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BEKKA-TYPE AMENABILITIES FOR UNITARY COREPRESENTATIONS OF LOCALLY COMPACT QUANTUM GROUPS

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Abstract. In this short note, we further Ng's work by extending Bekka amenability and weak Bekka amenability to general locally compact quantum groups, and we generalize some of Ng's results to the general case. In particular, we show that a locally compact quantum group $\mathbb{G}$ is coamenable if and only if the contra-corepresentation of its fundamental multiplicative unitary $W_{\mathbb{G}}$ is Bekka-amenable, and that $\mathbb{G}$ is amenable if and only if its dual quantum group's fundamental multiplicative unitary $W_{\widehat{\mathbb{G}}}$ is weakly Bekka-amenable.

1. Introduction

The notion of amenability essentially begins with Lebesgue (1904). In 1929, von Neumann introduced and studied the class of amenable groups and used it to explain why the Banach–Tarski paradox occurs only for dimensions greater than or equal to 3. In 1950, Dixmier extended the concept of amenability to topological groups (see [12] and [14]). In the 1970s, the ideas of amenability and coamenability for Kac algebras were introduced by Voiculescu [16], studied further by Enock and Schwartz [5], and later researched by Ruan [13]. In [10], following a work by Bekka [4], Ng introduced Bekka amenability and weak Bekka amenability for unitary corepresentations of Kac algebras, and used them to characterize amenability and coamenability for Kac algebras. Later, Ng [9], [11] investigated amenability

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and coamenability for Hopf C^*-algebras. In 2003, Bédos and Tuset [3] (and then Bédos, Conti, and Tuset [2] in 2005) extended amenability and coamenability to algebraic quantum groups and locally compact quantum groups.

In this short note, we give some remarks on Ng's work in [10]. We extend Bekka amenability and weak Bekka amenability to general locally compact quantum groups. Furthermore, we prove that a locally compact quantum group $\mathbb{G}$ is coamenable if and only if the contra-corepresentation of its fundamental multiplicative unitary $W_{\mathbb{G}}$ is Bekka-amenable, and that $\mathbb{G}$ is amenable if and only if its dual group's fundamental multiplicative unitary $W_{\widehat{\mathbb{G}}}$ is weakly Bekka-amenable. These results generalize Ng's corresponding propositions for Kac algebras in [10].

The notions of Bekka-type amenabilities studied in this note originate from Bekka's work in [4]. In the case of locally compact groups, all Bekka-type amenabilities for unitary corepresentations are equal to amenability (introduced by Bekka in [4]) for unitary representations. Remarkably, Bekka [4] showed that amenability for a locally compact group is equivalent to the fact that every unitary representation is amenable. These results justify the use of the term "Bekka-type amenabilities."

This note is organized as follows. After some preliminaries in Section 2, we discuss Bekka amenability and weak Bekka amenability for locally compact quantum groups in Section 3.

2. Notation and definitions

2.1. **Some notation.** In this note, we use the convention that the inner product $\langle\cdot,\cdot\rangle$ of a complex Hilbert space $\mathfrak{H}$ is conjugate-linear in the second variable. We denote by $\mathcal{L}(\mathfrak{H})$ and $\mathcal{K}(\mathfrak{H})$ the set of bounded linear operators and that of compact operators on $\mathfrak{H}$, respectively. For any $x, y \in \mathfrak{H}$ and $T \in \mathcal{L}(\mathfrak{H})$, we denote by $\omega_{x,y}$ the normal functional given by

$$\omega_{x,y}(T) := \langle Tx, y\rangle.$$

The symbol $\otimes$ denotes either a minimal C^*-algebraic tensor product or a tensor product of Hilbert spaces, $\bar{\otimes}$ denotes a von Neumann algebraic tensor product, and id denotes the identity map. Finally, if X and Y are C^*-algebras or Hilbert spaces, we use the symbol Σ to denote the canonical flip map from $X \otimes Y$ to $Y \otimes X$ sending $x \otimes y$ onto $y \otimes x$ for all $x \in X$ and $y \in Y$. Note that $\Sigma^2 = \mathrm{id}$.

For a C^*-algebra A, we use $\mathrm{Rep}(A)$ to denote the collection of unitary equivalence classes of nondegenerate $*$-representations of A. Let us also recall some notation concerning $\mathrm{Rep}(A)$. Suppose that $(\mu, \mathfrak{H}), (\nu, \mathfrak{K}) \in \mathrm{Rep}(A)$. We write $\nu \prec \mu$ if $\ker\mu \subset \ker\nu$.

