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NONCOMMUTATIVE GEOMETRY OF RATIONAL ELLIPTIC CURVES

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Abstract. We study an interplay between operator algebras and the geometry of rational elliptic curves. Namely, let $\mathcal{O}_B$ be the Cuntz–Krieger algebra given by a square matrix $B = (b-1, 1, b-2, 1)$, where b is an integer greater than or equal to 2. We prove that there exists a dense, self-adjoint subalgebra of $\mathcal{O}_B$ which is isomorphic (modulo an ideal) to a twisted homogeneous coordinate ring of the rational elliptic curve $\mathcal{E}(\mathbb{Q}) = \{(x, y, z) \in \mathbb{P}^2(\mathbb{C}) \mid y^2 z = x(x-z)(x - \frac{b-2}{b+2} z)\}$.

1. Introduction

In the 1950s, due to the work of J.-P. Serre and P. Gabriel, it became apparent that algebraic geometry could be recast in terms of noncommutative algebra (we refer the reader to an excellent survey by Stafford and van den Bergh [7]). The following simple example illustrates the idea. If X is a Hausdorff topological space and $C(X)$ is the C^*-algebra of continuous complex-valued functions on X, then by the Gelfand theorem the topology of X is determined by the commutative algebra $C(X)$. This fact can be written as $K_0^{\mathrm{top}}(X) \cong K_0^{\mathrm{alg}}(C(X))$, where K_0^{top} and K_0^{alg} are the topological and algebraic K_0-groups, respectively (see Blackadar [1]). Now consider the algebra $C(X) \otimes M_2(\mathbb{C})$ consisting of 2×2 matrices with entries in $C(X)$. Since the algebraic K-theory is stable under the tensor products, one gets an isomorphism $K_0^{\mathrm{alg}}(C(X)) \cong K_0^{\mathrm{alg}}(C(X) \otimes M_2(\mathbb{C}))$ (see [1, Section 5]). In other words, the topology of the space X is determined by the tensor product

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$C(X) \otimes M_2(\mathbb{C})$ which is no longer a commutative algebra. In the context of algebraic geometry, one replaces the space X by a projective variety V, the algebra $C(X)$ by the coordinate ring of V, the tensor product $C(X) \otimes M_2(\mathbb{C})$ by a twisted coordinate ring of V, and the group $K^{\mathrm{top}}(X)$ by a category of quasicoherent sheaves on V (see [7, p. 173]). Below we give a brief review of this construction when V is an elliptic curve (we refer the reader to Sklyanin [5], Smith and Stafford [6, pp. 265–268], and Stafford and van den Bergh [7, p. 197] for a detailed account).

Let k be a field of $\mathrm{char}(k) \neq 2$. The *Sklyanin algebra* $\mathfrak{S}_{\alpha,\beta,\gamma}(k)$ is a free k-algebra on four generators x_i and six quadratic relations,

$$\begin{cases} x_1x_2 - x_2x_1 = \alpha(x_3x_4 + x_4x_3), \\ x_1x_2 + x_2x_1 = x_3x_4 - x_4x_3, \\ x_1x_3 - x_3x_1 = \beta(x_4x_2 + x_2x_4), \\ x_1x_3 + x_3x_1 = x_4x_2 - x_2x_4, \\ x_1x_4 - x_4x_1 = \gamma(x_2x_3 + x_3x_2), \\ x_1x_4 + x_4x_1 = x_2x_3 - x_3x_2, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma \in k$ and $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$. If $\alpha \notin \{0; \pm 1\}$, then algebra $\mathfrak{S}_{\alpha,\beta,\gamma}(k)$ defines a nonsingular elliptic curve $\mathcal{E} \subset \mathbb{P}^3(k)$ given by an intersection of the quadrics $u^2 + v^2 + w^2 + z^2 = \frac{1-\alpha}{1+\beta}v^2 + \frac{1+\alpha}{1-\gamma}w^2 + z^2 = 0$ together with an automorphism $\sigma : \mathcal{E} \to \mathcal{E}$. We will use the following isomorphism (see [5], [6]):

