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ESSENTIAL NORM OF THE COMPOSITION OPERATORS ON THE GENERAL SPACES $H_{\omega,p}$ OF HARDY SPACES

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Abstract. We obtain estimates for the essential norm of the composition operators acting on the general spaces $H_{\omega,p}$ of Hardy spaces. Our characterization is given in terms of generalized Nevanlinna counting functions.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions on $\mathbb{D}$. For $0 < p < \infty$, the Hardy space H^p is the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0 \le r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right\}^{1/p}.$$

For $-1 < \alpha < \infty$ and $0 < p < \infty$, the classical weighted Bergman space $\mathcal{A}^p_\alpha$ consists of functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{\mathcal{A}^p_\alpha} = (\alpha + 1) \int_{\mathbb{D}} \left| f(z) \right|^p \left(1 - |z|^2\right)^\alpha dA(z) < \infty,$$

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where $dA(z)$ is the area measure on $\mathbb{D}$. The following generalization of the Littlewood–Paley formula was first used by Stanton [11]:

$$\|f\|_{H^p}^p = \left|f(0)\right|^p + \frac{p^2}{2}\int_{\mathbb{D}} \left|f(z)\right|^{p-2}\left|f'(z)\right|^2\left(1-|z|^2\right)^2 dA(z). \tag{1.1}$$

There is also an analogue (see [10, Lemma 2.3]):

$$\|f\|_{\mathcal{A}_\alpha^p}^p \asymp \left|f(0)\right|^p + \int_{\mathbb{D}} \left|f(z)\right|^{p-2}\left|f'(z)\right|^2\left(1-|z|^2\right)^{\alpha+2} dA(z). \tag{1.2}$$

For two positive real-valued functions f_1 and f_2, we write $f_1 \preceq f_2$ if there exists a positive constant C independent of the argument such that $f_1 \leq Cf_2$. Similarly, $f_1 \asymp f_2$ means that $f_1 \preceq f_2$ and that $f_2 \preceq f_1$.

In the present article, *weight function* means a positive integrable function $\omega \in C^2[0,1)$ which is radial, $\omega(z) = \omega(|z|)$. In order to state our results, first let us generalize [4, Definition 1.1].

Definition 1.1. For $0 < p < \infty$, a weight function ω is called *admissible* if

- (ω_1) ω is nonincreasing;
- (ω_2) $\frac{\omega(r)}{(1-r)^{(1+\delta)\frac{p}{2}}}$ is nondecreasing for some $\delta > 0$;
- (ω_3) $\lim_{r\to 1^-}\omega(r) = 0$;
- (ω_4) one of the two properties of convexity is fulfilled:
  - (i) ω is convex and $\lim_{r\to 1}\omega'(r) = 0$, or
  - (ii) ω is concave.

If ω satisfies conditions (ω_1)–(ω_3) and (ω_4)(i) (resp., (ω_4)(ii)), then we say that ω is (i)-*admissible* (resp., (ii)-*admissible*).

In view of results (1.1) and (1.2), the general space $H_{\omega,p}$ of the Hardy space is defined as follows (see [5]). For a weight function ω, $H_{\omega,p}$ denotes the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\omega,p}^p = \left|f(0)\right|^p + p^2\int_{\mathbb{D}} \left|f(z)\right|^{p-2}\left|f'(z)\right|^2\omega(z)\, dA(z) < \infty.$$

It is worth pointing out that, for $p \geq 2$, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$, then

$$\|f\|_{\omega,p}^p \asymp \sum_{n=0}^{\infty} |a_n|^p \omega_n, \tag{1.3}$$

where $\omega_0 = 1$ and, for $n \geq 1$,

$$\omega_n = 2\pi n^p \int_0^1 r^{pn-p+1}\omega(r)\, dr.$$

This is because we first use the Holder inequality to obtain

$$\begin{aligned}\left|\sum_{n=1}^{\infty} a_n z^n\right|^{p-2} &\leq \left\{\sum_{n=1}^{\infty} |na_n z^n|^p\right\}^{\frac{p-2}{p}} \left\{\sum_{n=1}^{\infty} \frac{1}{n^{\frac{p}{p-1}}}\right\}^{\frac{(p-1)(p-2)}{p}} \\ &\leq C\left\{\sum_{n=1}^{\infty} n^p|a_n|^p|z|^{(n-1)p}\right\}^{1-\frac{2}{p}}.\end{aligned}$$