2.2. **Locally compact quantum group.** Let $(C_0(\mathbb{G}), \Delta, \varphi, \psi)$ be a reduced locally compact quantum group as introduced in [7, Definition 4.1] (for simplicity, we denote it by $\mathbb{G}$). The dual locally compact quantum group of $\mathbb{G}$ is denoted by $(C_0(\widehat{\mathbb{G}}), \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$ (or simply, $\widehat{\mathbb{G}}$). We use $L^2(\mathbb{G})$ to denote the Hilbert space given by the GNS construction of the left-invariant Haar weight φ, and we consider both $C_0(\mathbb{G})$ and $C_0(\widehat{\mathbb{G}})$ as C^*-subalgebras of $\mathcal{L}(L^2(\mathbb{G}))$. Note that $L^2(\mathbb{G}) = L^2(\widehat{\mathbb{G}})$.

Let $\mathbf{1}$ be the identity of $M(C_0(\mathbb{G}))$. There is a unitary

$$W_{\mathbb{G}} \in M\big(C_0(\mathbb{G}) \otimes C_0(\widehat{\mathbb{G}})\big) \subseteq \mathcal{L}\big(L^2(\mathbb{G}) \otimes L^2(\mathbb{G})\big),$$

called the *fundamental multiplicative unitary*, that implements the comultiplication

$$\Delta(x) = W_{\mathbb{G}}^*(\mathbf{1} \otimes x)W_{\mathbb{G}}\big(x \in C_0(\mathbb{G})\big).$$

We denote by $W_{\widehat{\mathbb{G}}}$ the fundamental multiplicative unitary for the dual quantum group $\widehat{\mathbb{G}}$ given by $\Sigma W_{\mathbb{G}}^* \Sigma$, where Σ is the flip map as defined above. (Interested readers may refer to [7] and [15] for more details.)

The von Neumann subalgebra $L^\infty(\mathbb{G})$ generated by $C_0(\mathbb{G})$ in $\mathcal{L}(L^2(\mathbb{G}))$ is a Hopf–von Neumann algebra under a comultiplication $\widetilde{\Delta}$ defined by $W_{\mathbb{G}}$ as in the above (see [8], [15, Section 8.3.4]). We usually call $L^\infty(\mathbb{G})$ the *von Neumann algebraic quantum group* of $\mathbb{G}$. Then $L^1(\mathbb{G})$ denotes the predual of $L^\infty(\mathbb{G})$, and $L^1_*(\mathbb{G}) := \{\omega \in L^1(\mathbb{G}) \mid \exists \eta \in L^1(\mathbb{G}) \text{ s.t. } (\omega \otimes \mathrm{id})(W_{\mathbb{G}})^* = (\eta \otimes \mathrm{id})(W_{\mathbb{G}})\}$ is a dense *-subalgebra of $L^1(\mathbb{G})$ as introduced in [6, pp. 294–295].

2.3. **Corepresentation.** For any Hilbert space $\mathfrak{H}_U$, a unitary $U \in M(\mathcal{K}(\mathfrak{H}_U) \otimes C_0(\mathbb{G}))$ is called a *unitary corepresentation* of $\mathbb{G}$ on $\mathfrak{H}_U$ if

$$(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}, \tag{2.1}$$

where U_{ij} is the usual "leg notation" (see [7, p. 13], [15, Section 7.1.2]). Let $\mathrm{Corep}(\mathbb{G})$ denote the collection of unitary corepresentations of $\mathbb{G}$. For $U, V \in \mathrm{Corep}(\mathbb{G})$, T is called an *intertwiner* between U and V, and we write $T \in \mathrm{Intw}(U, V)$, if $T \in \mathcal{L}(\mathfrak{H}_U, \mathfrak{H}_V)$ such that

$$T(\mathrm{id} \otimes \omega)(U) = (\mathrm{id} \otimes \omega)(V)T \quad \text{for any } \omega \in L^1_*(\mathbb{G}).$$

We say that U is *unitarily equivalent* to V and we write $U \cong V$ if there exists $T \in \mathrm{Intw}(U, V)$ such that T is a unitary.

