Ann. Funct. Anal. 9 (2018), no. 2, 166–179
https://doi.org/10.1215/20088752-2017-0034
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

CONVOLUTION-CONTINUOUS BILINEAR OPERATORS ACTING ON HILBERT SPACES OF INTEGRABLE FUNCTIONS

EZGI ERDOĞAN,[1] JOSÉ M. CALABUIG,[2] and ENRIQUE A. SÁNCHEZ PÉREZ[2*]

Communicated by M. D. Contreras

Abstract. We study bilinear operators acting on a product of Hilbert spaces of integrable functions—zero-valued for couples of functions whose convolution equals zero—that we call *convolution-continuous bilinear maps*. We prove a factorization theorem for them, showing that they factor through ℓ^1. We also present some applications for the case when the range space has some relevant properties, such as the Orlicz or Schur properties. We prove that ℓ^1 is the only Banach space for which there is a norming bilinear map which equals zero exactly in those couples of functions whose convolution is zero. We also show some examples and applications to generalized convolutions.

1. Introduction

A main outcome of classical harmonic analysis is the deep result that establishes that the convolution $*$ of two functions in $L^2(G)$ of a compact group G has absolutely convergent Fourier series, and functions having this property are exactly the ones that can be written as a convolution of two functions in $L^2(G)$. Moreover, if we write $\mathfrak{R}(G)$ for the space of functions with absolutely convergent Fourier series and we define its norm as the ℓ^1-norm of the corresponding Fourier coefficients, then the resulting space is isometric to ℓ^1. Its proof goes back to Hewitt and former results, and can already be found in the classical book by Hewitt and Ross [12, Corollary 34.7]. Also, the characterization of convolution

Received Jan. 9, 2017; Accepted Mar. 2, 2017.
First published online Dec. 6, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47H60; Secondary 43A25, 46G25.
Keywords. convolution, bilinear operator, factorization, Fourier transform, summability.

by means of its properties as a bilinear operator acting on spaces of continuous functions is due to Edwards [7], who observed in his classical paper that certain commutation properties for a positive bilinear map imply that it is in fact the convolution operator. On the other hand, some new linearization procedures for analyzing summability and other properties of multilinear maps have been recently introduced. Concretely, factorization of bilinear maps through linear operators acting on classical Banach spaces has proved to be a useful technique for such an analysis (see, e.g., [13] and the references therein).

Bringing together these ideas, in the present paper we study bilinear maps acting on products of L^2-spaces that equal zero on convolution-null couples of functions. We will say that these bilinear operators are $*$-continuous, providing classical and current examples. We show that this apparently weak requirement for a bilinear map implies that it has to be necessarily the composition of convolution with a linear operator. Our main result (Theorem 3.1) proves this fact, opening in this way the door for providing some applications that have the specificity of convolution as an essential foundation.

Among them, we prove a characterization of ℓ^1—up to isomorphisms—as the unique space in which there exists a norming bilinear map with the same null-couples of that convolution (see Corollary 4.5). We also give some results about summability and topological properties of $*$-continuous maps when some Banach space properties are assumed on the range space Y. In particular, we consider finite cotype, the Orlicz property, and the Schur property for Y, showing that under these assumptions both summability and compactness properties improve. Finally, we explain some consequences of our results in a more applied context. We provide some applications to what are known as generalized convolutions of integral transforms of Fourier type. This topic has been recently developed to provide a unified context for studying integral equations of convolution type (see [9] and the references therein).

2. Preliminaries

Let X_1, X_2, Y be Banach spaces. We will say that a continuous bilinear map $B : X_1 \times X_2 \to Y$ is *weakly compact* if the set $B(B_{X_1} \times B_{X_2}) \subset Y$ is relatively weakly compact. As usual, we will write B_Y for the closed unit ball of the Banach space Y. If $A \subseteq Y$, we will denote by $\overline{A}$ the (norm) closure of A in Y. We will say that the bilinear map B is *equivalently norming* (or *norming* for short) for Y if there is a constant $k > 0$ such that

$$B_Y \subseteq k\overline{B(B_{X_1} \times B_{X_2})}.$$

In the same direction, we will consider that a set $A \subseteq (B_{X_1} \times B_{X_2})$ is norming if

$$B_Y \subseteq k\overline{B(A)}.$$

Let $1 \leq p, q \leq \infty$, and let X and Y be Banach spaces. As usual, we write $L(X, Y)$ for the space of linear and continuous operators from X to Y. Recall that an operator $T : X \to Y$ is said to be (p, q)-*summing* if there is a constant

$k > 0$ such that for every $x_1, \dots, x_n \in X$,

$$\Big(\sum_{i=1}^{n} \big\|T(x_i)\big\|_Y^p\Big)^{1/p} \le k \sup_{x^* \in B_{X^*}} \Big(\sum_{i=1}^{n} \big|\langle x_i, x^* \rangle\big|^q\Big)^{1/q}.$$

We will write $\Pi_{p,q}(X, Y)$ ($\Pi_p(X, Y)$ if $p = q$) for the operator ideal of all (p, q)-summing operators (see [5, Section 32]).

Throughout this article, we will repeatedly consider bilinear maps $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$. A relevant example of such a bilinear operator is the convolution map that is defined for Banach spaces Y containing the space of absolutely convergent Fourier series $\mathfrak{R}(\mathbb{T})$. If G is a locally compact Abelian group with Haar measure η, convolution is defined by the formula

$$f * g(x) = \int_G f(x - y) g(y)\, d\eta(y), \quad f, g \in L^1(G),$$

and is well defined and continuous among several domain and range function spaces of integrable functions. The study of the properties of convolution as a bilinear map—and in fact, its characterization by means of these properties—goes back to Edwards [7]. In that early paper, Edwards showed that the behavior with respect to invariance with respect to translations and other essential properties for a positive bilinear map actually determines that it must be the convolution operator (see [7, Theorem 1]; see also [8, Section 3.1]). That article assumed the domain space to be a space of continuous functions with compact support, but the main idea is the same one that we develop here; considerably more is now known about the topic.

