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# PERTURBATION BOUNDS FOR THE MOORE–PENROSE METRIC GENERALIZED INVERSE IN SOME BANACH SPACES

JIANBING CAO[1*] and WANQIN ZHANG[2]

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Abstract. Let $X, Y$ be Banach spaces, and let $T$, $\delta T : X \to Y$ be bounded linear operators. Put $\bar{T} = T + \delta T$. In this article, utilizing the gap between closed subspaces and the perturbation bounds of metric projections, we first present some error estimates of the upper bound of $\|\bar{T}^M - T^M\|$ in $L^p$ $(1 < p < +\infty)$ spaces. Then, by using the concept of strong uniqueness and modulus of convexity, we further investigate the corresponding perturbation bound $\|\bar{T}^M - T^M\|$ in uniformly convex Banach spaces.

## 1. Introduction

Let $X$ and $Y$ be Banach spaces, and let $B(X, Y)$ be the Banach space consisting of all bounded linear operators from $X$ to $Y$. For $T \in B(X, Y)$, let $\mathcal{N}(T)$ (resp., $\mathcal{R}(T)$) denote the kernel (resp., range) of $T$. It is well known that for $T \in B(X, Y)$, if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are topologically complemented in the spaces $X$ and $Y$, respectively, then there exists a linear projection generalized inverse $T^+ \in B(Y, X)$ defined by

$$T^+Tx = x, \quad x \in \mathcal{N}(T)^c \qquad \text{and} \qquad T^+y = 0, \quad y \in \mathcal{R}(T)^c,$$

where $\mathcal{N}(T)^c$ and $\mathcal{R}(T)^c$ are topologically complemented subspaces of $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. In this case, $T^+T$ is the projection from $X$ onto $\mathcal{N}(T)^c$ along $\mathcal{N}(T)$ and $TT^+$ is the projection from $Y$ onto $\mathcal{R}(T)^c$ along $\mathcal{R}(T)$. It is well known that when $X$ and $Y$ are Hilbert spaces, and if $\mathcal{N}(T)^c = \mathcal{N}(T)^\perp$ and


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*Corresponding author.
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$\mathcal{R}(T)^c = \mathcal{R}(T)^\perp$, then the corresponding generalized inverse $T^+$ defined above is called the *Moore–Penrose orthogonal projection generalized inverse* of $T$. In general, we denote $T^+$ by $T^\dagger$. In this case, $T^\dagger T$ is the orthogonal projection from $X$ onto $\mathcal{N}(T)$ and $TT^\dagger$ is the orthogonal projection from $Y$ onto $R(T)$. (We refer the reader to [16] and [17] for more information about generalized inverses.)

The linear projection generalized inverse and its perturbation analysis have applications and play an important role in many fields, such as computation, control theory, frame theory, and nonlinear analysis. In the Hilbert space case, the perturbation problems of the Moore–Penrose orthogonal projection generalized inverses of linear operators have been widely studied and numerous results have been obtained. Chen and Xue [5] introduced an important notation, the so-called *stable perturbation* of operators on Banach spaces: that is, if $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^+) = \{0\}$, then $T^+$ is said to be the *stable perturbation* of $T$. Lately, there has been increased interest in the stable perturbation theory of generalized inverses in the literature. In their previous work, Xue and Chen [18] further investigated this concept and some of its important applications in Hilbert spaces. As a result, they got the following important perturbation results. That is, when $X$, $Y$ are Hilbert spaces, if $T$, $\bar{T} = T + \delta T \in B(X, Y)$ with $\mathcal{R}(T)$ closed, $\|T^\dagger\|\|\delta T\| < 1$, and $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^\dagger) = \{0\}$, then $\bar{T}^\dagger$ exists and

$$\|\bar{T}^\dagger - T^\dagger\| \leq \frac{1+\sqrt{5}}{2} \frac{\|T^\dagger\|^2\|\delta T\|}{1 - \|T^\dagger\|\|\delta T\|}. \tag{1.1}$$

The perturbation bound (1.1) above has many applications, especially in solving the following so-called *least problem* and its perturbation:

$$\inf \|x\| \quad \text{subject to} \quad \|Tx - b\| = \inf_{z \in X} \|Tz - b\|. \tag{1.2}$$