Subsequently, since $\varphi(t) = t^p$ is convex for $p \geq 2$, by the Jensen inequality we have

$$\Big|\sum_{n=1}^{\infty} na_n z^{n-1}\Big|^2 \leq \Big\{\sum_{n=1}^{\infty} |na_n z^{n-1}|^p\Big\}^{\frac{2}{p}}.$$

Thus,

$$\begin{aligned}\|f\|_{\omega,p}^p &\leq |a_0|^p + C\int_{\mathbb{D}} \sum_{n=1}^{\infty} n^p |a_n|^p |z|^{(n-1)p}\omega(z)\, dA(z)\\ &= |a_0|^p + C\int_0^1 \sum_{n=1}^{\infty} n^p |a_n|^p r^{(n-1)p+1}\omega(r)\, dr.\end{aligned}$$

Conversely, let Δ be the Laplacian since

$$\Delta\big|f(z)\big|^p = p^2\big|f(z)\big|^{p-2}\big|f'(z)\big|^2, \tag{1.4}$$

and then whenever $f(z) \neq 0$, we have

$$r\frac{d}{dr}\Big(\frac{1}{2\pi}\int_0^{2\pi}\big|f(re^{i\theta})\big|^p\, d\theta\Big) = \frac{p^2}{2}\int_{|z|<r}\big|f(z)\big|^{p-2}\big|f'(z)\big|^2\, dA(z) \tag{1.5}$$

(see[12]). By (1.5), the proof of the lower estimate is straightforward (see [5]). The space $H_{\omega,p}$ is linear by virtue of (1.4). For $p \geq 2$, Lee in [5, Corollary 2.8] proved that every function in $H_{\omega,p}$ is the quotient of two bounded functions in $H_{\omega,p}$.

For $p = 2$, the space $H_{\omega,p}$ is the weighted Hilbert space H_ω (see [4]). Suppose that $p = 2$, $\omega_\alpha(r) = (1 - r^2)^\alpha$ for $\alpha > -1$, and denote $H_{\omega_\alpha,2}$ by H_α. Then the space H_1 can be identified with the Hardy space H^2. In the case $0 \leq \alpha < 1$, H_α is precisely the Dirichlet space $\mathcal{D}_\alpha$, and H_0 corresponds to the classical Dirichlet space $\mathcal{D}$.

An important ingredient in our study is the use of $N_{\varphi,\omega}$, the generalized Nevanlinna counting function associated with φ, ω, which is defined as follows. For a nonconstant analytic self-map φ of $\mathbb{D}$ and a weight ω, the generalized Nevanlinna counting function associated to φ, ω is defined by

$$N_{\varphi,\omega}(\xi) = \sum_{\varphi(z)=\xi, z\in\mathbb{D}} \omega(z), \quad \xi \in \mathbb{D}\diagdown\big\{\varphi(0)\big\}.$$

Note that $N_{\varphi,\omega}(\xi) = 0$ when $\xi \notin \varphi(\mathbb{D})$. By convention, we define $N_{\varphi,\omega}(z) = 0$ when $z = \varphi(0)$. In the special case when $\omega(r) = \log\frac{1}{r}$, $r \in [0, 1)$, $N_{\varphi,\omega} = N_\varphi$ is the usual Nevanlinna counting function associated to φ. The generalized Nevanlinna counting function is considered in the special case of weighted Bergman spaces with standard weights (see, e.g., [7]). By the following general change-of-variable formula, and in view of generalized Nevanlinna counting functions, we obtain an equivalent form of the norm on $H_{\omega,p}$.

Lemma 1.2 ([1, Proposition 2.1]). *Let φ be a nonconstant analytic function in $\mathbb{D}$, and let u, ν be nonnegative measurable functions on $\mathbb{C}$ with respect to area measure. Then*

$$\int_{\mathbb{D}} (u\circ\varphi)\nu|\varphi'|^2\, dA = \int_{\varphi(\mathbb{D})} u(\xi)\Big(\sum_{\varphi(z)=\xi}\nu(z)\Big)\, dA(\xi). \tag{1.6}$$

Replacing $u(\xi) = |\xi|^{p-2}$, $\nu(z) = \omega(z)$, and $\varphi(z) = f(z)$ in (1.6) with a nonconstant function $f \in H_{\omega,p}$, we have

$$\|f\|_{\omega,p}^p = \big|f(0)\big|^p + p^2 \int_{f(\mathbb{D})} |\xi|^{p-2} N_{\omega,f}(\xi)\, dA(\xi).$$

