}, y_0) \in C_2$ and $(\frac{9}{16}, y) \notin C_2$ for $y \in [0, y_0)$.

Now, consider the straight line $S_0 = \{(1-\lambda)(\frac{-1}{64}, 0) + \lambda(\frac{9}{64}, y_0) : \lambda \in \mathbb{R}\}$. Let H_0 be the open half-space in $\mathbb{R}^2$, which contains $(\frac{1}{8}, 0) \in C_2$, determined by S_0. Note that $(0,0)$ and $(\frac{1}{8}, 0) \in C_2 \cap H_0$ and $\overline{C_2 \cap H_0} = \overline{\text{co}}\{(\frac{-1}{64}, 0), (\frac{1}{8}, 0), (\frac{9}{64}, y_0)\}$. Then it is easy to see that $\delta(C_2 \cap H_0) = \|(\frac{9}{64}, y_0) - (\frac{-1}{64}, 0)\| = \frac{10}{64} < \frac{11}{64} \leq \frac{\delta(C_2)}{4}$.

Since $\delta(C_2 \cap H_0) \leq \frac{\delta(C_2)}{4}$, we have from the properties P1, P2, and P3 of K_3 that $K_3 \subseteq C_2 \cap H_0^c$. Consequently, $C_3 \subseteq C_2 = \overline{\text{co}}(K_3) \cap H_0^c$, as $C_2 \cap H_0^c$ is a closed convex set in $\mathbb{R}^2$.

Note that $(0,0) \notin C_3$ as $(0,0) \in C_2 \cap H_0$. This implies that $(0,0) \notin \bigcap_{\alpha \text{ is ordinal}} C_\alpha$. Therefore, $(0,0)$ is not the Lim's center of C_0.

Remark 2.4. Example 2.3 shows that the Chebyshev center of a weakly compact convex set C_0 in a Banach space need not contain the Lim's center of C_0 even if $r(C_0) = \frac{\delta(C_0)}{2}$.

However, for the following class of sets, the Lim's center is a Chebyshev center.

Definition 2.5 ([3, p. 904]). A nonempty subset K of a normed linear space X is said to be a *centrally symmetric set* if there exists an $a_0 \in X$ such that $K = 2a_0 - K$.

Proposition 2.6. *Let C_0 be a weakly compact convex set in a Banach space X. Assume that $C_0 = 2a - C_0$, for some $a \in X$. Then the Lim's center of C_0 is a, which is also a Chebyshev center of C_0.*

Proof. Note that for every $x \in C_0$, we have $2a - x \in C_0$. Therefore, $\delta(C_0) \geq r(x, C_0) \geq \|x-(2a-x)\| = 2\|a-x\|$ for all $x \in C_0$. Hence $r(a, C_0) = \sup\{\|a-x\| : x \in C_0\} \leq \frac{d}{2}$, where $d = \delta(C_0)$. It is also easy to see that $r(y, C_0) \geq \frac{d}{2}$, for any $y \in C_0$. Thus $r(a, C_0) = \frac{d}{2}$. Consequently, a is a Chebyshev center of C_0.

We claim that C_α is centrally symmetric about a, for every ordinal α. Let $K_1 := \{z \in C_0 : z = \frac{x+y}{2}$ for some $x, y \in C_0$ with $\|x - y\| = \frac{d}{2}\}$. Note that $K_1 = 2a - K_1$. For, if $z \in K_1$, then $z = \frac{x+y}{2}$ for some $x, y \in C_0$ with $\|x - y\| = \frac{d}{2}$. Hence, $2a - z = \frac{2a-x+2a-y}{2}$ and $\|2a - x - (2a - y)\| = \|x - y\|$. Consequently, $2a - z \in K_1$, $a \in C_1 := \overline{\mathrm{co}}(K_1)$, and $C_1 = 2a - C_1$. In a similar manner it can be shown that $a \in C_\alpha$ for every ordinal number α which is not a limit ordinal. Suppose that β_0 is the first limit ordinal number. Then, as $C_{\beta_0} = \bigcap_{\alpha<\beta_0} C_\alpha$ and $C_\alpha = 2a - C_\alpha$, it is easy to see that $C_{\beta_0} = 2a - C_{\beta_0}$. Hence, C_{β_0} is centrally symmetric about a, and a is a Chebyshev center of C_{β_0}.