2.4. **Universal quantum group.** The universal quantum group C^*-algebra of $\widehat{\mathbb{G}}$ is denoted by $(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}), \widehat{\Delta}^{\mathrm{u}})$ (see [6, Sections 4 and 5]). As shown in [6, Proposition 5.2], there exists a unitary

$$V_{\mathbb{G}}^{\mathrm{u}} \in M\big(C_0^{\mathrm{u}}(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G})\big)$$

that implements a bijection between unitary corepresentations U of $\mathbb{G}$ on $\mathfrak{H}$ and nondegenerate $*$-representations π_U of $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$ on $\mathfrak{H}$ through the correspondence

$$U = (\pi_U \otimes \mathrm{id})(V_{\mathbb{G}}^{\mathrm{u}}).$$

The identity $\mathbf{1}_{\mathbb{G}} = 1 \otimes \mathbf{1}$ of $\mathcal{L}(\mathbb{C}) \otimes M(C_0(\mathbb{G})) \cong M(\mathcal{K}(\mathbb{C}) \otimes C_0(\mathbb{G})) \cong M(C_0(\mathbb{G}))$ is a trivial unitary corepresentation of $\mathbb{G}$ on $\mathbb{C}$, and $\pi_{\mathbf{1}_{\mathbb{G}}}$ is a character of $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$. As in the literature, we write $U \prec W$ when $\pi_U \prec \pi_W$ (see, e.g., [3, Section 5] and Section 2.1).

2.5. **Contra-corepresentation.** Let U be a unitary corepresentation of $\mathbb{G}$ on a Hilbert space $\mathfrak{H}_U$. As in [3, p. 871], we define the *contra-corepresentation* $\overline{U}$ of U by

$$\overline{U} := (\tau \otimes R)(U),$$

where τ is the *canonical anti-isomorphism* from $\mathcal{L}(\mathfrak{H}_U)$ to $\mathcal{L}(\overline{\mathfrak{H}}_U)$ (with $\overline{\mathfrak{H}}_U$ being the conjugate Hilbert space of $\mathfrak{H}_U$) and R is the unitary antipode on $C_0(\mathbb{G})$. Then $\overline{U}$ is a unitary corepresentation of $\mathbb{G}$ on $\overline{\mathfrak{H}}_U$. Note that it is unique up to equivalence $\cong$ and that

$$\overline{\overline{U}} \cong U.$$

If W is another unitary corepresentation of $\mathbb{G}$ on a Hilbert space $\mathfrak{K}$, then we denote by $U \textcircled{\scriptsize$\top$} W$ the unitary corepresentation $U_{13}W_{23}$ on $\mathfrak{H} \otimes \mathfrak{K}$ and call it the *tensor product* of U and W. In this case,

$$\pi_{U \textcircled{\scriptsize$\top$} W} = (\pi_U \otimes \pi_W) \circ \widehat{\Delta}^{\mathrm{u}}. \tag{2.2}$$

3. Amenability, coamenability, and Bekka-type amenabilities

Let us first recall the following definitions of *amenability* and *coamenability* of a locally compact quantum group.

Definition 3.1 ([3, Definitions 3.1, 3.2]). Let $\mathbb{G}$ be a locally compact quantum group.

(a) We say that $\mathbb{G}$ is *coamenable* if there exists a state ϵ of $C_0(\mathbb{G}))$ such that $(\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}$.

(b) A *left-invariant mean* for a locally compact quantum group $\mathbb{G}$ is a state m on $L^\infty(\mathbb{G})$ such that $m(\omega \mathbin{\overline{\otimes}} \mathrm{id})\Delta = \omega(\mathbf{1})m$, for all $\omega \in L^1(\mathbb{G})$. We say that a locally compact quantum group $\mathbb{G}$ is *amenable* if it has a left-invariant mean.

Remark 3.2. Similarly, we can also define the following. A *right-invariant mean* for $\mathbb{G}$ is a state m on $L^\infty(\mathbb{G})$ such that $m(\mathrm{id} \mathbin{\overline{\otimes}} \omega)\Delta = \omega(\mathbf{1})m$ for all $\omega \in L^1(\mathbb{G})$. Clearly, m is a right-invariant mean if and only if $m \circ R$ is a left-invariant mean. Thus, $\mathbb{G}$ is amenable if and only if it has a right-invariant mean.

Coamenability may be characterized by the following equivalent formulations, which were obtained by Bédos and Tuset in [3].