Every analytic self-map φ of $\mathbb{D}$ induces a composition operator C_φ on $\mathcal{H}(\mathbb{D})$, defined by $(C_\varphi f)(z) = f(\varphi(z))$. (Some results of the composition operators can be found in [2] and [9], for example.) Pau and Perez studied the essential norm of composition operators on weighted Dirichlet spaces in [6]. Hassanlou generalized the resullts of [6] to weighted Hilbert spaces of analytic functions [3]. The purpose of the present paper is to generalize the results of [4] and [6] to the $H_{\omega,p}$ spaces and to present the characterization of the essential norm of the composition operator on the $H_{\omega,p}$ space by using the generalized Nevanlinna counting function. Note that, throughout the remainder of this paper, constants are denoted by C; they are positive and may differ from one occurrence to the other.

2. Preliminaries

In this section we give some lemmas which will be used in our characterizations. Per the following lemma, the generalized Nevanlinna counting function has the submean value property.

Lemma 2.1 ([4, Lemmas 2.2, 2.3])**.** *Let ω be an admissible weight, and let φ be an analytic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$. Then for every $r > 0$ and every $z \in \mathbb{D}$ such that $D(z, r) \subset \mathbb{D} \diagdown D(0, \frac{1}{2})$, we have*

$$N_{\varphi,\omega}(z) \le \frac{2}{r^2} \int_{D(z,r)} N_{\varphi,\omega}(\xi)\, dA(\xi),$$

where $D(z, r)$ denotes the disk of radius r centered at z.

By the same method used in the proof of [4, Lemma 2.5], we have the following lemma.

Lemma 2.2. *Let ω be a weight satisfying (ω_1) and (ω_2). For $a \in \mathbb{D}$, define*

$$f_a(z) = \frac{1}{\sqrt[p]{\omega(a)}} \frac{(1 - |a|^2)^{1+\delta}}{(1 - \bar{a}z)^{1+\delta}}, \quad z \in \mathbb{D}.$$

Then $\|f_a\|_{H_{\omega,p}\preceq 1}$.

Proof. By virtue of (ω_1) and (ω_2), $f_a(0) = \frac{(1-|a|^2)^{1+\delta}}{\sqrt[p]{w(a)}}$ is bounded by $\frac{2^{1+\delta}}{\sqrt[p]{w(0)}}$. Using simple computation,we have $f_a'(z) = \frac{(1+\delta)\bar{a}}{\sqrt[p]{w(a)}} \frac{(1-|a|^2)^{1+\delta}}{(1-\bar{a}z)^{2+\delta}}$. Thus,

$$\begin{aligned} &\int_{\mathbb{D}} \big|f_a(z)\big|^{p-2} \big|f_a'(z)\big|^2 \omega(z)\, dA(z) \\ &\quad \le \frac{(1+\delta)^2 |a|^2}{\omega(a)} \big(1 - |a|^2\big)^{(1+\delta)p} \int_{\mathbb{D}} \frac{\omega(z)}{|1 - \bar{a}z|^{p(1+\delta)+2}}\, dA(z). \end{aligned}$$

On one hand, applying (ω_1) and the following well-known estimate (see [12, Theorem 1.12]):

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^c}{|1-\bar{a}z|^{2+c+d}} \asymp \frac{1}{(1-|a|^2)^d}, \quad d>0, c>-1, \tag{2.1}$$

we obtain

$$\begin{aligned}\int_{|z|>|a|} \frac{\omega(z)}{|1-\bar{a}z|^{p(1+\delta)+2}}\, dA(z) &\le \omega(a) \int_{\mathbb{D}} \frac{1}{|1-\bar{a}z|^{p(1+\delta)+2}}\, dA(z) \\ &\asymp \frac{\omega(a)}{(1-|a|^2)^{p(1+\delta)}}.\end{aligned}$$

On the other hand, by (ω_2) and (2.1), we have

$$\begin{aligned}\int_{|z|\le|a|} \frac{\omega(z)}{|1-\bar{a}z|^{p(1+\delta)+2}}\, dA(z) &\le \frac{\omega(a)}{(1-|a|^2)^{\frac{p}{2}(1+\delta)}} \int_{|z|\le|a|} \frac{(1-|z|^2)^{\frac{p}{2}(1+\delta)}}{|1-\bar{a}z|^{p(1+\delta)+2}}\, dA(z) \\ &\asymp \frac{\omega(a)}{(1-|a|^2)^{p(1+\delta)}}.\end{aligned}$$

□

Lemma 2.3. *Let σ_a be the automorphism of the unit disk given by*

$$\sigma_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z\in\mathbb{D},$$

and let φ be an analytic self-map of $\mathbb{D}$. If ω satisfies (ω_1) and (ω_2), then

$$\omega(z) \asymp \omega\big(\sigma_{\varphi(0)}(z)\big).$$

Proof. The proof is similar to the proof of [4, Lemma 2.1]. □

Throughout this paper, by Lemma 2.3, we will assume that $\varphi(0)=0$.

