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ON THE PERTURBATION OF OUTER INVERSES OF LINEAR OPERATORS IN BANACH SPACES

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Abstract. The main concern of this article is the perturbation problem for outer inverses of linear bounded operators in Banach spaces. We consider the following perturbed problem. Let $T \in B(X,Y)$ with an outer inverse $T^{\{2\}} \in B(Y,X)$ and $\delta T \in B(X,Y)$ with $\|\delta T T^{\{2\}}\| < 1$. What condition on the small perturbation δT can guarantee that the simplest possible expression $B = T^{\{2\}}(I+\delta T T^{\{2\}})^{-1}$ is a generalized inverse, Moore–Penrose inverse, group inverse, or Drazin inverse of $T+\delta T$? In this article, we give a complete solution to this problem. Since the generalized inverse, Moore–Penrose inverse, group inverse, and Drazin inverse are outer inverses, our results extend and improve many previous results in this area.

1. Introduction and preliminaries

Let X and Y be Banach spaces. Let $B(X,Y)$ denote the Banach space of all bounded linear operators from X into Y. We write $B(X)$ as $B(X,X)$. For any $T \in B(X,Y)$, we denote by $N(T)$ and $R(T)$ the null space and the range of T, respectively. The identity operator will be denoted by I.

Recall that an operator $S \in B(Y,X)$ is said to be an *inner inverse* of $T \in B(X,Y)$ if $TST = T$ and an *outer inverse* if $STS = S$. If S is both an inner inverse and outer inverse of T, then S is called a *generalized inverse* of T, which is denoted by T^+. As is well known, the nonzero outer inverse of any bounded

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linear operator always exists, while an inner inverse or generalized inverse may not exist and it is not unique even if it does exist. In order to force its uniqueness, some further conditions have to be imposed. Let us recall some definitions.

Definition 1.1. Let X and Y be Hilbert spaces. An operator $S \in B(Y,X)$ is called the *Moore–Penrose inverse* of $T \in B(X,Y)$ if S satisfies the Penrose equations

$$(1)\ TST = T, \qquad (2)\ STS = S, \qquad (3)\ (TS)^* = TS, \qquad (4)\ (ST)^* = ST,$$

where T^* denotes the adjoint operator of T. The Moore–Penrose inverse of T is always written by $T^\dagger$, which is uniquely determined if it exists.

Definition 1.2. Let X be a Banach space. An operator $S \in B(X)$ is said to be the *Drazin inverse* of $T \in B(X)$ if S satisfies

$$(1^k)\ T^kST = T^k, \qquad (2)\ STS = S, \qquad (5)\ TS = ST$$

for some positive integer k. The Drazin inverse of T is always denoted by T^D, and the least such k is called the *index* of T. When $k = 1$, the corresponding Drazin inverse is called the *group inverse*, denoted by $T^\sharp$.

Let $\theta \subset \{1,2,3,4,5\}$ be a nonempty set. If S satisfies the equation (i) in Definitions 1.1 and 1.2 for all $i \in \theta$, then S is said to be a *θ-inverse* of T, which is denoted by T^θ. As we all know, each kind of θ-inverse has its own property, and many important generalized inverses, such as the Moore–Penrose inverse, the Drazin inverse, and the group inverse, belong to outer inverses which play a prominent role in numerical analysis, optimization, mathematical statistics, and so on (see [1], [9], [12]–[18]). The major reasons why the outer inverse has important practical value include the existence of the nonzero outer inverse of any bounded linear operator and the stability of the outer inverse. Nashed and Chen [16] gave the following stability theorem of the outer inverses, and Nashed [15] indicated the instability for the inner inverses.

Theorem 1.3 ([16, Lemma 2.2])**.** *Let $T \in B(X,Y)$ with an outer inverse $T^{\{2\}} \in B(Y,X)$ and $\delta T \in B(X,Y)$ with $\|\delta T T^{\{2\}}\| < 1$. Then*

$$B = T^{\{2\}}(I + \delta T T^{\{2\}})^{-1} = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$$

is an outer inverse of $\overline{T} = T + \delta T$ with $R(B) = R(T^{\{2\}})$ and $N(B) = N(T^{\{2\}})$.