We will say that a bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$ is *continuous with respect to convolution* (or $*$-*continuous* for short) if for every $f, g \in L^2(\mathbb{T})$,

$$f * g = 0 \quad \text{implies that} \quad B(f, g) = 0.$$

Also, we will say that such a bilinear map is $*$-*equivalent* if $f * g = 0$ if and only if $B(f, g) = 0$ for every $f, g \in L^2(\mathbb{T})$.

Let us recall some Banach space properties that we will use in the article; ℓ^1 satisfies all of them. A Banach space Y has the Dunford–Pettis property if every weakly compact linear operator $T : Y \to Z$ into another Banach space Z—or only for $Z = c_0$—is completely continuous (i.e., it transforms weakly compact sets in Y into norm-compact sets in Z; see, e.g., [6, Section VI]). A Banach space Y has the Schur property if weakly convergent sequences and norm-convergent sequences coincide in it. The results on this property that we use can be found in [14]; the interested reader can find updated information about variants of this property in [15]. A Banach space has the Orlicz property if the identity map in it is $(2, 1)$-summing.

Let us finish this section by providing some information about the space $\mathfrak{R}(G)$ of functions with absolutely summing Fourier coefficients. Consider the family $\mathfrak{R}(G)$ of the linear combinations (closure) of continuous positive definite functions on a topological group G. Recall that a function ϕ defined on G is said to be *positive definite* if the inequality $\sum_{n,m=1}^{N} c_n \overline{c_m} \phi(x_n - x_m) \ge 0$ holds for every choice of $x_1, x_2, \dots, x_N$ in G and for every choice of complex numbers $c_1, c_2, \dots, c_N$. It is known that this family and the space of functions with absolutely summing Fourier coefficients are exactly the same for a compact group G (see [12, Theorem 34.13]).

For a compact group G, $L^2(G) * L^2(G)$ gives the space $\mathfrak{R}(G)$ (see [12, Corollary 34.16]). Moreover, if G is a compact Abelian group, $L^2(G) * L^2(G)$ gives the space of functions which has absolutely convergent Fourier series. Under pointwise operations and with an appropriate norm, $\mathfrak{R}(G)$ is a commutative Banach algebra (see [12, Remark 34.34(a)]) with unit, and so for the compact Abelian group G with character group Γ we get that $\mathfrak{R}(G)$ is isomorphic to the Banach algebra $\ell_1(\Gamma)$ via the Fourier transform (see also [12, Corollary 34.7]). Recall that the Fourier transform $\widehat{\ }$ is a linear isometry from $L^2(G)$ onto $L^2(\Gamma)$ and that the inverse Fourier transform is a linear isometry from $L^2(\Gamma)$ onto $L^2(G)$; these two transformations are inverses of each other (see [12, Theorem 31.18]).

In particular, consider the compact Abelian group G as $\mathbb{T}$—the real line mod 2π. Then $\mathfrak{R}(\mathbb{T})$ is the unital Banach algebra known as the *Wiener algebra*. By the isomorphism given by Fourier transform, it is isomorphic to the Banach algebra $\ell^1(\mathbb{Z})$.

We note here the obvious fact that the range of a bilinear map is not in general a linear space. However, concerning this aspect, convolution is also a special bilinear map. Indeed, as we said before, for a compact Abelian group G we have that $L^2(G) * L^2(G) = \mathfrak{R}(G)$ holds, and so $L^2(G) * L^2(G)$ is a linear space. Let us recall also the following known fact about the normability of convolution for the algebra $\mathfrak{R}(G)$. This result is fundamental for our purposes.

Remark 2.1. Let G be a compact Abelian group. Convolution is norming for $\mathfrak{R}(G)$ with equivalence constants equal to 1. That is, $*(B_{L^2(G)} \times B_{L^2(G)}) = B_{\mathfrak{R}(G)}$. Following the notation of [12], recall that the norm in $\mathfrak{R}(G)$ is defined by

$$\|f\|_{\varphi_1} := \|\widehat{f}\|_1, \quad f \in \mathfrak{R}(G).$$

Proof. If $(f, g) \in B_{L^2(G)} \times B_{L^2(G)}$, we get $\|f * g\|_{\varphi_1} \leq \|f\|_2 \|g\|_2 \leq 1$ (see, e.g., [12, Theorem 34.14]). So, $f * g \in B_{\mathfrak{R}(G)}$. For the converse inequality, assume that $f \in B_{\mathfrak{R}(G)}$. Then there are $h, g \in L^2(G)$ such that $f = h * g$; this fact can be found in [12, Theorem 34.15(i)]. By using [12, Theorem 34.15(iv)], we also obtain that these functions h and g can be chosen in such a way that $1 \geq \|f\|_{\varphi_1} = \|g\|_2^2 = \|h\|_2^2$; that is, $(h, g) \in B_{L^2(G)} \times B_{L^2(G)}$. □

3. A commutative factorization diagram through convolution

This section is devoted to showing our main result: a factorization theorem for $*$-continuous bilinear operators. We start by providing some examples of this class of operators. For the sake of clarity, we will write our results for $\mathbb{T}$; the same proofs work for any compact Abelian group G with the usual changes.