3. Main results

The main result of the paper will concern the essential norm of C_φ on $H_{\omega,p}$. Nevertheless, we have to ensure the boundedness of C_φ.

For the case of (i)-admissible weights, C_φ is a bounded operator on $H_{\omega,p}$. Indeed, if we assume that $\varphi_r(z)=\varphi(rz)$, for every $0<r<1$ and $\frac{d^2\omega}{dr^2}=\sigma$, then by the proof of Lemma 2.2 in [4], we have

$$N_{\varphi,\omega}(z) \le \int_0^1 N_{\varphi_r}(z)\sigma(r)\, dr \le 2N_{\varphi,\omega}(z). \tag{3.1}$$

Using the classical Littlewood inequality, for the function $r^{-1}\varphi_r$, we get $N_{\varphi_r}(z) \le \log(\frac{r}{|z|})$. So by (3.1), ω_1 and ω_3, we get

$$N_{\varphi,\omega}(z) \le \int_0^1 N_{\varphi_r}(z)\sigma(r)\, dr = \int_{|z|}^1 N_{\varphi_r}(z)\sigma(r)\, dr \le \int_{|z|}^1 \log\Big(\frac{r}{|z|}\Big)\sigma(r)\, dr \le 2\omega(z).$$

Therefore, using the change-of-variable formula (1.6), since $N_{\varphi,\omega}(z) = 0$ when $z \notin \varphi(\mathbb{D})$, we have

$$\begin{aligned}
\|C_\varphi f\|^p_{H_{\omega,p}} &= \big|f\big(\varphi(0)\big)\big|^p + p^2\int_{\mathbb{D}} \big|f\big(\varphi(z)\big)\big|^{p-2}\big|f'\big(\varphi(z)\big)\big|^2\big|\varphi'(z)\big|^2\omega(z)\,dA(z)\\
&= \big|f(0)\big|^p + p^2\int_{\varphi(\mathbb{D})} \big|f(z)\big|^{p-2}\big|f'(z)\big|^2 N_{\varphi,\omega}(z)\,dA(z)\\
&= \big|f(0)\big|^p + p^2\int_{\mathbb{D}} \big|f(z)\big|^{p-2}\big|f'(z)\big|^2 N_{\varphi,\omega}(z)\,dA(z)\\
&\le \big|f(0)\big|^p + 2p^2\int_{\mathbb{D}} \big|f(z)\big|^{p-2}\big|f'(z)\big|^2 \omega(z)\,dA(z)\\
&\le 2\|f\|^p_{H_{\omega,p}},
\end{aligned} \tag{3.2}$$

which shows that for the (i)-admissible weight ω, C_φ is a bounded operator on $H_{\omega,p}$.

For the case of (ii)-admissible weights, we have the following theorem.

Theorem 3.1. *Let ω be a (ii)-admissible weight. Then C_φ is bounded on $H_{\omega,p}$ if and only if*

$$\sup_{|z|<1} \frac{N_{\varphi,\omega}(z)}{\omega(z)} < \infty. \tag{3.3}$$

Proof. Assume that (3.3) holds. It is clear that, in a way similar to (3.2), for each (ii)-admissible weight, C_φ is a bounded operator on $H_{\omega,p}$.

Conversely, assume that C_φ is a bounded operator on $H_{\omega,p}$. Let f_a be the test function defined as Lemma 2.2. In the case where $|a|$ is close enough to 1, $D(a, \frac{1-|a|}{2}) \subset \mathbb{D} \setminus D(0, \frac{1}{2})$. Using the change-of-variable formula (1.6), Lemma 2.1, and the well-known fact that $|1 - \bar{a}z| \asymp (1 - |a|)$ for $z \in D(a, \frac{1-|a|}{2})$, we have