$$\mathbf{QGr}\big(\mathfrak{S}_{\alpha,\beta,\gamma}(k)/\Omega\big) \cong \mathbf{Qcoh}(\mathcal{E}), \tag{1.2}$$

where $\mathbf{QGr}$ is a category of the quotient graded modules over the algebra $\mathfrak{S}_{\alpha,\beta,\gamma}(k)$ modulo torsion, $\mathbf{Qcoh}$ is a category of the quasicoherent sheaves on $\mathcal{E}$, and $\Omega \subset \mathfrak{S}_{\alpha,\beta,\gamma}(k)$ is a two-sided ideal generated by the central elements $\Omega_1 = -x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $\Omega_2 = x_2^2 + \frac{1+\beta}{1-\gamma}x_3^2 + \frac{1-\beta}{1+\alpha}x_4^2$ (see [6, p. 276]). The quotient of the Sklyanin algebra by the ideal Ω is called a *twisted homogeneous coordinate ring* of the elliptic curve $\mathcal{E}$.

Let A be a 2×2 matrix with nonnegative integer entries a_{ij} such that every row and every column of A is nonzero. The 2-dimensional *Cuntz–Krieger algebra* $\mathcal{O}_A$ is a C^*-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ generated by the partial isometries s_1 and s_2, and relations

$$\begin{cases} s_1^*s_1 = a_{11}s_1s_1^* + a_{12}s_2s_2^*, \\ s_2^*s_2 = a_{21}s_1s_1^* + a_{22}s_2s_2^*, \\ \mathrm{Id} = s_1s_1^* + s_2s_2^*, \end{cases} \tag{1.3}$$

where Id is the identity operator on $\mathcal{H}$. Occasionally, the algebra $\mathcal{O}_A$ will be written as $\mathcal{O}_{a_{11},a_{12},a_{21},a_{22}}$. If one defines $x_1 = s_1, x_2 = s_1^*, x_3 = s_2$, and $x_4 = s_2^*$, then it is easy to see that $\mathcal{O}_A$ contains a dense subalgebra $\mathcal{O}_A^0$, which is a free $\mathbb{C}$-algebra on four generators x_i and three quadratic relations

$$\begin{cases} x_2x_1 = a_{11}x_1x_2 + a_{12}x_3x_4, \\ x_4x_3 = a_{21}x_1x_2 + a_{22}x_3x_4, \\ 1 = x_1x_2 + x_3x_4, \end{cases} \tag{1.4}$$

along with an involution acting by the formula

$$x_1^* = x_2, \qquad x_3^* = x_4. \tag{1.5}$$

Note that equations (1.4) are invariant under this involution.

It is known that the ideal (1.1) is stable under involution (1.5) if and only if $\bar{\alpha} = \alpha, \beta = 1$, and $\gamma = -1$ (see Lemma 2.1); the involution turns the Sklyanin algebra $\mathfrak{S}_{\alpha,1,-1}(\mathbb{C})$ into a $*$-algebra (i.e., a self-adjoint algebra). Denote by $\mathcal{I}_0$ a nonhomogeneous two-sided ideal of $\mathfrak{S}_{\alpha,1,-1}(\mathbb{C})$ generated by the relation $x_1x_2 + x_3x_4 = 1$. Let $\mathcal{J}_0$ be a two-sided ideal of $\mathcal{O}_A^0$ generated by the four relations $x_4x_2 - x_1x_3 = x_3x_1 + x_2x_4 = x_4x_1 - x_2x_3 = x_3x_2 + x_1x_4 = 0$. The following theorem and corollary describe a family of Cuntz–Krieger algebras which are twisted homogeneous coordinate rings of rational elliptic curves.