Therefore $a \in C_\alpha$, for every ordinal number α. Hence, a is the Lim's center of C_0 as $\bigcap C_\alpha$ is a singleton, where the intersection is taken over all the ordinal numbers. □

3. Fixed-point theorems for commuting families

The following observation leads to the existence of common fixed points for a commuting family of isometry mappings.

Lemma 3.1. *Let K be a nonempty weakly compact convex set in a Banach space X. Suppose that for $i = 1, 2, \ldots, m$, $T_i : K \to K$ is a nonexpansive map such that $T_i \circ T_j(x) = T_j \circ T_i(x)$, for all $x \in K$ and $i, j \in \{1, 2, \ldots, m\}$. Let F_0 be the asymptotic center of the sequence $\{(T_1 \circ T_2 \circ \cdots \circ T_m)^n(K)\}$ with respect to K. Then $T_i(F_0) \subseteq F_0$, for $i = 1, 2, \ldots, m$.*

Proof. The proof we give here is for the case $m = 3$, which can be carried over for any integer m.

Note that for all $n \in \mathbb{N}$,

$$T_1^n \circ T_2^{n+1} \circ T_3^{n+1}(K) \subseteq (T_1 \circ T_2 \circ T_3)^n(K) \qquad \text{and}$$
$$(T_1 \circ T_2 \circ T_3)^{n+1}(K) \subseteq T_1^n \circ T_2^{n+1} \circ T_3^{n+1}(K).$$

Now, we claim that $T_1(F_0) \subseteq F_0$. Suppose that $x \in F_0$. Then

$$\begin{aligned} r_{n+1}\big(T_1(x)\big) &= r\big(T_1(x), (T_1 \circ T_2 \circ T_3)^{n+1}(K)\big) \\ &= r\big(x, T_1^n \circ T_2^{n+1} \circ T_3^{n+1}(K)\big) \leq r_n(x). \end{aligned}$$

Therefore, $r(T_1(x)) \leq r(x)$. As $x \in F_0$, $r(x) = r \leq r(T_1(x))$. Hence, $T_1(F_0) \subseteq F_0$. In a similar manner, it can be proved that $T_i(F_0) \subseteq F_0$, for $i = 2, 3$. ☐

Next, we prove that every finite family of commuting isometry maps has a common fixed point in $C(K)$.

Theorem 3.2. *Let K be a nonempty weakly compact convex set in a Banach space X such that K has the hereditary FPP. Let $\mathfrak{F}$ be a finite family of commuting isometry mappings on K. Then there exists $x_0 \in C(K)$ such that $T(x_0) = x_0$, for every $T \in \mathfrak{F}$.*

Proof. Suppose that $\mathfrak{F} = \{T_i : i = 1, 2, \ldots, m\}$. Then from Lemma 3.1, it follows that $F_0 = \mathrm{AC}(\{(T_1 \circ T_2 \circ \cdots \circ T_m)^n(K)\}, K)$ is invariant under each $T_i \in \mathfrak{F}$. Then from Theorem 1.9, it follows that there exists an $x_0 \in F_0$ such that $T_i(x_0) = x_0$, for $i = 1, 2, \ldots, m$.

Now we claim that $x_0 \in C(K)$. Note that for each $n \in \mathbb{N}$,

$$r_n(x_0) = r\big(x_0, (T_1 \circ T_2 \circ \cdots \circ T_m)^n(K)\big) = r(x_0, K).$$

Thus $r(x_0) = \lim_n r_n(x_0) = r(x_0, K)$. Also, since $x_0 \in F_0$, $r(x_0) \leq r(x)$ for all $x \in K$. But $r(x) \leq r_n(x) \leq r(x, K)$, for all $x \in K$. Hence $r(x_0, K) \leq r(x, K)$, for all $x \in K$. Therefore, $x_0 \in C(K)$. ☐

Remark 3.3. The previous theorem (Theorem 3.2) holds for a finite family $\mathcal{F}$ of commuting nonexpansive maps in which every member T satisfies, for every common fixed point x_0, $\|Tx_0 - Ty\| = \|x_0 - y\|$, for all $y \in K$.

Also, note that from Theorem 1.9 it follows that the set of all common fixed points of the family $\mathcal{F}$ is nonempty whenever K is a nonempty weakly compact convex set in a Banach space such that K has the hereditary FPP.

Next, we prove a common fixed-point theorem for an arbitrary family in which any two members commute.