Theorem 3.3 ([3, Theorem 3.1])**.** *For a locally compact quantum group $\mathbb{G}$, the following statements are equivalent.*

(a) *$\mathbb{G}$ is coamenable.*

(b) *The canonical surjective homomorphism $\Lambda : C_0^{\mathrm{u}}(\mathbb{G}) \to C_0(\mathbb{G})$ is an isomorphism.*

(c) *There exists a *-character on the C^*-algebra $C_0(\mathbb{G})$.*

(d) *There exists a net of unit vectors $\{\xi_i\}$ in $L^2(\mathbb{G})$ such that*

$$\lim_i \big\| W_{\mathbb{G}}(\xi_i \otimes v) - (\xi_i \otimes v) \big\| = 0, \quad \forall v \in L^2(\mathbb{G}).$$

Remark 3.4. Comparing Theorem 3.3 with [10, Theorem 2.3], we easily see that the presumed "amenability" for a Kac algebra in Ng [9] is actually the coamenability for its dual in the sense of Definition 3.1.

We now extend Bekka amenability and weakly Bekka amenability introduced in [10] to the general case.

Definition 3.5. For any $U \in \mathrm{Corep}(\mathbb{G})$, we say that

(a) U has the *weak containment property* (WCP) if $\mathbf{1}_{\mathbb{G}} \prec U$ (equivalently, $\pi_{\mathbf{1}_{\mathbb{G}}} \prec \pi_U$; see Sections 2.1 and 2.4); the WCP is actually the property (A) introduced in [10, Proposition and Definition 2.4];

(b) U is *Bekka-amenable* if $\pi_{\mathbf{1}_{\mathbb{G}}} \prec \pi_{U \textcircled{\scriptsize T} \overline{U}}$ (equivalently, $\mathbf{1}_{\mathbb{G}} \prec U \textcircled{\scriptsize T} \overline{U}$), that is, $U \textcircled{\scriptsize T} \overline{U}$ has the WCP;

(c) U is *weakly Bekka-amenable* if there exists a positive functional M on $\mathcal{L}(\mathfrak{H}_U)$ with $M(\mathrm{id}_{\mathfrak{H}_U}) = 1$ such that

$$M\big[(\mathrm{id}_{\mathfrak{H}_U} \otimes \omega)\big(\alpha_U(T)\big)\big] = M(T),$$

for any positive functional $\omega \in L^1(\mathbb{G})$ with $\omega(\mathbf{1}) = 1$ and $T \in \mathcal{L}(\mathfrak{H}_U)$, where

$$\alpha_U(T) := U(T \otimes \mathbf{1})U^*$$

is called a *coaction* of $\mathbb{G}$ on $\mathcal{L}(\mathfrak{H}_U)$; those M satisfying the above condition are called α_U*-invariant means.*

Remark 3.6.

(a) Let $\mathbb{G}$ be a locally compact quantum group of Kac type, and let U be an arbitrary finite-dimensional unitary corepresentation of $\mathbb{G}$. Ng [10, Proposition 3.10] proved that U is Bekka-amenable. Clearly, so is $\overline{U}$, since $\overline{U}$ is also finite-dimensional.

(b) When $\mathbb{G}$ is actually a locally compact group G, its reduced C^*-algebraic quantum group $C_0(G)$ is commutative. It can be obviously seen from the commutativity that, for any two $U, V \in \mathrm{Corep}(\mathbb{G})$, we have $U \textcircled{\scriptsize T} V = V \textcircled{\scriptsize T} U$. So, we have $U \textcircled{\scriptsize T} \overline{U} = \overline{U} \textcircled{\scriptsize T} U$, which implies that U is Bekka-amenable if and only if $\overline{U}$ is Bekka-amenable. In this case, Bekka amenability is in fact the amenability for unitary representations in Bekka [4].

Theorem 3.7 ([3, Theorem 5.2], [10, Proposition and Definition 2.4])**.** *Let $\mathbb{G}$ be a locally compact quantum group, and consider $U \in \mathrm{Corep}(\widehat{\mathbb{G}})$. Then the following are equivalent.*

(a) *U has the WCP.*

(b) *There exists a state ψ on $\mathcal{L}(\mathfrak{H}_U)$ such that $\psi(\mathrm{id} \overline{\otimes} \omega)(U) = \omega(\mathbf{1})$, for $\omega \in L^1(\widehat{\mathbb{G}})$.*

(c) *There exists a net $\{\xi_i\}$ of unit vectors in $\mathfrak{H}_U$ such that*

$$\lim_i \big\|U(\xi_i \otimes v) - (\xi_i \otimes v)\big\| = 0, \quad \textit{for all } v \in L^2(\mathbb{G}).$$

Corollary 3.8. *A locally compact quantum group $\mathbb{G}$ is coamenable if and only if $W_{\mathbb{G}}$ has the WCP as a unitary corepresentation of $\widehat{\mathbb{G}}$.*