Theorem 1.1. *For every integer $b \geq 2$, there exists a $*$-isomorphism*

$$\mathfrak{S}_{\frac{b-2}{b+2},1,-1}(\mathbb{C})/\mathcal{I}_0 \cong \mathcal{O}_B^0/\mathcal{J}_0, \quad \text{where } B = \begin{pmatrix} b-1 & 1 \\ b-2 & 1 \end{pmatrix}. \tag{1.6}$$

Corollary 1.2. *For every integer $b \geq 2$, there exists a dense, self-adjoint subalgebra of the Cuntz–Krieger algebra $\mathcal{O}_B$ isomorphic modulo the ideal $\mathcal{I}_0$ to the twisted homogeneous coordinate ring of the rational elliptic curve*

$$\mathcal{E}_b(\mathbb{Q}) = \left\{(x, y, z) \in \mathbb{P}^2(\mathbb{C}) \;\middle|\; y^2z = x(x-z)\left(x - \frac{b-2}{b+2}z\right)\right\}. \tag{1.7}$$

Remark 1.3. There exists a canonical isomorphism

$$\mathcal{O}_B \otimes \mathcal{K} \cong \mathbb{A}_B \rtimes_\sigma \mathbb{Z}, \tag{1.8}$$

where $\mathbb{A}_B$ is an *AF-algebra* with the incidence matrix B introduced by Effros and Shen [3], σ is the shift automorphism of $\mathbb{A}_B$, and $\mathcal{K}$ is the C^*-algebra of compact operators (see [1, Exercise 10.11.9]). Thus the algebra $\mathbb{A}_B$ is an analogue of the coordinate ring of the curve $\mathcal{E}_b(\mathbb{Q})$. This observation can be used to calculate traces of the Frobenius endomorphisms in terms of the algebra $\mathbb{A}_B$ (see [4]).

Our note is organized as follows. Theorem 1.1 is proved in Section 2. The proof of Corollary 1.2 can be found in Section 3. All preliminary facts have been introduced in Section 1 (we refer the reader to Cuntz and Krieger [2] and Stafford and van den Bergh [7] for more details).

2. Proof of Theorem 1.1

We will split the proof into a series of lemmas.

Lemma 2.1. *The ideal of free algebra $\mathbb{C}\langle x_1, x_2, x_3, x_4\rangle$ generated by equations (1.1) is stable under involution (1.5) if and only if $\bar{\alpha} = \alpha$, $\beta = 1$, and $\gamma = -1$.*

Proof. (i) Let us consider the first two equations in (1.1); this pair is invariant under involution (1.5). Indeed, by the rules of composition for an involution, we have

$$\begin{cases} (x_1x_2)^* = x_2^*x_1^* = x_1x_2, \\ (x_2x_1)^* = x_1^*x_2^* = x_2x_1, \\ (x_3x_4)^* = x_4^*x_3^* = x_3x_4, \\ (x_4x_3)^* = x_3^*x_4^* = x_4x_3. \end{cases} \tag{2.1}$$

Since $\alpha^* = \bar{\alpha} = \alpha$, the first two equations in (1.1) remain invariant under involution (1.5).

(ii) Let us consider the middle pair of equations in (1.1); by the rules of composition for an involution, we have

$$\begin{cases} (x_1x_3)^* = x_3^*x_1^* = x_4x_2, \\ (x_3x_1)^* = x_1^*x_3^* = x_2x_4, \\ (x_2x_4)^* = x_4^*x_2^* = x_3x_1, \\ (x_4x_2)^* = x_2^*x_4^* = x_1x_3. \end{cases} \tag{2.2}$$

One can apply the involution to the first equation $x_1x_3 - x_3x_1 = \beta(x_4x_2 + x_2x_4)$; then one gets $x_4x_2 - x_2x_4 = \bar{\beta}(x_1x_3 + x_3x_1)$. But the second equation says that $x_1x_3 + x_3x_1 = x_4x_2 - x_2x_4$; the last two equations are compatible if and only if $\bar{\beta} = 1$. Thus, $\beta = 1$.