(1) The first example is constructed by using translation-invariant linear operators, sometimes called *convolution operators* or *multipliers*—the operators that commute with translations. Because the bibliography on this topic is deep and wide, we mention here only the classical paper by Cowling and Fournier [4]. Thus, consider an operator $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ that

satisfies that $Tf * g = T(f * g)$. For example, we can take a convolution operator $T_k : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ with convolution kernel $k \in L^2(\mathbb{T})$, that is, $T_k f := k * f$. Consider now the bilinear map $B_k : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by convolution of $B_k(\cdot,\cdot) := T_k(\cdot) * (\cdot)$. Then we have

$$B_k(f,g) = T_k(f) * g = (k * f) * g = k * (f * g) = T_k(f * g),$$

for $f, g \in L^2(\mathbb{T})$. Although we note that for the convolution map defined in products of spaces of continuous functions, these arguments for translation-invariant bilinear maps can already be found in the paper by Edwards [7, proof of Proposition 1]. If $P(D)$ is a linear partial differential operator, then bilinear maps as $P(D)(f * g)$ are also usual in applications of harmonic analysis.

(2) Consider a Banach space Z and a Bochner 2-integrable function $\Phi : [0, 2\pi] \to Z$. Consider the vector-valued kernel bilinear operator $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Z$ defined by

$$B(f,g) := \int_0^{2\pi} \int_0^{2\pi} \Phi(x) f(x-y) g(y)\, dy\, dx.$$

Indeed, it can be written as $(f,g) \mapsto \int_0^{2\pi} \Phi(x)(f * g)(x)\, dx$, and then it is zero when $f * g = 0$.

Theorem 3.1. *Let $\mathbb{T}$ be the real line mod 2π, and let Y be an arbitrary Banach space. For a bilinear continuous operator $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$, the following statements are equivalent.*

(i) *B is $*$-continuous.*

(ii) *There is a linear and continuous map $T : \mathfrak{R}(\mathbb{T}) \to Y$ such that $B = T \circ *$.*

Proof. Assume first that B is $*$-continuous. Since the linear map $\check{}: \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ is an isometric isomorphism, we can define the bilinear operator

$$\widetilde{B}\big((a_n),(b_n)\big) := B\big((a_n)^{\vee},(b_n)^{\vee}\big), \quad (a_n),(b_n) \in \ell^2(\mathbb{Z}), \tag{3.1}$$

which clearly provides $\widetilde{B}(\widehat{f},\widehat{g}) = B(f,g)$ and the commutativity of the diagram

$$\begin{tikzcd} L^2(\mathbb{T}) \times L^2(\mathbb{T}) \arrow[r, "B"] \arrow[d, "\widehat{}\times\widehat{}"'] & Y \\ \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \arrow[ur, "\widetilde{B}"'] & \end{tikzcd} \tag{3.2}$$

Now, we prove that there is a bounded linear map $\widetilde{T} : \ell^1(\mathbb{Z}) \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \arrow[r, "\widetilde{B}"] \arrow[d, "\odot"'] & Y \\ \ell^1(\mathbb{Z}) \arrow[ur, "\widetilde{T}"'] & \end{tikzcd} \tag{3.3}$$

where $\odot$ is the pointwise product of sequences.

For each $N \in \mathbb{N}$, we define the linear map $\widetilde{T}_N : \ell^1(\mathbb{Z}) \to Y$ by

$$\widetilde{T}_N\big((a_n)\big) := \widetilde{B}\big((a_n), \chi_{[-N,N]\cap\mathbb{Z}}\big), \quad (a_n) \in \ell^1(\mathbb{Z}). \tag{3.4}$$

We claim that, since B is $*$-continuous, we have that $\widetilde{B}(\chi_{\{i\}}, \chi_{\{j\}}) = 0$ when $i \neq j$. Indeed, since

$$0 = \chi_{\{i\}} \cdot \chi_{\{j\}} = \widehat{\widecheck{\chi_{\{i\}}}} \cdot \widehat{\widecheck{\chi_{\{j\}}}} = (\widecheck{\chi_{\{i\}}} * \widecheck{\chi_{\{j\}}})\widehat{\ },$$

the $*$-continuity of B together with (3.2) gives $\widetilde{B}(\chi_{\{i\}}, \chi_{\{j\}}) = B(\widecheck{\chi_{\{i\}}}, \widecheck{\chi_{\{j\}}}) = 0$. Using this remark, it is easy to see that

$$\widetilde{T}_N\big((a_n)\big) = \sum_{|j|\leq N} a_j \widetilde{B}(\chi_{\{j\}}, \chi_{\{j\}}). \tag{3.5}$$

Therefore

$$\begin{aligned}
\big\|\widetilde{T}_N\big((a_n)\big)\big\|_Y &\leq \sum_{|j|\leq N} \big\|a_j \widetilde{B}(\chi_{\{j\}}, \chi_{\{j\}})\big\|_Y = \sum_{|j|\leq N} \big\|\widetilde{B}(a_j\chi_{\{j\}}, \chi_{\{j\}})\big\|_Y \\
&\leq \|\widetilde{B}\| \sum_{|j|\leq N} \|a_j\chi_{\{j\}}\|_{\ell^2(\mathbb{Z})} \|\chi_{\{j\}}\|_{\ell^2(\mathbb{Z})} \\
&= \|\widetilde{B}\| \sum_{|j|\leq N} |a_j| \leq \|\widetilde{B}\| \big\|(a_n)\big\|_{\ell^1(\mathbb{Z})},
\end{aligned}$$