Proof. Since $W_{\mathbb{G}}$ can be viewed as an element in $\mathrm{Corep}(\widehat{\mathbb{G}})$, the corollary easily follows from Theorem 3.3(d) and Theorem 3.7(c). □

The WCP is stable under some operations, for example, contragredient and tensor product. For any $U \in \mathrm{Corep}(\mathbb{G})$, we denote by $\mathbf{1}_U$ the trivial unitary corepresentation $\mathrm{id}_{\mathfrak{H}_U} \otimes \mathbf{1}$ of $\mathbb{G}$ on $\mathfrak{H}_U$.

Proposition 3.9 ([3, Proposition 5.3])**.** *Suppose that $\mathbb{G}$ is a locally compact quantum group, and consider $U, V \in \mathrm{Corep}(\mathbb{G})$.*

(a) *If U has the WCP, then so does $\overline{U}$.*
(b) *If both of U and V have the WCP, then so does $U \textcircled{\top} V$.*
(c) *If $U \textcircled{\top} \mathbf{1}_V$ or $\mathbf{1}_V \textcircled{\top} U$ has the WCP, then so does U.*

Next, we present a lemma and a proposition. These results are probably known. Since we have not found them or their proof explicitly stated in the literature, we give complete arguments for the benefit of the reader.

Lemma 3.10. *Let U and V be two unitary corepresentations of $\mathbb{G}$. Then one has*

$$\overline{U \textcircled{\top} V} \cong \overline{V} \textcircled{\top} \overline{U}.$$

Proof. Since the unitary antipode R is a *-antiautomorphism, one has

$$\begin{aligned}\overline{U \textcircled{\top} V} &= (\tau \otimes \tau \otimes R)(U_{13} V_{23}) \\ &= (\tau \otimes R)(V)_{23} (\tau \otimes R)(U)_{13} \\ &= \overline{V}_{23} \overline{U}_{13} = (\Sigma_{12} \otimes \mathbf{1}) \overline{V}_{13} \overline{U}_{23} (\Sigma_{12} \otimes \mathbf{1}) \\ &= (\Sigma_{12} \otimes \mathbf{1})(\overline{V} \textcircled{\top} \overline{U})(\Sigma_{12} \otimes \mathbf{1}).\end{aligned}$$

So, it follows that for any $\omega \in L^1_*(\mathbb{G})$, we have

$$\Sigma (\mathrm{id} \otimes \omega)(\overline{U \textcircled{\top} V}) = (\mathrm{id} \otimes \omega)(\overline{V} \textcircled{\top} \overline{U}) \Sigma,$$

since $\Sigma^2 = \mathrm{id}$. This implies that the unitary Σ lies in $\mathrm{Intw}(\overline{U \textcircled{\top} V}, \overline{V} \textcircled{\top} \overline{U})$. Hence, the lemma holds. □

The following proposition is usually called the *absorption principle*, which is the generalization of Fell's absorption principle for locally compact groups. Bédos, Conti, and Tuset [2, Proposition 3.4] proved the analogue in algebraic quantum groups.

Proposition 3.11. *Let $\mathbb{G}$ be a locally compact quantum group. For any $U \in \mathrm{Corep}(\widehat{\mathbb{G}})$, one has*

$$U \textcircled{\top} W_{\mathbb{G}} \cong \mathbf{1}_U \textcircled{\top} W_{\mathbb{G}}.$$

Proof. Let U be an arbitrary unitary corepresentation of $\widehat{\mathbb{G}}$. Set T to be the image of U on $\mathcal{L}(\mathfrak{H}_U \otimes L^2(\mathbb{G}))$.

For any $\omega \in L^1_*(\widehat{\mathbb{G}})$, one has

$$\begin{aligned}T(\mathrm{id} \otimes \omega)(U \textcircled{\top} W_{\mathbb{G}}) &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)\big(U_{12} U_{13} (W_{\mathbb{G}})_{23}\big) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)\big((W_{\mathbb{G}})_{23} U_{12}\big)\end{aligned}$$

$$= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)\big((\mathrm{id}_{\mathfrak{H}_U} \otimes 1)_{13}(W_{\mathbb{G}})_{23}U_{12}\big)$$
$$= (\mathrm{id} \otimes \omega)(\mathbf{1}_U \textcircled{\top} W_{\mathbb{G}})T,$$