The second equation in involution writes as $x_4x_2 + x_2x_4 = x_1x_3 - x_3x_1$; the last equation coincides with the first equation for $\beta = 1$. Therefore, $\beta = 1$ is necessary and sufficient for invariance of the middle pair of equations in (1.1) with respect to involution (1.5).

(iii) Let us consider the last pair of equations in (1.1); by the rules of composition for an involution, we have

$$\begin{cases} (x_1x_4)^* = x_4^*x_1^* = x_3x_2, \\ (x_4x_1)^* = x_1^*x_4^* = x_2x_3, \\ (x_2x_3)^* = x_3^*x_2^* = x_4x_1, \\ (x_3x_2)^* = x_2^*x_3^* = x_1x_4. \end{cases} \tag{2.3}$$

One can apply the involution to the first equation $x_1x_4 - x_4x_1 = \gamma(x_2x_3 + x_3x_2)$; then one gets $x_3x_2 - x_2x_3 = \bar{\gamma}(x_4x_1 + x_1x_4)$. But the second equation says that $x_1x_4 + x_4x_1 = x_2x_3 - x_3x_2$; the last two equations are compatible if and only if $\bar{\gamma} = -1$. Thus, $\gamma = -1$.

The second equation in involution writes as $x_3x_2 + x_2x_3 = x_4x_1 - x_1x_4$; the last equation coincides with the first equation for $\gamma = -1$. Therefore, $\gamma = -1$ is necessary and sufficient for invariance of the last pair of equations in (1.1) with respect to involution (1.5).

(iv) It remains to verify that condition $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ is satisfied by $\beta = 1$ and $\gamma = -1$ for any $\alpha \in k$. Lemma 2.1 follows. □

Lemma 2.2. *Whenever $\alpha \neq 1$, there exists an invertible linear transformation with rational coefficients which brings the system of equations*

$$\begin{cases} x_1x_2 - x_2x_1 = \alpha(x_3x_4 + x_4x_3), \\ x_1x_2 + x_2x_1 = x_3x_4 - x_4x_3 \end{cases} \tag{2.4}$$

to the form

$$\begin{cases} x_2x_1 = (b-1)x_1x_2 + x_3x_4, \\ x_4x_3 = (b-2)x_1x_2 + x_3x_4, \end{cases} \tag{2.5}$$

where $\alpha = \frac{b-2}{b+2}$.

Proof. (i) Let us isolate x_2x_1 and x_4x_3 in (2.4). For that, we will write (2.4) in the form

$$\begin{cases} x_2x_1 + \alpha x_4x_3 = x_1x_2 - \alpha x_3x_4, \\ x_2x_1 + x_4x_3 = -x_1x_2 + x_3x_4. \end{cases} \tag{2.6}$$

Consider (2.6) as a linear system of equations relatively x_2x_1 and x_4x_3; since $\alpha \neq 1$, it has a unique solution:

$$\begin{cases} x_2x_1 = \frac{1}{1-\alpha}\begin{vmatrix} x_1x_2-\alpha x_3x_4 & \alpha \\ -x_1x_2+x_3x_4 & 1 \end{vmatrix} = \frac{1+\alpha}{1-\alpha}x_1x_2 - \frac{2\alpha}{1-\alpha}x_3x_4, \\ \\ x_4x_3 = \frac{1}{1-\alpha}\begin{vmatrix} 1 & x_1x_2-\alpha x_3x_4 \\ 1 & -x_1x_2+x_3x_4 \end{vmatrix} = \frac{-2}{1-\alpha}x_1x_2 + \frac{1+\alpha}{1-\alpha}x_3x_4. \end{cases} \tag{2.7}$$

(ii) Let us substitute $\alpha = \frac{b-2}{b+2}$ in (2.7). Then one arrives at the following system of equations given in the matrix form

$$\begin{pmatrix} x_2x_1 \\ x_4x_3 \end{pmatrix} = \begin{pmatrix} \frac{b}{2} & 1-\frac{b}{2} \\ -1-\frac{b}{2} & \frac{b}{2} \end{pmatrix} \begin{pmatrix} x_1x_2 \\ x_3x_4 \end{pmatrix}. \tag{2.8}$$