Theorem 3.4. *Let K be a nonempty weakly compact convex set in a Banach space X such that K has the hereditary FPP. Let $\mathfrak{F}$ be a commuting family of isometry mappings on K. Furthermore, assume that $C(K)$ is a compact subset of K. Then there exists an $x_0 \in C(K)$ such that $T(x_0) = x_0$, for every $T \in \mathfrak{F}$.*

Proof. Suppose that $F_T = \{x \in C(K) : Tx = x\}$, for $T \in \mathfrak{F}$. Then from Theorem 3.2, it follows that F_T is a nonempty closed set.

Let $\mathcal{S} = \{F_T : T \in \mathfrak{F}\}$. As $C(K)$ is a compact set, it is enough to prove that $\mathcal{S}$ has the finite intersection property. Now from Theorem 3.2, it follows that every finite subset of $\mathcal{S}$ has nonempty intersection. Therefore, $\bigcap_{T\in\mathfrak{F}} F_T \neq \emptyset$. That is, there exists an $x_0 \in C(K)$ such that $Tx_0 = x_0$, for all $T \in \mathfrak{F}$. ☐

Note that if K has normal structure, then K has the hereditary FPP. Hence we have the following result.

Corollary 3.5. *Let K be a nonempty weakly compact convex set having normal structure in a Banach space X such that $C(K)$ is a compact set. Let $\mathfrak{F}$ be a commuting family of isometry mappings on K. Then there exists an $x_0 \in C(K)$ such that $T(x_0) = x_0$, for every $T \in \mathfrak{F}$.*

In the case of Banach spaces with uniformly Kadec–Klee (UKK) norm, it is known from [9] that $C(K)$ is a compact convex set whenever K is a nonempty weakly compact convex set. The notion of Banach spaces with UKK norm is defined as follows.

Definition 3.6 (see [4]). A Banach space X is said to have *uniformly Kadec–Klee* (UKK) norm if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\{x_n\} \subseteq B[0,1], \qquad x_n \text{ converges weakly to } x_0, \qquad \text{and}$$
$$\operatorname{sep}\{x_n\} := \inf\big\{\|x_n - x_m\| : n \neq m\big\} > \epsilon$$

imply that

$$\|x_0\| \leq 1 - \delta.$$

We obtain the following result from Theorem 3.4.

Corollary 3.7. *Let K be a nonempty weakly compact convex set in a Banach space X with UKK norm. Let $\mathfrak{F}$ be a commuting family of isometry mappings on K. Then there exists an $x_0 \in C(K)$ such that $T(x_0) = x_0$, for every $T \in \mathfrak{F}$.*

The following example illustrates that $C(K)$ need not be invariant under isometry maps.

Example 3.8. Consider the Hilbert space $l_2(\mathbb{N}) = \{x : \mathbb{N} \to \mathbb{R} : \sum_{i \in \mathbb{N}} |x(i)|^2 < \infty\}$. Let X_λ be the reflexive Banach space $l_2(\mathbb{N})$ with the norm $\|x\|_\lambda = \max\{\|x\|_\infty, \frac{1}{\lambda}\|x\|_2\}$, for $\lambda \geq 1$. It is known from [4] that X_λ has normal structure whenever $\lambda \in [1, \sqrt{2})$.

Suppose that $\lambda = \frac{\sqrt{5}}{2}$ and that K is the intersection of the closed balls $B[x_0, 1]$ and $B[-x_0, 1]$ in X_λ, where $x_0 = (\frac{1}{2}, 0, 0, \ldots)$. Then it is easy to see that $K = -K$ and $e_n \in K$, for $n \geq 2$. Moreover, $x \in K$ implies that $|x(1)| \leq \frac{1}{2}$ and $|x(n)| \leq 1$, for all $n \geq 2$. Also, for $x, y \in K$ $\|x - y\|_\lambda \leq \|x - x_0\|_\lambda + \|x_0 - y\|_\lambda \leq 2$. But $\|e_n - (-e_n)\|_\lambda = 2\|e_n\|_\lambda = 2$. Hence, $\delta(K) = 2$. Since $K = -K$ and $\frac{\delta(K)}{2} \leq r(x, K)$, for $x \in K$, we have $r(0, K) = \frac{\delta(K)}{2} = 1$. Therefore, $0 \in C(K)$.

Now, we claim that $C(K) = \{(1-t)x_0 + t(-x_0) : t \in [0,1]\}$. It is easy to see that $C(K) \subseteq \{(1-t)x_0 + t(-x_0) : t \in [0,1]\}$. For suppose that $x \in K$ such that $x \notin \{(1-t)x_0 + t(-x_0) : t \in [0,1]\}$. Then $x(n) \neq 0$ for some $n \geq 2$. Thus $r(x, K) \geq \|x - (-\operatorname{sgn}(x(n)))e_n\|_\lambda \geq |x(n) + \operatorname{sgn}(x(n))| > 1 = r(K)$.