where the second $=$ comes from the pentagonal relation (see [1, Définition A.1]): $U_{12}U_{13}(W_{\mathbb{G}})_{23} = (W_{\mathbb{G}})_{23}U_{12}$ for any $U \in \mathrm{Corep}(\mathbb{G})$. The above calculation implies that T is a unitary intertwiner between $U \textcircled{\top} W_{\mathbb{G}}$ and $\mathbf{1}_U \textcircled{\top} W_{\mathbb{G}}$. Hence, the equivalence of $U \textcircled{\top} W_{\mathbb{G}}$ and $\mathbf{1}_U \textcircled{\top} W_{\mathbb{G}}$, as two elements in $\mathrm{Corep}(\widehat{\mathbb{G}})$, is obtained. $\square$

Corollary 3.12. *Let $\mathbb{G}$ be a locally compact quantum group. For any $U \in \mathrm{Corep}(\widehat{\mathbb{G}})$, one has*

$$\overline{W}_{\mathbb{G}} \textcircled{\top} U \cong \overline{W}_{\mathbb{G}} \textcircled{\top} \mathbf{1}_U.$$

Proof. For any $U \in \mathrm{Corep}(\widehat{\mathbb{G}})$, it is obvious that $\overline{U}$ is also in $\mathrm{Corep}(\widehat{\mathbb{G}})$. By Proposition 3.11, one has

$$\overline{U} \textcircled{\top} W_{\mathbb{G}} \cong \mathbf{1}_{\overline{U}} \textcircled{\top} W_{\mathbb{G}}.$$

Hence, since $\mathbf{1}_{\overline{U}} \cong \overline{\mathbf{1}_U}$, by Lemma 3.10, we have

$$\overline{W}_{\mathbb{G}} \textcircled{\top} U \cong \overline{\overline{U} \textcircled{\top} W_{\mathbb{G}}} \cong \overline{\mathbf{1}_{\overline{U}} \textcircled{\top} W_{\mathbb{G}}} \cong \overline{\overline{\mathbf{1}_U} \textcircled{\top} W_{\mathbb{G}}} \cong \overline{W}_{\mathbb{G}} \textcircled{\top} \mathbf{1}_U. \qquad \square$$

Corollary 3.13. *Let $\mathbb{G}$ be a locally compact quantum group. If $\overline{W}_{\mathbb{G}}$ is Bekka-amenable as a unitary corepresentation of $\widehat{\mathbb{G}}$, then $W_{\mathbb{G}}$ is also Bekka-amenable.*

Proof. Consider $W_{\mathbb{G}}$ as a unitary corepresentation of $\widehat{\mathbb{G}}$. If $\overline{W}_{\mathbb{G}}$ is Bekka-amenable, then $\overline{W}_{\mathbb{G}} \textcircled{\top} W_{\mathbb{G}}$ has WCP. Hence, combining Corollary 3.12 and Proposition 3.9(c), we have that $\overline{W}_{\mathbb{G}}$ has WCP as a unitary corepresentation of $\widehat{\mathbb{G}}$. Hence, by Proposition 3.9(a) and (b), both $W_{\mathbb{G}}$ and $W_{\mathbb{G}} \textcircled{\top} \overline{W}_{\mathbb{G}}$ also have the WCP; that is, $W_{\mathbb{G}}$ is Bekka-amenable. $\square$

Using the above results and the concept of the WCP, we can get a characterization of coamenability for locally compact quantum groups.

Proposition 3.14. *Let $\mathbb{G}$ be a locally compact quantum group. The following statements are equivalent.*

(a) *$\mathbb{G}$ is coamenable.*
(b) *$\overline{W}_{\mathbb{G}}$ is Bekka-amenable as a unitary corepresentation of $\widehat{\mathbb{G}}$.*

Proof. First, assume that $\mathbb{G}$ is coamenable, that is, that $W_{\mathbb{G}}$ has the WCP by Corollary 3.8. Using assertions (a) and (b) of Proposition 3.9, we know that $\overline{W}_{\mathbb{G}}$ has the WCP and so does $\overline{W}_{\mathbb{G}} \textcircled{\top} W_{\mathbb{G}}$. Hence, by Definition 3.5(b), $\overline{W}_{\mathbb{G}}$ is Bekka-amenable as a unitary corepresentation of $\widehat{\mathbb{G}}$.