and so $\widetilde{T}_N$ is continuous; in fact, the family $\{\widetilde{T}_N : N \in \mathbb{N}\}$ is uniformly bounded, since $\|\widetilde{T}_N\| \leq \|\widetilde{B}\|$ for all $N \in \mathbb{N}$. Moreover, for each fixed $(a_n) \in \ell^1(\mathbb{Z})$, $(\widetilde{T}_N((a_n)))$ is a Cauchy sequence in the Banach space Y, and so it is convergent. Indeed, for a given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\sum_{|j|>N} |a_j| < \varepsilon/\|\widetilde{B}\|$ for all $N \geq k$. By using (3.5), we have, for all $M > N \geq k$,

$$\begin{aligned}
\big\|\widetilde{T}_M\big((a_n)\big) - \widetilde{T}_N\big((a_n)\big)\big\|_Y &= \Big\|\sum_{|j|\leq M} a_j \widetilde{B}(\chi_{\{j\}}, \chi_{\{j\}}) - \sum_{|j|\leq N} a_j \widetilde{B}(\chi_{\{j\}}, \chi_{\{j\}})\Big\|_Y \\
&\leq \sum_{N<|j|\leq M} \big\|\widetilde{B}(a_j\chi_{\{j\}}, \chi_{\{j\}})\big\|_Y \\
&\leq \|\widetilde{B}\| \sum_{|j|>N} |a_j| < \varepsilon.
\end{aligned}$$

Therefore, by using the Banach–Steinhaus theorem—the uniform boundedness principle—the map $\widetilde{T} : \ell^1(\mathbb{Z}) \to Y$ given by

$$\widetilde{T}\big((a_n)\big) := \lim_{N\to\infty} \widetilde{T}_N\big((a_n)\big) \tag{3.6}$$

is linear and bounded. Finally, by the continuity of $\widetilde{B}$ and again (3.5), we have

$$\widetilde{B}\big((a_n), (b_n)\big) = \sum_{j=-\infty}^{\infty} a_j b_j \widetilde{B}(\chi_{\{j\}}, \chi_{\{j\}})$$

$$= \lim_{N\to\infty} \sum_{|j|\le N} a_j b_j \widetilde{B}(\chi_{\{j\}}, \chi_{\{j\}})$$
$$= \lim_{N\to\infty} \widetilde{T}_N\big((a_n b_n)\big) = \widetilde{T} \circ \odot\big((a_n), (b_n)\big),$$

and so the commutativity of (3.3) follows.

On the other hand, given a linear map $\widetilde{T} : \ell^1(\mathbb{Z}) \to Y$, we can use the fact that the Fourier transform $\widehat{\ } : \mathfrak{R}(\mathbb{T}) \to \ell^1(\mathbb{Z})$ is an isometric isomorphism (see [12, Corollary 34.7]) to define an operator $T : \mathfrak{R}(\mathbb{T}) \to Y$ by

$$T(f) := \widetilde{T}(\widehat{f}), \quad f \in \mathfrak{R}(\mathbb{T}). \tag{3.7}$$

This gives the factorization

$$\begin{tikzcd} \ell^1(\mathbb{Z}) \arrow[r, "\widetilde{T}"] \arrow[d, "\check{\ }"'] & Y \\ \mathfrak{R}(\mathbb{T}) \arrow[ur, "T"'] & \end{tikzcd} \tag{3.8}$$

Finally, the classical identity $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ that works in general for $f, g \in L^1(\mathbb{T})$ allows us to write $* = \check{\ } \circ \odot \circ (\widehat{\ } \times \widehat{\ })$. Hence, we obtain the commutativity of the diagram

$$\begin{tikzcd} L^2(\mathbb{T}) \times L^2(\mathbb{T}) \arrow[r, "\widehat{\ } \times \widehat{\ }"] \arrow[rrr, bend right, "*"'] & \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \arrow[r, "\odot"] & \ell^1(\mathbb{Z}) \arrow[r, "\check{\ }"] & \mathfrak{R}(\mathbb{T}), \end{tikzcd} \tag{3.9}$$

and (ii) holds.

Conversely, assume that there is a linear and continuous operator $T : \mathfrak{R}(\mathbb{T}) \to Y$ such that $B = T \circ *$. Then if $f * g = 0$, obviously $B(f, g) = T(f * g) = 0$ and B is $*$-continuous. The proof is done. $\square$

Remark 3.2. Seeing the proof of our result, we can give an explicit formula for the operator T in terms of the classical Dirichlet kernel. We claim that the map T of the theorem is given by

$$T(f) = \lim_{N\to\infty} B(f, D_N),$$

where D_N stands for the *Dirichlet kernel* which is given by the formula

$$D_N(x) = \sum_{|j|\le N} e^{ijx}.$$

Indeed, just observe that

$$\big(\chi_{[-N,N]\cap\mathbb{Z}}\big)^{\check{}} = \sum_{|j|\le\infty} \chi_{[-N,N]\cap\mathbb{Z}}(j) e^{ijx} = D_N(x),$$

and use (3.7), (3.6), (3.4), and (3.1) to obtain

$$T(f) = \widetilde{T}(\widehat{f}) = \lim_{N\to\infty} \widetilde{T}_N(\widehat{f}) = \lim_{N\to\infty} \widetilde{B}(\widehat{f}, \chi_{[-N,N]\cap\mathbb{Z}})$$
$$= \lim_{N\to\infty} B\big(f, (\chi_{[-N,N]\cap\mathbb{Z}})^{\check{}}\big) = \lim_{N\to\infty} B(f, D_N).$$

Actually, putting together the commutativity of diagrams (3.2), (3.3), (3.8), and (3.9), we have proof of the following.