$$\begin{aligned}
\|C_\varphi f_a\|^p_{H_{\omega,p}} &\ge p^2\int_{\mathbb{D}} \big|f_a\big(\varphi(z)\big)\big|^{p-2}\big|f_a'\big(\varphi(z)\big)\big|^2\big|\varphi'(z)\big|^2\omega(z)\,dA(z)\\
&= p^2\int_{\varphi(\mathbb{D})} \big|f_a(z)\big|^{p-2}\big|f_a'(z)\big|^2 N_{\varphi,\omega}(z)\,dA(z)\\
&\ge C\frac{(1-|a|^2)^{p(1+\delta)}}{\omega(a)}\int_{\varphi(\mathbb{D})} \frac{N_{\varphi,\omega}(z)}{|1-\bar{a}z|^{p(1+\delta)+2}}\,dA(z)\\
&\ge C\frac{(1-|a|^2)^{p(1+\delta)}}{\omega(a)}\int_{D(a,\frac{1-|a|}{2})} \frac{N_{\varphi,\omega}(z)}{|1-\bar{a}z|^{p(1+\delta)+2}}\,dA(z)\\
&\ge C\frac{1}{(1-|a|^2)^2\omega(a)}\int_{D(a,\frac{1-|a|}{2})} N_{\varphi,\omega}(z)\,dA(z)\\
&\ge C\frac{N_{\varphi,\omega}(a)}{\omega(a)},
\end{aligned} \tag{3.4}$$

where C does not depend on the point a. This gives

$$\sup_{a\in\mathbb{D}}\frac{N_{\varphi,\omega}(a)}{\omega(a)}\leq C\sup_{a\in\mathbb{D}}\|C_\varphi f_a\|^p_{H_{\omega,p}}\leq C\|C_\varphi\|^p\sup_{a\in\mathbb{D}}\|f_a\|^p_{H_{\omega,p}}.$$

Thus (3.3) holds by virtue of Lemma 2.2 and the boundedness of C_φ on $H_{\omega,p}$. $\square$

Recall that the essential norm $\|T\|_e$ of a bounded linear operator T is its distance (in the operator norm) from compact operators; that is,

$$\|T\|_e=\inf_K\|T-K\|,$$

where K is compact. In [6, Theorem 3.2], Pau and Perez estimated the essential norm of C_φ on $\mathcal{D}_\alpha$, $0<\alpha<1$, as follows:

$$\|C_\varphi\|_e^2\asymp\limsup_{|z|\to 1}\frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha}. \tag{3.5}$$

This result was later generalized by Hassanlou to the weighted Hilbert spaces H_ω in [3] as well:

$$\|C_\varphi\|_e^2\asymp\limsup_{|z|\to 1}\frac{N_{\varphi,\omega}(z)}{\omega(z)}, \tag{3.6}$$

where ω is an admissible weight.

We generalize the results (3.5) and (3.6) for the spaces $H_{\omega,p}$ in the following theorem.

Theorem 3.2. *Let ω be an admissible weight, and let C_φ be a bounded operator on $H_{\omega,p}$. Then*

$$\|C_\varphi\|_e^p\asymp\limsup_{|z|\to 1}\frac{N_{\varphi,\omega}(z)}{\omega(z)}. \tag{3.7}$$

Proof. For the lower estimate, we use the same technique used in the proof of [8, Theorem 2.1]. Suppose that

$$L:=\limsup_{|z|\to 1}\frac{N_{\varphi,\omega}(z)}{\omega(z)}.$$

For an analytic function $f(z)=\sum_{n=0}^\infty a_nz^n$ in $\mathbb{D}$, let

$$T_kf(z)=\sum_{n=0}^k a_nz^n,\qquad R_kf(z)=\sum_{n=k+1}^\infty a_nz^n.$$

Since T_k is compact and C_φ is bounded, we have

$$\|C_\varphi\|_e=\big\|C_\varphi(T_k+R_k)\big\|_e\leq\|C_\varphi T_k\|_e+\|C_\varphi R_k\|_e\leq\|C_\varphi R_k\|,$$

for each $k \in \mathbb{N}$. It follows that

$$\begin{aligned}\|C_\varphi\|_e^p &\leq \liminf_{k\to\infty} \|C_\varphi R_k\|^p \\ &\leq \liminf_{k\to\infty} \sup_{\|f\|_{H_{\omega,p}}\leq 1} \big\|(C_\varphi R_k)(f)\big\|^p \\ &= p^2 \liminf_{k\to\infty} \sup_{\|f\|_{H_{\omega,p}}\leq 1} \int_{\mathbb{D}} \big|(R_k f)(z)\big|^{p-2} \big|(R_k f)'(z)\big|^2 N_{\varphi,\omega}(z)\, dA(z) \\ &\leq p^2 L \liminf_{k\to\infty} \sup_{\|f\|_{H_{\omega,p}}\leq 1} \int_{\mathbb{D}} \big|(R_k f)(z)\big|^{p-2} \big|(R_k f)'(z)\big|^2 \omega(z)\, dA(z) \\ &= p^2 L \liminf_{k\to\infty} \sup_{\|f\|_{H_{\omega,p}}\leq 1} \|R_k f\|_{H_{\omega,p}}^p \\ &\leq CL.\end{aligned}$$