Conversely, if $\overline{W}_{\mathbb{G}}$ is Bekka-amenable, then $\overline{W}_{\mathbb{G}} \textcircled{\top} W_{\mathbb{G}}$ has the WCP. Considering $\overline{W}_{\mathbb{G}}$ as a unitary corepresentation of $\widehat{\mathbb{G}}$, it follows from Proposition 3.11 that $\mathbf{1}_{\overline{W}_{\mathbb{G}}} \textcircled{\top} W_{\mathbb{G}}$ has the WCP. Consequently, by Proposition 3.9(c), we know that $W_{\mathbb{G}}$ has the WCP. Therefore, using Corollary 3.8 again, we know that $\mathbb{G}$ is coamenable. $\square$

Corollary 3.15. *Let $\mathbb{G}$ be a locally compact quantum group. If $\mathbb{G}$ is coamenable, then $W_{\mathbb{G}}$ is Bekka-amenable as a unitary corepresentation of $\widehat{\mathbb{G}}$.*

Proof. The proof follows directly from Corollary 3.13 and Proposition 3.14. □

Ng [10], using Bekka amenability of the fundamental multiplicative unitary, gave a characterization of amenability of a Kac algebra. Using our terminology, we rewrite Ng's proposition as follows.

Proposition 3.16 ([10, Proposition 3.6]). *Let $\mathbb{G}$ be a locally compact quantum group of Kac type. Then $\mathbb{G}$ is coamenable if and only if $W_{\mathbb{G}}$ is Bekka-amenable as a unitary corepresentation of $\widehat{\mathbb{G}}$.*

As a direct consequence of Propositions 3.14 and 3.16, the following corollary implies that, in the Kac case, $W_{\mathbb{G}}$ is Bekka-amenable if and only if $\overline{W}_{\mathbb{G}}$ is Bekka-amenable. Note that the equivalence of (a) and (b) in this corollary is in fact Proposition 3.16 proved by Ng in [10, Proposition 3.6]. We list these statements here just for comparison with the other results.

Corollary 3.17. *Let $\mathbb{G}$ be a locally compact quantum group of Kac type. Consider $W_{\mathbb{G}}$ and $\overline{W}_{\mathbb{G}}$ as two unitary corepresentations of $\widehat{\mathbb{G}}$. The following statements are equivalent.*

(a) $\mathbb{G}$ *is coamenable.*
(b) $W_{\mathbb{G}}$ *is Bekka-amenable.*
(c) $\overline{W}_{\mathbb{G}}$ *is Bekka-amenable.*

In the following, we focus on weak Bekka amenability of unitary corepresentations. Using this property, we give another characterization for amenability, and generalize some of Ng's results on weak Bekka amenability (see [10, Proposition 3.4]). Some proofs of the results shown below follow from lines of argument similar to those of [10, Proposition 3.4]. For completeness, we present the argument here.

Proposition 3.18. *Let G be a locally compact quantum group. The following statements are equivalent.*

(a) $\mathbb{G}$ *is amenable.*
(b) *The fundamental multiplicative unitary $W_{\widehat{\mathbb{G}}}$ of its dual group is weakly Bekka-amenable as an element in* Corep($\mathbb{G}$)*.*
(c) *Every $U \in$* Corep($\mathbb{G}$) *is weakly Bekka-amenable.*

Proof. To obtain that (a) implies (c), we first note that, by Remark 3.2 and amenability of $\mathbb{G}$, there exists a right-invariant mean m on $\mathbb{G}$. Let U be an arbitrary unitary corepresentation of $\mathbb{G}$. For any positive functional ω on $\mathcal{L}(\mathfrak{H}_U)$ with $\omega(\mathrm{id}_{\mathfrak{H}_U}) = 1$, we can define a linear map Φ_ω from $\mathcal{L}(\mathfrak{H}_U)$ to $C_0(\mathbb{G})$ by

$$\Phi_\omega(T) = (\omega \otimes \mathrm{id})\alpha_U(T) \quad \text{for any } T \in \mathcal{L}(\mathfrak{H}_U).$$

Furthermore, one can easily show that Φ_ω is a completely positive map such that $\Delta \circ \Phi_\omega = (\Phi_\omega \otimes \mathrm{id}) \circ \alpha_U$ and $\Phi_\omega(\mathrm{id}_{\mathfrak{H}_U}) = 1$. Thus, we have that $M = m \circ \Phi_\omega$ is an α_U-invariant mean for U, and so U is weakly Bekka-amenable. By arbitrariness of the choice of U, statement (c) holds.