However, it is generally well known that not every closed subspace in a Banach space is complemented; thus, the linear generalized inverse $T^+$ of $T$ may not exist. In this case, in order to solve some approximation problems in Banach spaces, we may seek other types of generalized inverses for $T$. For example, generally speaking, the linear projection generalized inverse cannot deal with the extremal solution, or the best approximation solution of an ill-posed operator equation in Banach spaces. To solve the best approximation problems for an ill-posed linear operator equation in Banach spaces, Nashed and Votruba [12] introduced the concept of the (set-valued) metric generalized inverse of a linear operator in Banach spaces. Later, in 2003, Wang and Wang [15] defined the Moore–Penrose metric generalized inverse for a linear operator with closed range in Banach spaces and gave some useful characterizations. Then, Ni [13] defined and characterized the Moore–Penrose metric generalized inverse for an arbitrary linear operator in a Banach space.

Let $X$, $Y$ be Banach spaces, and let $T$, $\delta T \in B(X, Y)$ with $\mathcal{R}(T)$ closed. Put $\bar{T} = T + \delta T$. Suppose that the Moore–Penrose metric generalized inverse $T^M$ of $T$ exists. Then, motivated by some results obtained in Hilbert spaces, it is natural to ask the following questions: *If* $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$*, then what additional* *conditions can guarantee that* $\bar{T}^M$ *exists? If* $\bar{T}^M$ *exists, then what is its expression and can we give any error estimations of the upper bound of* $\|\bar{T}^M - T^M\|$*? Furthermore, how does one solve the least problem (1.2) and its perturbation in Banach spaces?*

In recent years, the perturbation problems of the Moore–Penrose metric generalized inverse have been widely studied, with some important progress made in this direction by using the concept of quasiadditivity and the theory of stable perturbation. In a number of recent papers (see [4], [6], [7], [11]), the authors presented some perturbation results of the Moore–Penrose metric generalized inverse under certain additional assumptions, and also obtained some descriptions of the Moore–Penrose metric generalized inverse in Banach spaces. All these results give some partial answers to the perturbation problems of the Moore–Penrose metric generalized inverse. However, perturbation problems for nonlinear generalized inverses are still far from being completely solved.

In this article, utilizing the gap between closed subspaces and the perturbation bounds of metric projections, then under the stable perturbation and some quasiadditivity assumption, we will further study the perturbation problems for the Moore–Penrose metric generalized inverses in some Banach spaces, especially for the $L^p$ $(1 < p < +\infty)$ spaces and uniformly convex Banach spaces. In the next section, we give some necessary concepts and preliminary results. We prove our main results in Sections 3 and 4. Finally, we end with a concluding remark in Section 5.

## 2. Preliminaries

In this section, we recall some concepts and results used in this article. Let $T : X \to Y$ be a mapping, and let $D$ be a subset of $X$. Recall from [16] that $D$ is said to be *homogeneous* if $\lambda x \in D$ whenever $x \in D$ and $\lambda \in \mathbb{R}$, and that a mapping $T\colon X \to Y$ is said to be a *bounded homogeneous operator* if $T$ maps every bounded set in $X$ into a bounded set in $Y$ and $T(\lambda x) = \lambda T(x)$ for every $x \in X$ and every $\lambda \in \mathbb{R}$. Let $H(X, Y)$ denote the set of all bounded homogeneous operators from $X$ to $Y$. Equipped with the usual linear operations on $H(X, Y)$ and the norm on $T \in H(X, Y)$ defined by $\|T\| = \sup\{\|Tx\| \mid \|x\| = 1, x \in X\}$, we can easily prove that $(H(X, Y), \|\cdot\|)$ is a Banach space. Obviously, $B(X, Y) \subset H(X, Y)$. For a bounded homogeneous operator $T \in H(X, Y)$, we always denote by $\mathcal{D}(T)$, $\mathcal{N}(T)$, and $\mathcal{R}(T)$ the domain, the null space, and the range of $T$, respectively.

*Definition* 2.1. Let $M \subset X$ be a subset, and let $T : X \to Y$ be a mapping. Then we call $T$ *quasiadditive* on $M$ if $T$ satisfies

$$T(x + z) = T(x) + T(z), \quad \forall x \in X, \forall z \in M.$$

For a homogeneous operator $T \in H(X, X)$, if $T$ is quasiadditive on $\mathcal{R}(T)$, then we simply say that $T$ is a *quasilinear operator*.