Proof of the upper estimate is similar to [6, Theorem 3.2]. Consider the functions f_a defined in Lemma 2.2. Applying (ω_2), we conclude that $f_a \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Hence for every compact operator K on $H_{\omega,p}$, we have $\lim_{|a|\to 1^-} \|K f_a\|_{H_{\omega,p}} = 0$. Thus

$$\begin{aligned}\|C_\varphi - K\| &\geq \limsup_{|a|\to 1} \|C_\varphi f_a - K f_a\|_{H_{\omega,p}} \\ &\geq \limsup_{|a|\to 1} \|C_\varphi f_a\|_{H_{\omega,p}} - \limsup_{|a|\to 1} \|K f_a\|_{H_{\omega,p}} \\ &= \limsup_{|a|\to 1} \|C_\varphi f_a\|_{H_{\omega,p}}.\end{aligned}$$

Moreover, it follows that

$$\|C_\varphi\|_e^p \geq \limsup_{|a|\to 1} p^2 \int_{\mathbb{D}} \big|f_a\big(\varphi(z)\big)\big|^{p-2} \big|f_a'\big(\varphi(z)\big)\big|^2 \big|\varphi'(z)\big|^2 \omega(z)\, dA(z)$$

since $|f_a(\varphi(0))| \to 0$ as $|a| \to 1$. Therefore, from (3.4), it holds that

$$\|C_\varphi\|_e^p \geq C \limsup_{|a|\to 1} \frac{N_{\varphi,\omega}(a)}{\omega(a)}.$$

□

Corollary 3.3. *Let ω be an admissible weight. Then C_φ is compact on $H_{\omega,p}$ if and only if*

$$\lim_{|z|\to 1} \frac{N_{\varphi,\omega}(z)}{\omega(z)} = 0.$$

In the following example we are going to characterize the weight function ω_μ associated to μ, which ensures that $\mathcal{A}_\mu^p(\mathbb{D}) \supseteq H_{\omega_\mu,p}$, for $p \geq 2$.

Example 3.4. For $0 < p < \infty$ and a continuous function $\mu : [0, 1) \to (0, \infty)$ such that $\mu \in L^1(0, 1)$, the weighted Bergman space $\mathcal{A}_\mu^p(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$ such that

$$\|f\|_\mu^p = \int_{\mathbb{D}} \big|f(z)\big|^p \mu\big(|z|\big)\, dA(z) < \infty.$$

For $p \geq 2$, by the Jensen inequality, a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathcal{A}_\mu^p(\mathbb{D})$ if and only if

$$\|f\|_\mu^p \asymp \sum_{n\geq 0} |a_n|^p \mu_n < \infty,$$

where

$$\mu_n = \int_0^1 r^{np+1}\mu(r)\, dr, \quad n \geq 0.$$

Using the same techniques used in [4], the weight function associated to μ is defined by

$$\omega_\mu(r) = \int_r^1 (t-r)\mu(t)\, dt.$$

Since $\mu \in L^1(0,1)$, we deduce that

$$\lim_{r\to 1^-} \omega'_\mu(r) = \lim_{r\to 1^-} -\int_r^1 \mu(t)\, dt = 0.$$

Note that $\omega''_\mu(r) = \mu(r)$. We have

$$\frac{\mu_{n+1}}{(n+1)^p} \preceq \int_0^1 r^{np+1}\omega_\mu(r)\, dr, \quad n \geq 0.$$

Thus for every $f \in \mathcal{A}_\mu^p(\mathbb{D})$, we have

$$\|f\|_\mu^p \preceq \|f\|_{\omega_\mu,p}^p,$$

which ensures that $\mathcal{A}_\mu^p(\mathbb{D}) \supseteq H_{\omega_\mu,p}$.

Moreover, the weight ω_μ always satisfied $(\omega_1), (\omega_3)$, and (i). Thus ω_μ is (i)-admissible if and only if it satisfies (ω_2).

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