It is verified directly that

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{b}{2} & 1-\frac{b}{2} \\ -1-\frac{b}{2} & \frac{b}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} b-1 & 1 \\ b-2 & 1 \end{pmatrix}. \tag{2.9}$$

In other words, matrices (2.5) and (2.8) are similar in the matrix group $GL_2(\mathbb{Q})$. Lemma 2.2 is proved. $\square$

Lemma 2.3. *If $b \geq 2$ is an integer, then there exists a $*$-isomorphism*

$$\mathfrak{S}_{\frac{b-2}{b+2},1,-1}(\mathbb{C})/\mathcal{I}_0 \cong \mathcal{O}^0_{b-1,1,b-2,1}/\mathcal{J}_0; \tag{2.10}$$

the isomorphism is given by identification of generators x_i of the respective algebras.

Proof. Since b is an integer number, one gets that $\alpha = \frac{b-2}{b+2}$ is a rational number. In particular, α is real; that is, $\bar{\alpha} = \alpha$. Thus, by Lemma 2.1, algebra $\mathfrak{S}_{\frac{b-2}{b+2},1,-1}(\mathbb{C})$ is a self-adjoint Sklyanin algebra.

Recall that the ideal $\mathcal{I}_0$ is generated by the relation

$$x_1x_2 + x_3x_4 = 1, \tag{2.11}$$

while the ideal $\mathcal{J}_0$ is generated by the system of relations

$$\begin{cases} x_1x_3 = x_4x_2, \\ x_3x_1 = -x_2x_4, \\ x_1x_4 = -x_3x_2, \\ x_4x_1 = x_2x_3. \end{cases} \tag{2.12}$$

Note that ideals $\mathcal{I}_0$ and $\mathcal{J}_0$ are stable under involution (1.5).

By Lemma 2.2, the first pair of equations in the system (1.1) with $\alpha = \frac{b-2}{b+2}$ coincides with the first pair of equations in the system (1.4) with $a_{11} = b-1, a_{12} = 1, a_{21} = b-2$, and $a_{22} = 1$. Thus, if one complements system (1.1) with equation (2.11) and system (1.4) with the system of equations (2.12), then one obtains the required $*$-isomorphism (2.10). Lemma 2.3 is proved. □

Theorem 1.1 follows from Lemma 2.3.

Remark 2.4. The ideals $\mathcal{I}_0$ and $\mathcal{J}_0$ do not depend on "modulus" b of the Sklyanin algebra $\mathfrak{S}_{\frac{b-2}{b+2},1,-1}(\mathbb{C})$; therefore, algebra $\mathcal{O}^0_{b-1,1,b-2,1}$ can be viewed as a twisted homogeneous coordinate ring of the elliptic curve $\mathcal{E} \subset \mathbb{P}^3(\mathbb{C})$.

3. Proof of Corollary 1.2

We will split the proof into a series of lemmas, starting with the following elementary result.

Lemma 3.1. *If α is a real number different from* 0 *and* 1, *then the algebra $\mathfrak{S}_{\alpha,1,-1}(\mathbb{C})/\Omega_0$ is the coordinate ring of a nonsingular elliptic curve $\mathcal{E}(\mathbb{C}) = \{(x, y, z) \in \mathbb{P}^2(\mathbb{C}) \mid y^2z = x(x-z)(x-\alpha z)\}$.*

Proof. Recall that the Sklyanin algebra $\mathfrak{S}_{\alpha,1,-1}(\mathbb{C})$ defines an elliptic curve $\mathcal{E} \subset \mathbb{P}^3(\mathbb{C})$ given by the intersection of two quadrics (see [6, p. 267]):

$$\begin{cases} (1-\alpha)v^2 + (1+\alpha)w^2 + 2z^2 = 0, \\ u^2 + v^2 + w^2 + z^2 = 0. \end{cases} \tag{3.1}$$