*Definition* 2.2. Let $P \in H(X, X)$. If $P^2 = P$, then we call $P$ a *homogeneous projection*. In addition, if $P$ is also quasiadditive on $\mathcal{R}(P)$, that is, for any $x \in X$

and any $z \in \mathcal{R}(P)$,

$$P(x+z) = P(x) + P(z) = P(x) + z,$$

then we call $P$ a *quasilinear projection*.

The following concept of bounded homogeneous generalized inverse is also a generalization of bounded linear generalized inverse.

*Definition* 2.3 ([1, Definition 3.1]). Let $T \in B(X,Y)$. If there is $T^h \in H(Y,X)$ such that

$$TT^hT = T, \qquad T^hTT^h = T^h,$$

then we call $T^h$ a *bounded homogeneous generalized inverse* of $T$.

The following lemma characterizes the existence of a homogeneous generalized inverse of $T \in B(X,Y)$, which turns out to be very useful in our analysis.

**Lemma 2.4** ([3, Proposition 2.4])**.** *Let $T \in B(X,Y)$. Then $T$ has a homogeneous generalized inverse $T^h \in H(Y,X)$ if and only if $\mathcal{R}(T)$ is closed and there exist a bounded quasilinear projection $P_{\mathcal{N}(T)}\colon X \to \mathcal{N}(T)$ and a bounded homogeneous projection $Q_{\mathcal{R}(T)} : Y \to \mathcal{R}(T)$.*

*Definition* 2.5 ([14, Definition 4.1]). Let $G$ be a subset of $X$. The set-valued mapping $P_G : X \to G$ defined by

$$P_G(x) = \big\{ s \in G \;\big|\; \|x-s\| = \mathrm{dist}(x,G) \big\}, \quad \forall x \in X$$

is called the *set-valued metric projection*, where $\mathrm{dist}(x,G) = \inf_{z\in X} \|x-z\|$.

For a subset $G \subset X$, if $P_G(x) \neq \emptyset$ for each $x \in X$, then $G$ is said to be *approximal*; if $P_G(x)$ is at most a singleton for each $x \in X$, then $G$ is said to be *semi-Chebyshev*; if $G$ is simultaneously approximal and a semi-Chebyshev set, then $G$ is called a *Chebyshev set*. We denote by $\pi_G$ any selection for the set-valued mapping $P_G$, that is, any single-valued mapping $\pi_G : \mathcal{D}(\pi_G) \to G$ with the property that $\pi_G(x) \in P_G(x)$ for any $x \in \mathcal{D}(\pi_G)$, where $\mathcal{D}(\pi_G) = \{x \in X : P_G(x) \neq \emptyset\}$. For the particular case when $G$ is a Chebyshev set, then $\mathcal{D}(\pi_G) = X$ and $P_G(x) = \{\pi_G(x)\}$. In this case, the mapping $\pi_G$ is called the *metric projection* from $X$ onto $G$.

*Remark* 2.6 ([14, Section 3.3]). Let $G \subset X$ be a closed convex subset. It is well known that if $X$ is reflexive, then $G$ is a proximal set, and if $X$ is strictly convex, then $G$ is a semi-Chebyshev set. Thus, every closed convex subset in a reflexive and strictly convex Banach space is a Chebyshev set.

The following lemma gives some important properties of the metric projections.

**Lemma 2.7** ([14, Theorem 4.1])**.** *Let $X$ be a Banach space, and let $L$ be a subspace of $X$. Then*

(1) $\pi_L^2(x) = \pi_L(x)$ *for any* $x \in \mathcal{D}(\pi_L)$*, that is,* $\pi_L$ *is idempotent;*
(2) $\|x - \pi_L(x)\| \leq \|x\|$ *for any* $x \in \mathcal{D}(\pi_L)$*, that is,* $\|\pi_L\| \leq 2$.

*In addition, if $L$ is a semi-Chebyshev subspace, then*

(3) $\pi_L(\lambda x) = \lambda \pi_L(x)$ *for any* $x \in X$ *and* $\lambda \in \mathbb{R}$*, that is,* $\pi_L$ *is homogeneous;*

(4) $\pi_L(x+z) = \pi_L(x) + \pi_L(z) = \pi_L(x) + z$ *for any* $x \in \mathcal{D}(\pi_L)$ *and* $z \in L$, *that is,* $\pi_L$ *is quasiadditive on* $L$.

Now we present the definition of the Moore–Penrose metric generalized inverse (see [16] for more information about the Moore–Penrose metric generalized inverses and related knowledge).