Corollary 3.3. *Let $\mathbb{T}$ be the real line mod 2π, and let Y be an arbitrary Banach space. For a bilinear continuous operator $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$, the following statements are equivalent.*

(i) *B is $*$-continuous.*

(ii) *There are linear and bilinear continuous operators such that the following diagram commutes:*

$$\begin{tikzcd}
L^2(\mathbb{T}) \times L^2(\mathbb{T}) \arrow[r, "B"] \arrow[d, "\widehat{}\times\widehat{}"'] \arrow[ddd, bend right=60, "*"'] & Y \\
\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \arrow[ur, "\widetilde{B}"'] \arrow[d, "\odot"'] & \\
\ell^1(\mathbb{Z}) \arrow[uur, "\widetilde{T}"'] \arrow[d, "\check{}"'] & \\
\mathfrak{R}(\mathbb{T}) \arrow[uuur, "T"'] &
\end{tikzcd}$$

where $\widehat{}$ and $\check{}$ stand for the Fourier and the inverse Fourier transforms, respectively, and $\odot$ is the pointwise product of sequences.

We show now two examples of $*$-continuous bilinear operators concerning two recent developments in bilinear Fourier analysis.

(A) Let us explain some relations of our class with a genuine bilinear version of convolution, that is, given by the so-called *translation-invariant bilinear operators*. A considerable effort has been made recently for understanding this class of maps in the setting of multilinear harmonic analysis; we refer to [10] and the references therein for information on the topic. They are given—in the case in which we consider $\mathbb{R}$ as measurable space and the operator is defined by a nonnegative regular Borel measure μ—by the formula

$$B_\mu(f, g) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(x-z)\, d\mu(y,z), \quad f, g \in L^2(\mathbb{R})$$

(see [10] and the references therein). We consider the "compact group version" (see [10, (1)]) of this definition with a slight modification. Take $\mu = k(z)\, dy\, dz$ for $k \in L^2[0, 2\pi]$, and consider the map

$$\begin{aligned} B_k(f, g) &:= \int_0^{2\pi} \int_0^{2\pi} f(y-x) g(x-z) k(z)\, dy\, dz \\ &= \int_0^{2\pi} f(y-x)(k * g)(x)\, dy, \quad f, g \in L^2[0, 2\pi]. \end{aligned}$$

Using this, and if Ψ is a Z-valued Bochner 2-integrable function—Z is a Banach space—then we can define the Z-valued kernel bilinear map by

$$B_{\Psi,k}(f,g)=\int_0^{2\pi}\Psi(y)\Big(\int_0^{2\pi}f(y-x)(k*g)(x)\,dx\Big)\,dy,\quad f,g\in L^2[0,2\pi].$$

Clearly,

$$B_{\Psi,k}(f,g)=\int_0^{2\pi}\Psi(y)\big(k*(f*g)\big)(y)\,dy,$$

and this is zero if $f*g=0$.

(B) Let $1<p<\infty$, and consider the continuous bilinear map $u:\ell^p\times\ell^{p'}\to\ell^1$ given by the pointwise product $u((a_i),(b_i)):=(a_i)\odot(b_i)=(a_ib_i)\in\ell^1$. We will use for this example the u-convolution for spaces of Bochner integrable functions defined by Blasco in [1] (see also [2] and Blasco and Calabuig [3]). Following [1] and the notation in this paper, the u-convolution can be defined as a bilinear map $*_u:L^1(\mathbb{T},\ell^p)\times L^1(\mathbb{T},\ell^{p'})\to L^1(\mathbb{T},\ell^1)$ by the formula

$$\phi *_u \psi(t)=\int_0^{2\pi}u\big(\phi(e^{is}),\psi(e^{i(t-s)})\big)\frac{ds}{2\pi}\in L^1(\mathbb{T},\ell^1),$$

for $\phi\in L^1(\mathbb{T},\ell^p)$, $\psi\in L^1(\mathbb{T},\ell^{p'})$. Consider now two sequences of integrable functions (k_i) and (v_i), and assume that the linear maps $T_1:L^2(\mathbb{T})\to L^1(\mathbb{T},\ell^p)$ and $T_2:L^2(\mathbb{T})\to L^1(\mathbb{T},\ell^{p'})$ given by

$$T_1(f)(w):=\sum_{i=1}^{\infty}(k_i*f)(w)e_i\in\ell^p,\qquad T_2(g)(w):=\sum_{i=1}^{\infty}(v_i*g)(w)e_i\in\ell^{p'}$$

are well defined for all $f,g\in L^2(\mathbb{T})$ and continuous. We consider the bilinear map $B:=*_u\circ(T_1,T_2):L^2(\mathbb{T})\times L^2(\mathbb{T})\to L^1(\mathbb{T},Z)$. Let us show that it is $*$-continuous. Indeed, for a fixed couple of functions $f,g\in L^2(\mathbb{T})$, we have

$$\begin{aligned}B(f,g)(t)&=\int_0^{2\pi}\Big(\sum_{i=1}^{\infty}(k_i*f)(e^{is})(v_i*g)(e^{i(t-s)})e_i\Big)\frac{ds}{2\pi}\\&=\sum_{i=1}^{\infty}\Big(\int_0^{2\pi}(k_i*f)(e^{is})(v_i*g)(e^{i(t-s)})\Big)\frac{ds}{2\pi},\quad e_i\in\ell^1.\end{aligned}$$

Thus, $B(f,g)=\sum_{i=1}^{\infty}(k_i*v_i)*(f*g)e_i$, and so it is $*$-continuous.