This says that the outer inverse of the perturbed operator $\overline{T} = T + \delta T$ possesses the simplest possible expression $\overline{T}^{\{2\}} = T^{\{2\}}(I + \delta T T^{\{2\}})^{-1} = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$, whose null space and range are identical with $T^{\{2\}}$'s and obviously, $\overline{T}^{\{2\}} \to T^{\{2\}}$ as $\delta T \to 0$. Characterizations for the simplest possible expressions of the generalized inverse, Moore–Penrose inverse, group inverse, and Drazin inverse appear in [2], [4]–[6], [8], [9], and [12]. In particular, Castro-González and Vélez-Cerrada [2] gave the equivalent conditions for $B = [I + T^D(\overline{T} - T)]^{-1}T^D$ to be a generalized inverse of $\overline{T}$ under the assumption that T is Drazin invertible.

Motivated by these results, we will consider the following perturbed problem. Let $T \in B(X,Y)$ with an outer inverse $T^{\{2\}} \in B(Y,X)$ and $\delta T \in B(X,Y)$ with $\|\delta T T^{\{2\}}\| < 1$. What condition on the small perturbation δT can guarantee that the simplest possible expression $B = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$ is a generalized inverse, Moore–Penrose inverse, group inverse, or Drazin inverse of $\overline{T} = T + \delta T$? It should be pointed out that if $2 \in \theta$ and the θ-inverse $\overline{T}^{\theta}$ preserves the null space and range of T^{θ}, then $\overline{T}^{\theta} = (I + T^{\theta}\delta T)^{-1}T^{\theta}$ (see [9]). This makes the above problem more meaningful. We give a complete solution to this problem below. Since the generalized inverse, Moore–Penrose inverse, group inverse, and Drazin inverse are outer inverses, the results obtained in this article extend and improve many previous results in this area.

2. Main results

The first theorem below gives the characterizations for $B = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$ to be a generalized inverse of $\overline{T} = T + \delta T$, which is an extension of the main results in [3], [8], [9], [12], and [14].

Theorem 2.1. *Let $T \in B(X, Y)$ with an outer inverse $T^{\{2\}} \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta T T^{\{2\}}\| < 1$, then the following statements are equivalent:*

(1) $B = T^{\{2\}}(I + \delta T T^{\{2\}})^{-1} = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$ *is a generalized inverse of* $\overline{T} = T + \delta T$;
(2) $R(\overline{T}) \cap N(T^{\{2\}}) = \{0\}$;
(3) $X = N(\overline{T}) \oplus R(T^{\{2\}})$ *or* $X = N(\overline{T}) + R(T^{\{2\}})$;
(4) $Y = R(\overline{T}) \oplus N(T^{\{2\}})$;
(5) $R(\overline{T}) = R(\overline{T}T^{\{2\}})$ *or* $R(\overline{T}) \subset R(\overline{T}T^{\{2\}})$;
(6) $N(T^{\{2\}}\overline{T}) = N(\overline{T})$ *or* $N(T^{\{2\}}\overline{T}) \subset N(\overline{T})$;
(7) $(I + \delta T T^{\{2\}})^{-1}R(\overline{T}) = R(TT^{\{2\}})$ *or* $(I + \delta T T^{\{2\}})^{-1}R(\overline{T}) \subset R(TT^{\{2\}})$;
(8) $(I + T^{\{2\}}\delta T)^{-1}N(T^{\{2\}}T) = N(\overline{T})$ *or* $(I + T^{\{2\}}\delta T)^{-1}N(T^{\{2\}}T) \subset N(\overline{T})$;
(9) $(I + \delta T T^{\{2\}})^{-1}\overline{T}N(T^{\{2\}}T) \subset R(TT^{\{2\}})$.

Proof. It follows from Theorem 1.3 that $B = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$ is an outer inverse of $\overline{T}$ with $R(B) = R(T^{\{2\}})$ and $N(B) = N(T^{\{2\}})$. Then $\overline{T}B$ and $B\overline{T}$ are projectors with $R(B\overline{T}) = R(B)$, $N(\overline{T}B) = N(B)$, $R(B) \cap N(\overline{T}) = \{0\}$, and X and Y have the topological direct sum decompositions:

$$X = N(B\overline{T}) \oplus R(B) \qquad \text{and} \qquad Y = N(B) \oplus R(\overline{T}B).$$

(1) $\Rightarrow$ (2). If B is a generalized inverse of $\overline{T}$, then

$$Y = R(\overline{T}B) \oplus N(\overline{T}B) = R(\overline{T}) \oplus N(B) = R(\overline{T}) \oplus N(T^{\{2\}})$$

and thus $R(\overline{T}) \cap N(T^{\{2\}}) = \{0\}$.