We will pass in (3.1) from variables (u, v, w, z) to the new variables (X, Y, Z, T) given by the formulas

$$\begin{cases} u^2 = T^2, \\ v^2 = \frac{1}{2}Y^2 - \frac{1}{2}Z^2 - T^2, \\ w^2 = X^2 + \frac{1}{2}Y^2 - \frac{1}{2}Z^2 - T^2, \\ z^2 = Z^2. \end{cases} \tag{3.2}$$

Then equations (3.1) take the form

$$\begin{cases} \alpha X^2 + Z^2 - T^2 = 0, \\ X^2 + Y^2 - T^2 = 0. \end{cases} \tag{3.3}$$

Let us consider another (polynomial) transformation $(x, y) \mapsto (X, Y, Z, T)$ given by the formulas

$$\begin{cases} X = -2y, \\ Y = x^2 - 1 + \alpha, \\ Z = x^2 + 2(1-\alpha)x + 1 - \alpha, \\ T = x^2 + 2x + 1 - \alpha. \end{cases} \tag{3.4}$$

Then both of the equations in (3.3) give us the equation $y^2 = x(x+1)(x+1-\alpha)$, which after a shift $x' = x+1$ takes the canonical form

$$y^2 = x(x-1)(x-\alpha). \tag{3.5}$$

Using the projective transformation $x = \frac{x'}{z'}$ $y = \frac{y'}{z'}$ in (3.5), one gets the homogeneous equation of elliptic curve $\mathcal{E}$:

$$y^2 z = x(x-z)(x-\alpha z). \tag{3.6}$$

Lemma 3.1 follows. ☐

Lemma 3.2. *If $b \geq 2$ is an integer, then there exists a dense, self-adjoint subalgebra of the Cuntz–Krieger algebra $\mathcal{O}_{b-1,1,b-2,1}$ which is related (modulo ideal $\mathcal{I}_0$) to a twisted homogeneous coordinate ring of the rational elliptic curve $\mathcal{E}(\mathbb{Q}) = \{(x,y,z) \in \mathbb{P}^2(\mathbb{C}) \mid y^2 z = x(x-z)(x - \frac{b-2}{b+2}z)\}$; the curve is nonsingular unless $b = 2$.*

Proof. If one assumes that $\alpha = \frac{b-2}{b+2}$ in Lemma 3.1, then

$$\mathfrak{S}_{\frac{b-2}{b+2},1,-1}(\mathbb{C})/\mathcal{I}_0 \cong \mathcal{O}^0_{b-1,1,b-2,1}/\mathcal{J}_0. \tag{3.7}$$

The right-hand side of (3.7) is a subalgebra of the Cuntz–Krieger algebra $\mathcal{O}_{b-1,1,b-2,1}$; such an algebra is self-adjoint, since the ideal $\mathcal{J}_0$ is invariant under involution (1.5). The right-hand side of (3.7) is a dense subalgebra of the Cuntz–Krieger algebra $\mathcal{O}_{b-1,1,b-2,1}$, since $\mathcal{O}^0_{b-1,1,b-2,1}/\mathcal{J}_0$ is dense in $\mathcal{O}_{b-1,1,b-2,1}$.

On the other hand, if $b \neq 2$, then the algebra $\mathcal{O}^0_{b-1,1,b-2,1}/\mathcal{J}_0$ is related to the factor (by the ideal $\mathcal{I}_0$) of the coordinate ring $\mathfrak{S}_{\frac{b-2}{b+2},1,-1}(\mathbb{C})/\Omega$ of the nonsingular curve $\mathcal{E}(\mathbb{Q}) = \{(x,y) \in \mathbb{P}^2(\mathbb{C}) \mid y^2 z = x(x-z)(x - \frac{b-2}{b+2}z)\}$. It is easy to see that the curve $\mathcal{E}(\mathbb{Q})$ is singular if and only if $b = 2$. Lemma 3.2 is proved. ☐

Corollary 1.2 follows from Lemma 3.2.

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