Suppose that $x \in \{(1-t)x_0 + t(-x_0) : t \in [0,1]\}$. Then for $y \in K$, $\|y - x\|_\lambda \leq (1-t)\|y - x_0\|_\lambda + t\|y + x_0\|_\lambda \leq 1 = r(K)$. Hence, $r(x, K) \leq r(K)$. This shows that $C(K) = \{(1-t)x_0 + t(-x_0) : t \in [0,1]\}$.

Define $T(e_i) = e_{i+1}$ for all $i \in \mathbb{N}$. Then extend T linearly to the whole of K. Now, it is easy to see that $\|Tx - Ty\|_\lambda = \|x - y\|_\lambda$ and $\|Tx - x_0\|_2 = \|Tx - (-x_0)\|_2$, for $x, y \in K$.

Note that for $x \in K$, either $\|Tx - x_0\|_2 \leq \|x - x_0\|_2$ or $\|Tx - x_0\|_2 \leq \|x + x_0\|_2$ and $\|Tx \pm x_0\|_\infty \leq 1$. Hence T is a self-map on K. As $T(\alpha e_1) = \alpha e_2$ for all $\alpha \in \mathbb{R}$, we have $T(C(K)) \subsetneq C(K)$. This proves that $C(K)$ need not be invariant under isometry maps.

However, we claim that $0 \in K$ is a fixed point for every isometry self-map S on K. It is enough to prove that there exists an $x \in K$ such that

(a) $\|x-(\pm)x_0\|_\lambda = 1$, and
(b) $\|Sx - S(-x)\|_\lambda = \frac{1}{\lambda}\|Sx - S(-x)\|_2$.

For if there exists an $x \in K$ such that

(1) $\|x - \pm x_0\|_\lambda = 1$, and
(2) $\|Sx - S(-x)\|_\lambda = \frac{1}{\lambda}\|Sx - S(-x)\|_2$,

then $S(0) = 0$.

Assume that such an $x \in K$ exists. Then $\frac{1}{\lambda}\|Sx - S(-x)\|_2 = 2$ and

$$\begin{aligned} 2\lambda = \big\|Sx - S(-x)\big\|_2 &\le \|Sx - 0\|_2 + \big\|0 - S(-x)\big\|_2 \\ &\le \lambda\|Sx - 0\|_\lambda + \lambda\big\|0 - S(-x)\big\|_\lambda \\ &\le 2\lambda, \quad \text{as } 0 \in C(K) \text{ and } r(K) = 1. \end{aligned}$$

Hence, $2\lambda = \|Sx - S(-x)\|_2 \le \|Sx - 0\|_2 + \|0 - S(-x)\|_2 = 2\lambda$. Since $0 \in C(K)$ and $r(K) = 1$, we have $\|S(x) - 0\|_2 = \lambda = \|S(-x) - 0\|_2$. Further, since $\|Sx - 0 + 0 - S(-x)\|_2 = \|Sx - 0\|_2 + \|S(-x) - 0\|_2$ and $\|\cdot\|_2$ is strictly convex, we have $Sx - 0 = r(0 - S(-x))$ for some $r \ge 0$. This implies that $S(-x) = -S(x)$ as $\|S(x) - 0\|_2 = \lambda = \|S(-x) - 0\|_2$.

Now, note that for $z = (1-t)x + t(-x)$ with $t \in (0,1)$, we have

$$\begin{aligned} 2\lambda = \big\|Sx - S(-x)\big\|_2 &\le \|Sx - Sz\|_2 + \big\|Sz - S(-x)\big\|_2 \\ &\le \lambda\|Sx - Sz\|_\lambda + \lambda\big\|Sz - S(-x)\big\|_\lambda \\ &= \lambda\|x - z\|_\lambda + \lambda\big\|z - (-x)\big\|_\lambda, \quad \text{as } S \text{ is isometry} \\ &= 2\lambda. \end{aligned}$$

This implies that $\|Sx - S(-x)\|_2 = \|Sx - Sz\|_2 + \|Sz - S(-x)\|_2$. Now, by the strict convexity of $\|\cdot\|_2$, we have $S(z) - S(-x) = r(Sx - Sz)$ for some $r \ge 0$. Since S is an isometry, we have $2(1-t) = \|z - (-x)\|_\lambda = r\|z - x\|_\lambda = 2rt$.