4. Properties of $*$-continuous bilinear operators and applications

Some direct consequences on the properties of $*$-continuous bilinear maps can be fixed by using some classical properties. We will analyze separately the two main cases that are reasonable to consider in our context: when Y is a reflexive space, and when Y is a Banach space with the Schur property. In the first case, it will be shown that $*$-continuous operators have good summability properties in the event that Y has some suitable geometric properties. The second case—regarding topological properties—will provide some information in the case in which B is weakly compact. We will finish the section—and the article—by showing an application of our results to what is called *generalized convolution.*

4.1. **Summability properties of $*$-continuous bilinear maps: Hilbert spaces, finite cotype spaces, and ℓ^p-spaces.** Consider a $*$-continuous bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$. In this case, the factorization given by Theorem 3.1 gives that, for all $f_1, \dots, f_n, g_1, \dots, g_n \in L^2(\mathbb{T})$,

$$\sum_{i=1}^{n} \big\| B(f_i, g_i) \big\|_Y \le k \sum_{i=1}^{n} \| \widehat{f}_i \cdot \widehat{g}_i \|_{\ell^1} = k \sum_{i=1}^{n} \sum_{j=1}^{\infty} |a_j^i b_j^i|,$$

where (a_j^i) and (b_j^i) are the sequences of Fourier coefficients of f_i and g_i, respectively. This summability can be improved when we consider Y as some particular spaces.

The first application that we will show is when Y is a Hilbert space H; for instance, $H = L^2(\mathbb{T})$. In this case, an integral domination can even be obtained. By Grothendieck's theorem we have that $L(\ell^1, H) = \Pi_1(\ell^1, H)$, and so we directly obtain the following.

Corollary 4.1. *If H is a Hilbert space and $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to H$ is $*$-continuous, then B factors through a summing operator T as $B = T \circ *$. Consequently, there is a constant $k > 0$ such that the following equivalent assertions hold.*

(i) *For $f_1, \dots, f_n, g_1, \dots, g_n \in L^2(\mathbb{T})$,*

$$\sum_{i=1}^{n} \big\| B(f_i, g_i) \big\|_H \le k \sup_{\varphi \in B_{\ell^\infty}} \sum_{i=1}^{n} \big| \langle \widehat{f}_i \cdot \widehat{g}_i, \varphi \rangle \big| = k \sup_{(\varphi_j) \in B_{\ell^\infty}} \sum_{i=1}^{n} \Big| \sum_{j=1}^{\infty} a_j^i b_j^i \varphi_j \Big|,$$

where (a_j^i) and (b_j^i) are the sequences of Fourier coefficients of f_i and g_i, respectively.

(ii) *For $f, g \in L^2(\mathbb{T})$,*

$$\big\| B(f, g) \big\|_H \le k \int_{B_{\ell^\infty}} \big| \langle \widehat{f} \cdot \widehat{g}, \varphi \rangle \big| \, d\eta(\varphi) = k \int_{B_{\ell^\infty}} \Big| \sum_{j=1}^{\infty} a_j b_j \varphi_j \Big| \, d\eta(\varphi),$$

where η is a regular probability measure on the unit ball of ℓ^∞ given by Pietsch's domination theorem, and (a_j) and (b_j) are the sequences of Fourier coefficients of f and g, respectively.

Some similar (but weaker) results can also be obtained if we consider some cotype-related properties for the space Y. For instance, recall that cotype 2 for a Banach space implies the Orlicz property (see [5, Section 8.9]); if we ask Y to have the Orlicz property, we get a domination for any $*$-continuous bilinear map as follows: for $f_1, \dots, f_n, g_1, \dots, g_n \in L^2(\mathbb{T})$,

$$\Big(\sum_{i=1}^{n} \big\| B(f_i, g_i) \big\|_Y^2 \Big)^{1/2} \le k \sup_{\varepsilon_i \in \{-1,1\}} \Big\| \sum_{i=1}^{n} \varepsilon_i \widehat{f}_i \cdot \widehat{g}_i \Big\|_{\ell^1}.$$

The second case that we may consider is the one given when Y is an ℓ^p-space. We have that for $1 \leq p \leq \infty$ and r such that $1/r = 1 - |1/p - 1/2|$, $L(\ell^1, \ell^p) = \Pi_{r,1}(\ell^1, \ell^p)$. This result can be found in [5, Section 34.11]. We can use it to directly prove the following.

Corollary 4.2. *Let $1 \leq p \leq \infty$, and take r such that $1/r = 1 - |1/p - 1/2|$. Take a $*$-continuous bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to \ell^p$. Then there is a constant $k > 0$ such that for $f_1, \ldots, f_n, g_1, \ldots, g_n \in L^2(\mathbb{T})$,*

$$\Big(\sum_{i=1}^{n} \big\|B(f_i, g_i)\big\|_{\ell^p}^{r}\Big)^{1/r} \leq k \sup_{\varepsilon_i \in \{-1,1\}} \Big\|\sum_{i=1}^{n} \varepsilon_i \widehat{f}_i \cdot \widehat{g}_i\Big\|_{\ell^1} = k \sup_{(\varphi_j) \in B_{\ell^\infty}} \sum_{i=1}^{n} \Big|\sum_{j=1}^{\infty} a_j^i b_j^i \varphi_j\Big|,$$

where (a_j^i) and (b_j^i) are the sequences of Fourier coefficients of f_i and g_i, respectively.