(2) $\Rightarrow$ (1). If $R(\overline{T}) \cap N(T^{\{2\}}) = \{0\}$, then $R(\overline{T}) \cap N(B) = \{0\}$ and for all $x \in X$,

$$\overline{T}B\overline{T}x - \overline{T}x \in R(\overline{T}) \cap N(B),$$

that is, $\overline{T}B\overline{T}x = \overline{T}x$, which implies that B is also an inner inverse of $\overline{T}$. Thus B is a generalized inverse of $\overline{T}$.

(1) $\Rightarrow$ (3). If B is a generalized inverse of $\overline{T}$, then

$$X = R(B\overline{T}) \oplus N(B\overline{T}) = N(\overline{T}) \oplus R(B) = N(\overline{T}) \oplus R(T^{\{2\}}),$$

and therefore $X = N(\overline{T}) + R(T^{\{2\}})$.

(3) $\Rightarrow$ (1). If $X = N(\overline{T}) + R(T^{\{2\}})$, then for all $x \in X$, x can be expressed by $x = x_1 + x_2$, where $x_1 \in N(\overline{T})$ and $x_2 \in R(T^{\{2\}})$. Hence $x_2 \in R(B)$ and

$$(\overline{T}B\overline{T} - \overline{T})x = (\overline{T}B\overline{T} - \overline{T})x_2 = 0;$$

that is, B is an inner inverse of $\overline{T}$. Thus B is a generalized inverse of $\overline{T}$.

(1) $\Rightarrow$ (4). See (1) $\Rightarrow$ (2).

(4) $\Rightarrow$ (2). This is obvious.

(3) $\Rightarrow$ (5). We have $R(\overline{T}) = \overline{T}(X) = \overline{T}[N(\overline{T}) + R(T^{\{2\}})] = \overline{T}[R(T^{\{2\}})] = R(\overline{T}T^{\{2\}})$.

(5) $\Rightarrow$ (1). If $R(\overline{T}) \subset R(\overline{T}T^{\{2\}})$, then

$$R(\overline{T}) \subset \overline{T}R(T^{\{2\}}) = \overline{T}R(B) = R(\overline{T}B) = N(I - \overline{T}B),$$

and hence $(I - \overline{T}B)\overline{T} = 0$, which means that B is an inner inverse of $\overline{T}$.

(1) $\Rightarrow$ (6). If B is a generalized inverse of $\overline{T}$, then

$$N(\overline{T}) = N(B\overline{T}) = N\big((I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}\overline{T}\big) = N(T^{\{2\}}\overline{T}).$$

(6) $\Rightarrow$ (1). If $N(T^{\{2\}}\overline{T}) \subset N(\overline{T})$, then

$$R(I - B\overline{T}) = N(B\overline{T}) = N\big((I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}\overline{T}\big) = N(T^{\{2\}}\overline{T}) \subset N(\overline{T})$$

and $\overline{T}(I - B\overline{T}) = 0$, which implies that B is an inner inverse of $\overline{T}$.

(1) $\Rightarrow$ (7). If B is a generalized inverse of $\overline{T}$, then

$$\begin{aligned} R(\overline{T}) &= R(\overline{T}B) = \overline{T}R(B) = \overline{T}R(T^{\{2\}}) = \overline{T}R(T^{\{2\}}TT^{\{2\}}) = \overline{T}T^{\{2\}}R(TT^{\{2\}}) \\ &= (\overline{T}T^{\{2\}} + I - TT^{\{2\}})R(TT^{\{2\}}) = (I + \delta TT^{\{2\}})R(TT^{\{2\}}). \end{aligned}$$

(7) $\Rightarrow$ (8). Obviously, $(I + T^{\{2\}}\delta T)N(\overline{T}) = [I + T^{\{2\}}(\overline{T} - T)]N(\overline{T}) = (I - T^{\{2\}}T)N(\overline{T}) \subset N(T^{\{2\}}T)$. On the other hand, by (7), for any $x \in N(T^{\{2\}}T)$, we have

$$\overline{T}x \in R(\overline{T}) \subset (I + \delta TT^{\{2\}})R(TT^{\{2\}}) = \overline{T}R(T^{\{2\}}).$$