Thus $r = \frac{1-t}{t}$ and consequently $Sz = (1-t)Sx + tS(-x)$. This implies that $S(0) = 0$, as $0 = \frac{1}{2}x + \frac{1}{2}(-x)$ and $S(-x) = -S(x)$.

Now, we claim that there exists an $x \in K$ such that

(a) $\|x - (\pm)x_0\|_\lambda = 1$, and
(b) $\|Sx - S(-x)\|_\lambda = \frac{1}{\lambda}\|Sx - S(-x)\|_2$.

Suppose that $\|Sx - S(-x)\|_\lambda = \|Sx - S(-x)\|_\infty$ for all $x \in K$ satisfying $\|x - (\pm)x_0\|_\lambda = 1$.

Note that the uncountable set $F = \{x = (0, \cos\theta, \sin\theta, 0, 0, \ldots) : \theta \in [0, 2\pi]\}$ is a subset of K and that $\|x - (\pm x_0)\|_\lambda = 1$ for all $x \in K$. Then, by our assumption, we have $\|Sx - S(-x)\|_\lambda = \|Sx - S(-x)\|_\infty$ for all $x \in F$. This implies that, for every $x \in F$, there exists $j_0 \in \mathbb{N}$ such that $|S(x)(j_0) - S(-x)(j_0)| = 2$.

Now, since $S(\pm x) \in K$, it is easy to see that $j_0 \ge 2$, $S(\pm x)(j_0) = \pm 1$ and $S(\pm x)(i) = 0$ for all $i \ne j_0$. Hence, $S(-x) = -S(x)$, as $\|Sx - S(-x)\|_\infty = 2$. This implies that $Sx \in \{\pm e_n : n \ge 2\}$ for all $x \in F$. Therefore, the isometry map S maps the uncountable set F into a countable set $\{\pm e_n : n \ge 2\}$. This contradiction proves that there exists an $x \in K$ such that

(a) $\|x - (\pm)x_0\|_\lambda = 1$, and
(b) $\|Sx - S(-x)\|_\lambda = \frac{1}{\lambda}\|Sx - S(-x)\|_2$.

Consequently, we have $S(0) = 0$.

Therefore, $T(0) = 0$ for all isometry self-maps T on K.

Theorem 3.9. *Let K be a nonempty weakly compact convex set in a Banach space, and let $\mathfrak{F}$ be a finite commuting family of affine isometry maps on K. Then there exists an $x \in C(K)$ such that $Tx = x$, for all $T \in \mathfrak{F}$.*

Proof. Suppose that $\mathfrak{F} = \{T_i : i = 1, 2, \ldots, m\}$. Note that from Lemma 3.1, it follows that F_0, the asymptotic center of $\{(T_1 \circ T_2 \circ \cdots \circ T_m)^n(K)\}$ with respect to K, is invariant under each T_i, for $i = 1, 2, \ldots, m$.

Now by Theorem 1.4, we have that the center of F_0, say, x_0, is a fixed point for every T_i, for $i = 1, 2, \ldots, m$. Hence, $r(x_0) = \lim r_n(x_0) = r(x_0, K)$. Also as $x_0 \in F_0$ and $r(x) \leq r(x, K)$ for all $x \in K$, we have $r(x_0, K) = r(x_0) \leq r(x) \leq r(x, K)$, for all $x \in K$. Therefore, $x_0 \in C(K)$. □

Theorem 3.10. *Let K be a nonempty weakly compact convex set in a Banach space, and let $\mathfrak{F}$ be a commuting family of affine isometry maps on K. Then there exists an $x \in C(K)$ such that $Tx = x$, for all $T \in \mathfrak{F}$.*

Proof. Let $\mathcal{S} = \{F_T : T \in \mathfrak{F}\}$, where $F_T = \{x \in C(K) : Tx = x\}$. Since each $T \in \mathfrak{F}$ is an affine map, F_T is a convex set in $C(K)$. Hence, F_T is a weakly compact convex set in $C(K)$.

Note that from Theorem 3.9, it follows that $\mathcal{S}$ has the finite intersection property. Therefore, $\bigcap_{T \in \mathfrak{F}} F_T \neq \emptyset$. Thus there exists an $x_0 \in C(K)$ such that $T(x_0) = x_0$, for all $T \in \mathfrak{F}$. □

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