Note that for the case $p = 2$, we obtain the result given in Corollary 4.1. Another interesting case is the one provided by the classical Littlewood inequality, which can be written as $L(\ell^1, \ell^{4/3}) = \Pi_{4/3,1}(\ell^1, \ell^{4/3})$ (see [5, Section 34.12]): if B is defined on $\ell^{4/3}$, we obtain that

$$\Big(\sum_{i=1}^{n} \big\|B(f_i, g_i)\big\|_{\ell^{4/3}}^{4/3}\Big)^{3/4} \leq k \sup_{\varphi \in B_{\ell^\infty}} \sum_{i=1}^{n} \big|\langle \widehat{f}_i \cdot \widehat{g}_i, \varphi\rangle\big|.$$

4.2. **Weak compactness and bilinear maps on spaces with the Schur property.** In this section, we analyze some properties for Y that are typical for nonreflexive spaces, ones that would imply some concrete consequences on the factorization through the convolution map. (The assumption in this section is that $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$ is *weakly compact*; that is, $B(B_{L^2(\mathbb{T})} \times B_{L^2(\mathbb{T})})$ is a relatively weakly compact set in Y.)

Let us introduce some notation. We will say that a subset $A \subseteq L^2(\mathbb{T}) \times L^2(\mathbb{T})$ is $*$-*relatively weakly compact* if the set $\{f * g : (f, g) \in A\} \subseteq \mathfrak{R}(\mathbb{T})$ is relatively weakly compact. The following result shows that $*$-continuous operators satisfy a certain kind of Dunford–Pettis property.

Corollary 4.3. *Let $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$ be a $*$-continuous weakly compact bilinear map, and let A be a $*$-relatively weakly compact set. Then $B(A)$ is relatively compact. Moreover, if there is a $*$-relatively weakly compact bilinear operator B for Y, then Y is finite-dimensional.*

Proof. This is just a consequence of the fact that ℓ^1 has the Dunford–Pettis property. Since we have the factorization though ℓ^1 of B, and such factorization satisfies that

$$*(B_{L^2(\mathbb{T})} \times B_{L^2(\mathbb{T})}) = B_{\mathfrak{R}(\mathbb{T})},$$

we have that $\widehat{\ }(*(B_{L^2(\mathbb{T})} \times B_{L^2(\mathbb{T})})) = B_{\ell^1}$. Therefore, $T \circ \check{\ } : \ell^1 \to Y$ is weakly compact. Take a $*$-relatively weakly compact set A. We have that $\widehat{\ }(*A)$ is then a relatively weakly compact set of ℓ^1. The Dunford–Pettis property of ℓ^1 then gives that $T \circ \check{\ } \circ \widehat{\ } \circ *(A) = T \circ *(A) = B(A)$ is relatively compact. The last statement is then clear. $\square$

This theorem can be improved for the case in which Y has the Schur property; recall that this means that weakly and norm-convergent sequences coincide in it. Notice that there are spaces other than ℓ^1 that have the Schur property (e.g., some discrete Nakano spaces; see (IV) in [11]). The same kind of arguments that we use in the following result—with slightly more restrictive requirements on B—will be used in Corollary 4.5 to then prove that $*$-continuous bilinear maps can indeed characterize ℓ^1.

Corollary 4.4. *Let $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$ be a $*$-continuous weakly compact bilinear map. Let Y be a Banach lattice with the Schur property. Then $B(B_{L^2(\mathbb{T})} \times B_{L^2(\mathbb{T})})$ is a relatively compact set in Y. Consequently, if B is norming for Y, then Y is finite-dimensional.*

Proof. We use Theorem 2 in [14], establishing that a Banach space Y has the Schur property if and only if every weakly compact operator from ℓ^1 to Y is compact. As we showed in the proof of Corollary 4.3, we have that $\widehat{\ }(*(B_{L^2(\mathbb{T})} \times B_{L^2(\mathbb{T})})) = B_{\ell^1}$. Therefore, $T \circ \widecheck{\ } : \ell^1 \to Y$ is weakly compact, and so the Schur property for Y gives that it is also compact. Then obviously normability of B implies that Y has finite dimension. □

Recall that a bilinear map B is $*$-equivalent if for every $f, g \in L^2(\mathbb{T})$,

$$f * g = 0 \iff B(f, g) = 0.$$

Corollary 4.5. *A Banach space Y is isomorphic to ℓ^1 if and only if it admits a $*$-equivalent norming bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$.*

Proof. The direct implication is obvious: if $R : \ell^1 \to Y$ is an isomorphism, just take $B = R \circ \widehat{\ } \circ *$.

For the converse implication, suppose that B satisfies the requirements. Now take into account that we have a factorization of B as $B = T \circ *$ as a consequence of Theorem 3.1. Since we also have that $B(f, g) = 0$ implies $f * g = 0$, we have that T (and so $\widetilde{T}$) is injective. But B is norming, and so we have that

$$\widetilde{T}(B_{\ell^1}) \subseteq kB_Y \subseteq kKB(B_{L^2(\mathbb{T})} \times B_{L^2(\mathbb{T})}) = kK\widetilde{T}(B_{\ell^1}).$$

This gives the result. □

4.3. **Applications: $*$-continuous generalized convolutions.** Let us finish the article by remarking on a new construction that has shown itself to be useful for applications. It concerns what is called *generalized convolution* (we direct the reader to [9] for relevant information, as well as to references in this article for the original definitions). Let U_1, U_2, and U_3 be linear spaces (may be different) on the same field of scalars, and let V be a commutative algebra. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, and $K_3 \in L(U_3, V)$ are the linear operators from U_1, U_2, and U_3 to V, respectively.

Definition 4.6 ([9, Definition 2.3]). A bilinear map $* : U_1 \times U_2 \to U_3$ is called the *convolution with weight-element* δ—an element of the algebra V—for K_3, K_1, K_2 (in that order) if the identity

$$K_3\big(*(f, g)\big) = \delta K_1(f) K_2(g)$$

holds for any $f \in U_1$ and $g \in U_2$. The equality above is called the *factorization identity* of the convolution.