Then there exists a $y \in R(T^{\{2\}})$ such that $\overline{T}y = \overline{T}x$. Hence $x - y \in N(\overline{T})$ and

$$(I + T^{\{2\}}\delta T)(x - y) = (I - T^{\{2\}}T)(x - y) = (I - T^{\{2\}}T)x = x.$$

This implies that $N(T^{\{2\}}T) \subset (I + T^{\{2\}}\delta T)N(\overline{T})$.

(8) $\Rightarrow$ (2). Taking any $y \in R(\overline{T}) \cap N(T^{\{2\}})$, we can find an $x \in X$ satisfying $y = \overline{T}x$ and $T^{\{2\}}\overline{T}x = 0$. Hence

$$\begin{aligned} T^{\{2\}}T(I + T^{\{2\}}\delta T)x &= T^{\{2\}}Tx + T^{\{2\}}TT^{\{2\}}\delta Tx \\ &= T^{\{2\}}Tx + T^{\{2\}}\overline{T}x - T^{\{2\}}Tx = 0, \end{aligned}$$

implying that $(I + T^{\{2\}}\delta T)x \in N(T^{\{2\}}T)$. By (8), $x \in N(\overline{T})$ and so $y = \overline{T}x = 0$.

(7) $\Rightarrow$ (9). This is obvious.

(9) $\Rightarrow$ (2). Let $y \in R(\overline{T}) \cap N(T^{\{2\}})$. We can find an $x \in X$ satisfying $y = \overline{T}x$ and $T^{\{2\}}\overline{T}x = 0$. Since $X = N(T^{\{2\}}T) \oplus R(T^{\{2\}})$, $x = x_1 + x_2$, where $x_1 \in N(T^{\{2\}}T)$ and $x_2 \in R(T^{\{2\}})$. Then

$$(I + \delta T T^{\{2\}})Tx_2 = \big[I + (\overline{T} - T)T^{\{2\}}\big]Tx_2 = \overline{T}T^{\{2\}}Tx_2 = \overline{T}x_2.$$

Hence

$$(I + \delta T T^{\{2\}})^{-1}\overline{T}x_2 = Tx_2 \in R(TT^{\{2\}}),$$

and by (9),

$$(I + \delta T T^{\{2\}})^{-1}\overline{T}x_1 \in R(TT^{\{2\}}).$$

Noting that $y \in N(T^{\{2\}})$, we get $(I + \delta T T^{\{2\}})y = y = \overline{T}x$ and

$$y = (I + \delta T T^{\{2\}})^{-1}\overline{T}x = (I + \delta T T^{\{2\}})^{-1}\overline{T}(x_1 + x_2) \in R(TT^{\{2\}}).$$

Thus $y \in R(TT^{\{2\}}) \cap N(T^{\{2\}})$. It follows from $R(TT^{\{2\}}) \cap N(T^{\{2\}}) = \{0\}$ that $y = 0$. □

Assuming that the outer inverse $T^{\{2\}}$ is also a generalized inverse T^{+}, we get the following.

Corollary 2.2. *Let $T \in B(X, Y)$ with a generalized inverse $T^{+} \in B(Y, X)$ and $\delta T \in B(X, Y)$ with $\|\delta T T^{+}\| < 1$. Then the following statements are equivalent:*

(1) *$B = T^{+}(I + \delta T T^{+})^{-1} = (I + T^{+}\delta T)^{-1}T^{+}$ is a generalized inverse of $\overline{T} = T + \delta T$;*
(2) *$R(\overline{T}) \cap N(T^{+}) = \{0\}$;*
(3) *$X = N(\overline{T}) \oplus R(T^{+})$ or $X = N(\overline{T}) + R(T^{+})$;*
(4) *$Y = R(\overline{T}) \oplus N(T^{+})$;*
(5) *$R(\overline{T}) = R(\overline{T}T^{+})$;*
(6) *$N(T^{+}\overline{T}) = N(\overline{T})$;*
(7) *$(I + \delta T T^{+})^{-1}R(\overline{T}) = R(T)$;*
(8) *$(I + T^{+}\delta T)^{-1}N(T) = N(\overline{T})$;*
(9) *$(I + \delta T T^{+})^{-1}\overline{T}N(T) \subset R(T)$.*