*Definition* 2.8 ([16, Definition 4.3.1], [15, Definition 2.1]). Let $T \in B(X,Y)$. Suppose that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces of $X$ and $Y$, respectively. If there exists a bounded homogeneous operator $T^M : Y \to X$ such that

(1) $TT^MT = T$,
(2) $T^MTT^M = T^M$,
(3) $T^MT = I_X - \pi_{\mathcal{N}(T)}$, and
(4) $TT^M = \pi_{\mathcal{R}(T)}$,

then $T^M$ is called the *Moore–Penrose metric generalized inverse* of $T$, where $\pi_{\mathcal{N}(T)}$ and $\pi_{\mathcal{R}(T)}$ are the metric projections onto $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively.

Here we only need the following result which characterizes the existence of the Moore–Penrose metric generalized inverse.

**Lemma 2.9** ([16, Theorem 4.3.1], [15, Corollary 2.1]). *Let* $T \in B(X,Y)$. *If* $\mathcal{N}(T)$ *and* $\mathcal{R}(T)$ *are Chebyshev subspaces of* $X$ *and* $Y$*, respectively, then there exists a unique Moore–Penrose metric generalized inverse* $T^M$ *of* $T$.

The gap function is one of the main tools in this article. Here we only give the definition (see [8] for more information). Let $M$, $N$ be two closed subspaces in $X$. We denote by $S_N$ the unit sphere of $N$. Set

$$\delta(M,N) = \begin{cases} \sup\{\operatorname{dist}(x,N) \mid x \in M, \|x\| = 1\}, & M \neq \{0\}, \\ 0, & M = \{0\}. \end{cases}$$

We call $\hat{\delta}(M,N) = \max\{\delta(M,N), \delta(N,M)\}$ the *gap* between $M$ and $N$.

## 3. Stable perturbation analysis for $T^M$ on $L^p$-spaces

By using a bounded homogeneous generalized inverse, we first give the following result.

**Lemma 3.1.** *Let* $T \in B(X,Y)$ *such that* $T^h \in H(Y,X)$ *exists. Assume that* $\mathcal{N}(T)$ *and* $\mathcal{R}(T)$ *are Chebyshev subspaces in* $X$ *and* $Y$*, respectively. Then* $T^M = (I_X - \pi_{\mathcal{N}(T)})T^h\pi_{\mathcal{R}(T)}$.

*Proof.* Since $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces, it follows from Lemma 2.9 that $T$ has the unique Moore–Penrose metric generalized inverse $T^M$ which satisfies

$$TT^MT = T, \qquad T^MTT^M = T^M, \qquad TT^M = \pi_{\mathcal{R}(T)}, \qquad T^MT = I_X - \pi_{\mathcal{N}(T)}.$$

Set $T^\natural = (I_X - \pi_{\mathcal{N}(T)})T^h\pi_{\mathcal{R}(T)}$. Then $T^\natural = T^MTT^hTT^M = T^MTT^M = T^M$. □

**Lemma 3.2.** *Let* $X$, $Y$ *be reflexive and strictly convex Banach spaces, and let* $T \in B(X,Y)$ *with* $\mathcal{R}(T)$ *closed. Put* $\bar{T} = T + \delta T$. *Then*

(1) *the Moore–Penrose metric generalized inverse $T^M$ of $T$ exists.*

*In addition, if $T^M$ is quasiadditive on $\mathcal{R}(\delta T)$ and $\|T^M\|\|\delta T\| < 1$, $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, then*

(2) *the Moore–Penrose metric generalized inverse $\bar{T}^M$ of $\bar{T}$ exists. Moreover,*

$$\bar{T}^M = (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M \delta T)^{-1} T^M \pi_{\mathcal{R}(\bar{T})}; \tag{3.1}$$

(3) $\|\bar{T}^M\| \leq \frac{2\|T^M\|}{1-\|T^M\delta T\|}$.

*Proof.* (1) Note that $X, Y$ are reflexive and strictly convex Banach spaces. So, for $T \in B(X, Y)$ with $\mathcal{R}(T)$ closed, we get that $T^M$ uniquely exists from Remark 2.6 and Lemma 2.9.