Fix $U_1 = U_2 = L^2(\mathbb{T})$, $U_3 = V = \mathfrak{R}(\mathbb{T})$, and $K_3 = \mathrm{id}$, and consider $*$ as the usual convolution bilinear map. Let us now write a characterization of when a bilinear map—defined as a product in the algebra of two linear operators—defines a generalized convolution associated to $*$. Indeed, as a consequence of Theorem 3.1, we directly obtain the following.

Corollary 4.7. *Consider two operators $S_1, S_2 : L^2(\mathbb{T}) \to \mathfrak{R}(\mathbb{T})$ and $\delta \in \mathfrak{R}(\mathbb{T})$, and consider the bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to \mathfrak{R}(\mathbb{T})$ given by $B(\cdot,\cdot) = \delta S_1(\cdot)S_2(\cdot)$. Then the following assertions are equivalent.*

(i) *B is $*$-continuous.*

(ii) *There is an operator $T : \mathfrak{R}(\mathbb{T}) \to \mathfrak{R}(\mathbb{T})$ such that $*$ is a convolution with weight δ for T, S_1, S_2.*

In this case, the factorization identity is $T \circ * = B = \delta S_1 S_2$.

Acknowledgments. Erdoğan's work was supported by TÜBITAK, the Scientific and Technological Research Council of Turkey. Calabuig's work was supported by Ministerio de Economía, Industria y Competitividad (MINECO) grant MTM2014-53009-P. Sánchez Pérez's work was supported by MINECO grant MTM2016-77054-C2-1-P.

References

1. O. Blasco, "Convolution by means of bilinear maps" in *Function Spaces (Edwardsville, Ill., 1998)*, Contemp. Math. **232**, Amer. Math. Soc., Providence, 1999, 85–104. Zbl 0936.46026. MR1678323. DOI 10.1090/conm/232/03387. 174
2. O. Blasco, "Bilinear maps and convolutions" in *Nuclear Groups and Lie Groups (Madrid, 1999)*, Res. Exp. Math. **24**, Heldermann, Lemgo, 2001, 45–55. Zbl 1020.46002. MR1858142. 174
3. O. Blasco and J. M. Calabuig, *Fourier analysis with respect to bilinear maps*, Acta Math. Sin. (Engl. Ser.) **25** (2009), no. 4, 519–530. Zbl 1173.42316. MR2495504. DOI 10.1007/s10114-009-7399-x. 174
4. M. G. Cowling and J. J. F. Fournier, *Inclusions and noninclusion of spaces of convolution operators*, Trans. Amer. Math. Soc. **221** (1976), no. 1, 59–95. Zbl 0331.43007. MR0493164. DOI 10.2307/1997542. 169
5. A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Math. Stud. **176**, North-Holland, Amsterdam, 1993. Zbl 0774.46018. MR1209438. 168, 175, 176
6. J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math Surveys **15**, Amer. Math. Soc., Providence, 1977. Zbl 0369.46039. MR0453964. 168
7. R. E. Edwards, *Convolutions as bilinear and linear operators*, Canad. J. Math. **16** (1964), 275–285. Zbl 0128.35601. MR0162147. DOI 10.4153/CJM-1964-027-0. 167, 168, 170
8. R. E. Edwards, *Fourier Series: A Modern Introduction, Vol. 1*, 2nd ed., Grad. Texts in Math. **64**, Springer, New York, 1979. Zbl 0424.42001. MR0545506. 168
9. B. T. Giang and M. N. Tuan, *Generalized convolutions and the integral equations of the convolution type*, Complex Var. Elliptic Equ. **55** (2010), no. 4, 331–345. Zbl 1193.44029. MR2650006. DOI 10.1080/17476930902998886. 167, 177
10. L. Grafakos and J. Soria, *Translation-invariant bilinear operators with positive kernels*, Integral Equations Operator Theory **66** (2010), no. 2, 253–264. Zbl 1193.42067. MR2595656. DOI 10.1007/s00020-010-1746-2. 173

11. I. Halperin and H. Nakano, *Generalized l^p spaces and the Schur property*, J. Math. Soc. Japan **5** (1953), 50–58. Zbl 0050.33403. MR0058125. DOI 10.2969/jmsj/00510050. 177
12. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, Vol. II: Structure and Analysis for Compact Groups*, Grundlehren Math. Wiss. **152**, Springer, New York, 1970. Zbl 0213.40103. MR0262773. 166, 169, 172
13. D. Pellegrino, P. Rueda, and E. A. Sánchez-Pérez, *Surveying the spirit of absolute summability on multilinear operators and homogeneous polynomials*, RACSAM **110** (2016), no. 1, 285–302. Zbl 1350.46034. MR3462088. DOI 10.1007/s13398-015-0224-8. 167
14. W. Wnuk, *Some characterizations of Banach lattices with the Schur property*, Rev. Mat. Complut. **2** (1989), no. 3, 217–224. Zbl 0717.46018. MR1057221. 168, 177
15. W. Wnuk, *On the dual positive Schur property in Banach lattices*, Positivity **17** (2013), no. 3, 759–773. Zbl 1310.46023. MR3090691. DOI 10.1007/s11117-012-0203-7. 168

[1]Department of Mathematics, Faculty of Art and Science, University of Marmara, TR-34722, Kadiköy, Istanbul, Turkey.
E-mail address: ezgi.erdogan@marmara.edu.tr

[2]Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain.
E-mail address: jmcalabu@mat.upv.es; easancpe@mat.upv.es