Proof. Noting that $R(TT^{+}) = R(T)$ and $N(T^{+}T) = N(T)$, by Theorem 2.1, we can get the desired result. □

Remark 2.3. Corollary 2.2 extends the main results in [3], [8], [9], and [12]. It is worth mentioning that in [3], $\overline{T}$ is called a *stable perturbation* of T if $\overline{T}$ satisfies $R(\overline{T}) \cap N(T^{+}) = \{0\}$. This notion of stable perturbation is an extension of rank-preserving perturbation and has been used widely in perturbation theory of generalized inverses (see [7]–[10], [12], [19]).

Corollary 2.4 ([2, Theorem 3.2])**.** *Let $T \in B(X)$ be a Drazin invertible with* $\operatorname{ind}(T) = r$*. The following assertions on $\overline{T}$ such that $\|T^{D}(\overline{T} - T)\| < 1$ is invertible are equivalent:*

(1) *$B = [I + T^{D}(\overline{T} - T)]^{-1}T^{D} = T^{D}[I + (\overline{T} - T)T^{D}]^{-1}$ is a generalized inverse of $\overline{T}$;*
(2) *$\overline{T}[I + T^{D}(\overline{T} - T)]^{-1}T^{\pi} = 0$ or $T^{\pi}[I + (\overline{T} - T)T^{D}]^{-1}\overline{T} = 0$;*
(3) *$R(\overline{T}) \cap N(T^{r}) = \{0\}$;*

(4) $X = N(\overline{T}) + R(T^r)$;
(5) $R(\overline{T}T^D) = R(\overline{T})$;
(6) $N(T^D\overline{T}) = N(\overline{T})$;
(7) $T^\pi N(\overline{T}) = N(T^r)$.

Proof. Noting that $N(T^r) = N(T^D)$ and that $R(T^r) = R(T^D)$, by Theorem 2.1, we can get the equivalence between (1), (3), (4), (5), and (6). It follows from (7) and (8) in Theorem 2.1 and

$$\begin{aligned}\overline{T}\big[I + T^D(\overline{T} - T)\big]^{-1}T^\pi = 0 &\Leftrightarrow (I + T^D\delta T)^{-1}R(T^\pi) \subset N(\overline{T})\\ &\Leftrightarrow (I + T^D\delta T)^{-1}N(T^DT) \subset N(\overline{T}),\\ T^\pi\big[I + (\overline{T} - T)T^D\big]^{-1}\overline{T} = 0 &\Leftrightarrow (I + \delta TT^D)^{-1}R(\overline{T}) \subset N(T^\pi)\\ &\Leftrightarrow (I + \delta TT^D)^{-1}R(\overline{T}) \subset R(TT^D),\\ T^\pi N(\overline{T}) = N(T^r) &\Leftrightarrow (I - T^DT)N(\overline{T}) = N(T^D)\\ &\Leftrightarrow (I + T^D\delta T)N(\overline{T}) = N(T^DT)\end{aligned}$$

that (2) and (7) are equivalent to any one of the others. □

Remark 2.5. It should be noted that statement (2) above in [2, Theorem 3.2] is

$$\overline{T}\big[I + T^D(\overline{T} - T)\big]^{-1}T^\pi = T^\pi\big[I + (\overline{T} - T)T^D\big]^{-1}\overline{T} = 0.$$

If X and Y are Hilbert spaces and the orthogonal topological direct sum is considered, we have the following.

Theorem 2.6. *Let X and Y be Hilbert spaces. Let $T \in B(X, Y)$ with an outer inverse $T^{\{2\}} \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta TT^{\{2\}}\| < 1$, then*

$$B = T^{\{2\}}(I + \delta TT^{\{2\}})^{-1} = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$$

is a $\{1, 2, 3\}$-inverse of $\overline{T} = T + \delta T$ if and only if

$$Y = R(\overline{T}) \dot{+} N(T^{\{2\}}),$$

where $\dot{+}$ denotes the orthogonal topological direct sum.