It is clear that (c) implies (b), since $W_{\widehat{\mathbb{G}}}$ can be viewed as a unitary corepresentation of $\mathbb{G}$. To show that (b) implies (a), assume that $W_{\widehat{\mathbb{G}}}$ is weakly Bekka-amenable, and let ω be an $\alpha_{W_{\widehat{\mathbb{G}}}}$-invariant mean. Hence, statement (a) follows from the fact that the restriction $\omega|_{L^\infty(\mathbb{G})}$ is indeed a left-invariant mean for $\mathbb{G}$. □

As in the Kac case, Bekka amenability is still stronger than weak Bekka amenability in the general case.

Proposition 3.19. *Let $\mathbb{G}$ be a locally compact quantum group, and let U be any unitary corepresentation of $\mathbb{G}$. If U is Bekka-amenable, then U is weakly Bekka-amenable.*

Proof. (a) If U is Bekka-amenable, then we know that $U \textcircled{\scriptsize $\top$} \overline{U}$ has the WCP. Consequently, by Theorem 3.7(c), there exists a net of unit vectors $\{\xi_i\} \subset \mathfrak{H}_{U \textcircled{\scriptsize $\top$} \overline{U}}$ such that, for any $v \in L^2(\mathbb{G})$,

$$\lim_i \big\|(U \textcircled{\scriptsize $\top$} \overline{U})(\xi_i \otimes v) - \xi_i \otimes v\big\| = \lim_i \big\|(U \textcircled{\scriptsize $\top$} \overline{U})^*(\xi_i \otimes v) - \xi_i \otimes v\big\| = 0. \qquad (*)$$

Then the net of the vector states $\{\omega_{\xi_i,\xi_i}\}$ has a subnet weak*-convergent to some positive functional $m \in \mathcal{L}(\mathfrak{H}_{U \textcircled{\scriptsize $\top$} \overline{U}})^*$.

For any unit vector $v \in L^2(\mathbb{G})$ and $T \in \mathcal{L}(\mathfrak{H}_{U \textcircled{\scriptsize $\top$} \overline{U}})$, one has

$$\begin{aligned}
& m\big[(\mathrm{id}_{\mathfrak{H}_{U \textcircled{\scriptsize $\top$} \overline{U}}} \otimes \omega_{v,v})\big(\alpha_{U \textcircled{\scriptsize $\top$} \overline{U}}(T)\big)\big] \\
&\quad = \lim_i \omega_{\xi_i,\xi_i}\big[(\mathrm{id}_{\mathfrak{H}_U} \otimes \omega_{v,v})\big(\alpha_{U \textcircled{\scriptsize $\top$} \overline{U}}(T)\big)\big] \\
&\quad = \lim_i \big\langle (T \otimes \mathrm{id}_{L^2(\mathbb{G})})(U \textcircled{\scriptsize $\top$} \overline{U})^*(\xi_i \otimes v), (U \textcircled{\scriptsize $\top$} \overline{U})^*(\xi_i \otimes v)\big\rangle \\
&\quad = \lim_i \big\langle (T \otimes \mathrm{id}_{L^2(\mathbb{G})})(\xi_i \otimes v), \xi_i \otimes v\big\rangle \quad \big(\text{by equation } (*)\big) \\
&\quad = \lim_i \omega_{\xi_i,\xi_i}(T)\|\xi\|^2 = m(T).
\end{aligned}$$

Because every $\omega \in L^1(\mathbb{G})$ is a linear combination of $\omega_{v,v}$'s, the above equalities imply that m is an $\alpha_{U \textcircled{\scriptsize $\top$} \overline{U}}$-invariant mean. Define the positive functional M on $\mathcal{L}(\mathfrak{H}_U)$ by $M(T) = m(T \otimes \mathrm{id}_{\mathfrak{H}_{\overline{U}}})$ for any $T \in \mathcal{L}(\mathfrak{H}_U)$. Then, we can obtain weak Bekka amenability of U by checking that M is indeed an α_U-invariant mean as required. □

Finally, we conclude from the above results that for any locally compact quantum group $\mathbb{G}$, the following relation holds:

coamenability of $\widehat{\mathbb{G}}$ $\Leftrightarrow$ Bekka amenability of $\overline{W}_{\widehat{\mathbb{G}}}$ $\Rightarrow$ Bekka amenability of $W_{\widehat{\mathbb{G}}}$ $\Rightarrow$ weak Bekka amenability of $W_{\widehat{\mathbb{G}}}$ $\Leftrightarrow$ amenability of $\mathbb{G}$,

where $\Leftrightarrow$ means "equal to" and $\Rightarrow$ means "imply."

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