Proof. If B is a $\{1, 2, 3\}$-inverse of $\overline{T}$, then

$$Y = R(\overline{T}B) \dot{+} N(\overline{T}B) = R(\overline{T}) \dot{+} N(B) = R(\overline{T}) \dot{+} N(T^{\{2\}}).$$

Conversely, if $Y = R(\overline{T}) \dot{+} N(T^{\{2\}})$, then by Theorem 2.1, B is a generalized inverse of $\overline{T}$ and $Y = R(\overline{T}) \dot{+} N(B)$. Hence $\overline{T}B$ is the orthogonal projector from Y onto $R(\overline{T})$. Thus $(\overline{T}B)^* = \overline{T}B$ and B is a $\{1, 2, 3\}$-inverse of $\overline{T}$. □

Symmetrically, by Theorem 2.1(3), we can get the following result.

Theorem 2.7. *Let X and Y be Hilbert spaces. Let $T \in B(X, Y)$ with an outer inverse $T^{\{2\}} \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta TT^{\{2\}}\| < 1$, then*

$$B = T^{\{2\}}(I + \delta TT^{\{2\}})^{-1} = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$$

is a $\{1, 2, 4\}$-inverse of $\overline{T} = T + \delta T$ if and only if

$$X = N(\overline{T}) \dot{+} R(T^{\{2\}}).$$

Utilizing Theorems 2.6 and 2.7, we can obtain the equivalent condition that $B = T^{\{2\}}(I + \delta T T^{\{2\}})^{-1}$ is the Moore–Penrose inverse of $\overline{T}$.

Theorem 2.8. *Let X and Y be Hilbert spaces. Let $T \in B(X, Y)$ with an outer inverse $T^{\{2\}} \in B(Y, X)$. Let $\delta T \in B(X, Y)$ satisfy $\|\delta T T^{\{2\}}\| < 1$. Then*

$$B = T^{\{2\}}(I + \delta T T^{\{2\}})^{-1} = (I + T^{\{2\}}\delta T)^{-1} T^{\{2\}}$$

is the Moore–Penrose inverse of $\overline{T} = T + \delta T$ if and only if

$$X = N(\overline{T}) \dot{+} R(T^{\{2\}}) \qquad \text{and} \qquad Y = R(\overline{T}) \dot{+} N(T^{\{2\}}).$$

Corollary 2.9 ([4, Theorem 3.1])**.** *Let X and Y be Hilbert spaces, and let $T \in B(X, Y)$ with the Moore–Penrose inverse $T^\dagger \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta T T^\dagger\| < 1$, then*

$$B = T^\dagger(I + \delta T T^\dagger)^{-1} = (I + T^\dagger \delta T)^{-1} T^\dagger$$

is the Moore–Penrose inverse of $\overline{T} = T + \delta T$ if and only if

$$R(\overline{T}) = R(T) \qquad \text{and} \qquad N(\overline{T}) = N(T).$$

Proof. Since $T^\dagger$ is the Moore–Penrose inverse of T,

$$X = N(T) \dot{+} R(T^\dagger) \qquad \text{and} \qquad Y = R(T) \dot{+} N(T^\dagger).$$

Then by Theorem 2.8, B is the Moore–Penrose inverse of $\overline{T}$ if and only if

$$X = N(\overline{T}) \dot{+} R(T^\dagger) \qquad \text{and} \qquad Y = R(\overline{T}) \dot{+} N(T^\dagger)$$

if and only if

$$N(\overline{T}) = N(T) \qquad \text{and} \qquad R(\overline{T}) = R(T).$$ □

The next theorem concerns the characterization for $B = T^{\{2\}}(I + \delta T T^{\{2\}})^{-1}$ to be the group inverse of $\overline{T}$, which is an extension of the main results in [8], [9], and [11].

Theorem 2.10. *Let $T \in B(X)$ with an outer inverse $T^{\{2\}} \in B(X)$. If $\delta T \in B(X)$ satisfies $\|\delta T T^{\{2\}}\| < 1$, then the following statements are equivalent:*

1. *$B = T^{\{2\}}(I + \delta T T^{\{2\}})^{-1} = (I + T^{\{2\}}\delta T)^{-1} T^{\{2\}}$ is the group inverse of $\overline{T} = T + \delta T$;*
2. *$R(\overline{T}) \cap N(T^{\{2\}}) = \{0\}$ and $\overline{T} = T^{\{2\}} T \overline{T} = \overline{T} T T^{\{2\}}$;*
3. *$X = N(\overline{T}) + R(T^{\{2\}})$, $R(\overline{T}) \subseteq R(T^{\{2\}})$ and $N(T^{\{2\}}) \subseteq N(\overline{T})$.*

Proof. It can be verified that

$$\begin{aligned} B\overline{T} = \overline{T}B &\Leftrightarrow (I + T^{\{2\}}\delta T)^{-1} T^{\{2\}} \overline{T} = \overline{T} T^{\{2\}} (I + \delta T T^{\{2\}})^{-1} \\ &\Leftrightarrow T^{\{2\}} \overline{T}(I + \delta T T^{\{2\}}) = (I + T^{\{2\}}\delta T) \overline{T} T^{\{2\}} \\ &\Leftrightarrow T^{\{2\}} \overline{T} - T^{\{2\}} \overline{T} T T^{\{2\}} = \overline{T} T^{\{2\}} - T^{\{2\}} T \overline{T} T^{\{2\}} \\ &\quad\ \text{(right multiply with } T T^{\{2\}} \text{ and left multiply with } T^{\{2\}} T) \\ &\Leftrightarrow \overline{T} T^{\{2\}} = T^{\{2\}} T \overline{T} T^{\{2\}} \text{ and } T^{\{2\}} \overline{T} = T^{\{2\}} \overline{T} T T^{\{2\}}. \end{aligned}$$

(1) $\Rightarrow$ (2). If B is the group inverse of $\overline{T}$, then $B\overline{T} = \overline{T}B$ and B is a generalized inverse of $\overline{T}$. By Theorem 2.1,

$$R(\overline{T}) \cap N(T^{\{2\}}) = \{0\} \qquad \text{and} \qquad X = N(\overline{T}) + R(T^{\{2\}}).$$

Hence for all $x \in X$, $T^{\{2\}}\overline{T}(I - TT^{\{2\}})x = 0$ which implies that $\overline{T}(I - TT^{\{2\}})x \in N(T^{\{2\}})$. Thus

$$\overline{T}(I - TT^{\{2\}})x \in R(\overline{T}) \cap N(T^{\{2\}})$$

and so $\overline{T} = \overline{T}TT^{\{2\}}$. Noting that

$$\begin{aligned}(I - T^{\{2\}}T)\overline{T}X &= (I - T^{\{2\}}T)\overline{T}\big[N(\overline{T}) + R(T^{\{2\}})\big] \\ &= (I - T^{\{2\}}T)\overline{T}R(T^{\{2\}}) = \{0\},\end{aligned}$$

we get $\overline{T} = T^{\{2\}}T\overline{T}$.

(2) $\Rightarrow$ (3). By Theorem 2.1, we have $X = N(\overline{T}) + R(T^{\{2\}})$. If $\overline{T} = T^{\{2\}}T\overline{T} = \overline{T}TT^{\{2\}}$, then $R(\overline{T}) \subseteq R(T^{\{2\}})$ and $N(T^{\{2\}}) \subseteq N(\overline{T})$.

(3) $\Rightarrow$ (1). It follows from Theorem 2.1 that B is a generalized inverse of $\overline{T}$. By $R(\overline{T}) \subseteq R(T^{\{2\}})$ and $N(T^{\{2\}}) \subseteq N(\overline{T})$, we can get $\overline{T} = T^{\{2\}}T\overline{T}$ and $\overline{T} = \overline{T}TT^{\{2\}}$, respectively. Therefore, $B\overline{T} = \overline{T}B$. □

Corollary 2.11 ([9, Theorem 2.10])**.** *Let $T \in B(X)$ with the group inverse $T^\sharp \in B(X)$ and $\delta T \in B(X)$ with $\|\delta TT^\sharp\| < 1$. Then the following statements are equivalent:*

(1) *$B = T^\sharp(I+\delta TT^\sharp)^{-1} = (I+T^\sharp\delta T)^{-1}T^\sharp$ is the group inverse of $\overline{T} = T+\delta T$;*
(2) *$\overline{T} = \overline{T}T^\sharp T = TT^\sharp\overline{T}$;*
(3) *$R(\overline{T}) \subseteq R(T)$ and $N(T) \subseteq N(\overline{T})$;*
(4) *$R(\overline{T}) = R(T)$ and $N(T) = N(\overline{T})$.*

Proof. Obviously, (4) $\Rightarrow$ (3). Noting that $R(T) \cap N(T^\sharp) = \{0\}$ and $X = N(T^\sharp) \oplus R(T^\sharp)$, we get that $\overline{T} = TT^\sharp\overline{T}$ implies $R(\overline{T}) \subseteq R(T)$ and $R(\overline{T}) \cap N(T^\sharp) = \{0\}$, $N(T^\sharp) \subseteq N(\overline{T})$ implies $X = N(\overline{T}) + R(T^\sharp)$. Thus by Theorem 2.10, we can obtain the equivalence between (1), (2), and (3). To that end, we need to show (1) $\Rightarrow$ (4). In fact, if B is the group inverse of $\overline{T}$, then $R(\overline{T}) = R(B) = R(T^\sharp) = R(T)$ and $N(\overline{T}) = N(B) = N(T^\sharp) = N(T)$. □

Theorem 2.12. *Let $T \in B(X)$ with an outer inverse $T^{\{2\}} \in B(X)$. If $\delta T \in B(X)$ satisfies $\|\delta TT^{\{2\}}\| < 1$, then*

$$B = T^{\{2\}}(I + \delta TT^{\{2\}})^{-1} = (I + T^{\{2\}}\delta T)^{-1}T^{\{2\}}$$

is the Drazin inverse of $\overline{T} = T + \delta T$ if and only if the following statements hold:

(1) *$\overline{T}T^{\{2\}} = T^{\{2\}}T\overline{T}T^{\{2\}}$ and $T^{\{2\}}\overline{T} = T^{\{2\}}\overline{T}TT^{\{2\}}$;*
(2) *there exists a positive integer $k \in N$ such that*

$$\overline{T}^k(I - TT^{\{2\}}) = 0 \qquad \text{or} \qquad (I - T^{\{2\}}T)\overline{T}^k = 0.$$

Proof. If B is the Drazin inverse of $\overline{T}$, then $B\overline{T} = \overline{T}B$. As in the proof of Theorem 2.10, we can obtain $\overline{T}T^{\{2\}} = T^{\{2\}}T\overline{T}T^{\{2\}}$ and $T^{\{2\}}\overline{T} = T^{\{2\}}\overline{T}TT^{\{2\}}$. Let k be the index of $\overline{T}$. Then

$$0 = \overline{T}^k(I - \overline{T}B) = \overline{T}^k\left[I - \overline{T}T^{\{2\}}(I + \delta TT^{\{2\}})^{-1}\right] = \overline{T}^k(I - TT^{\{2\}})(I + \delta TT^{\{2\}})^{-1}$$

and hence $\overline{T}^k(I - TT^{\{2\}}) = 0$. Similarly, it follows from $(I - B\overline{T})\overline{T}^k = 0$ that $(I - T^{\{2\}}T)\overline{T}^k = 0$. Conversely, if $\overline{T}T^{\{2\}} = T^{\{2\}}T\overline{T}T^{\{2\}}$ and $T^{\{2\}}\overline{T} = T^{\{2\}}\overline{T}TT^{\{2\}}$, then $B\overline{T} = \overline{T}B$. Hence by $\overline{T}^k(I - TT^{\{2\}}) = 0$ or by $(I - T^{\{2\}}T)\overline{T}^k = 0$, we can get $\overline{T}^k = \overline{T}^k B\overline{T}$. Therefore, B is the Drazin inverse of $\overline{T}$. □

As an application, we can obtain Theorem 2.11 in [8] and [9].

Corollary 2.13 ([8, Theorem 2.11], [9, Theorem 2.11])**.** *Let $T \in B(X)$ with the Drazin inverse $T^D \in B(X)$ and $\delta T \in B(X)$ with $\|\delta TT^D\| < 1$. Then*

$$B = T^D(I + \delta TT^D)^{-1} = (I + T^D\delta T)^{-1}T^D$$

is the Drazin inverse of $\overline{T} = T + \delta T$ if and only if the following statements hold:

(1) $\overline{T}T^D = T^DT\overline{T}T^D$, $T^D\overline{T} = T^D\overline{T}TT^D$, *and*

(2) *there exists a positive integer $k \in N$ such that $\overline{T}^k(I - TT^D) = 0$.*

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