(2) Since $T^M$ is quasiadditive on $\mathcal{R}(\delta T)$ (i.e., $T^M \delta T \in B(X)$) and $\|T^M\|\|\delta T\| < 1$, we get that $I_X + T^M \delta T$ is invertible in $B(X)$. Now, by our results in [3, Theorem 3.4], the stable perturbation condition $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$ implies that $\bar{T}$ has a bounded homogeneous generalized inverse $\bar{T}^h = (I_X + T^M \delta T)^{-1} T^M$. Consequently, we get that $\mathcal{R}(\bar{T})$ is closed from Lemma 2.4, and thus $\bar{T}^M$ uniquely exists from Remark 2.6 and Lemma 2.9. Finally, by using Lemma 3.1, we can obtain that $\bar{T}^M$ has the following form:

$$\bar{T}^M = (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M \delta T)^{-1} T^M \pi_{\mathcal{R}(\bar{T})}.$$

(3) From Lemma 2.7, we know that $\|I_X - \pi_{\mathcal{N}(\bar{T})}\| \leq 1$ and $\|\pi_{\mathcal{R}(\bar{T})}\| \leq 2$. Therefore, by using the equality (3.1) we get

$$\|\bar{T}^M\| \leq \|I_X - \pi_{\mathcal{N}(\bar{T})}\|\|(I_X + T^M \delta T)^{-1} T^M\|\|\pi_{\mathcal{R}(\bar{T})}\| \leq \frac{2\|T^M\|}{1 - \|T^M \delta T\|}.$$

This completes the proof. □

The next lemma, which is an estimate for the stability of the metric projection in $L^p$ $(1 < p < +\infty)$ space, is one of the principal tools we use for establishing our main theorem.

**Lemma 3.3** ([9, Corollary 2.5])**.** *Let $M$ and $N$ be closed linear subspaces in $L^p$ $(1 < p < +\infty)$. Then we have*

$$\|\pi_M - \pi_N\|_p \leq \begin{cases} 10(p-1)^{-\frac{1}{2}} \hat{\delta}(M, N)^{\frac{1}{2}} & (1 < p \leq 2), \\ 10 C_p^{-\frac{1}{p}} \hat{\delta}(M, N)^{\frac{1}{p}} & (2 \leq p < +\infty), \end{cases}$$

*where $C_p = (p-1)(1+s)^{2-p}$ and $s$ is the unique positive zero of the function $t^{p-1} - (p-1)t - (p-2)$.*

Now, under the stable perturbation condition $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, we can estimate the upper bound of $\|\bar{T}^M - T^M\|$.

**Theorem 3.4.** *Let $X = L^p$ $(1 < p < +\infty)$. Let $T$, $\delta T \in B(X)$ with $\mathcal{R}(T)$ closed and $\|T^M\|\|\delta T\| < 1$. Suppose that $T^M$ is quasiadditive on $\mathcal{R}(T)$ and $\mathcal{R}(\delta T)$. Put*

$\bar{T} = T + \delta T$. *If* $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, *then* $\bar{T}^M$ *exists and*

$$\|\bar{T}^M - T^M\| \leq \begin{cases} \frac{\|T^M\|}{1-\|T^M\delta T\|}\{10(p-1)^{-\frac{1}{2}}\hat{\delta}(\mathcal{R}(T),\mathcal{R}(\bar{T}))^{\frac{1}{2}} \\ \quad + 10(1-\|T^M\delta T\|)(p-1)^{-\frac{1}{2}}\hat{\delta}(\mathcal{N}(\bar{T}),\mathcal{N}(T))^{\frac{1}{2}} \\ \quad + \|T^M\delta T\|\} & (1 < p \leq 2), \\ \frac{\|T^M\|}{1-\|T^M\delta T\|}\{10C_p^{-\frac{1}{p}}\hat{\delta}(\mathcal{R}(T),\mathcal{R}(\bar{T}))^{\frac{1}{p}} \\ \quad + 10(1-\|T^M\delta T\|)C_p^{-\frac{1}{p}}\hat{\delta}(\mathcal{N}(\bar{T}),\mathcal{N}(T))^{\frac{1}{p}} \\ \quad + \|T^M\delta T\|\} & (2 \leq p < +\infty). \end{cases}$$

*Proof.* It is well known that $L^p$ $(1 < p < +\infty)$ is a reflexive and strictly convex Banach space, so $T^M$ is well defined. Clearly, under our assumption, $\bar{T}^M$ also uniquely exists by Lemma 3.2.

It follows from Lemma 3.2 that $\bar{T}^M = (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M\delta T)^{-1}T^M\pi_{\mathcal{R}(\bar{T})}$. Note that we also have $T^M = (I_X - \pi_{\mathcal{N}(T)})T^M\pi_{\mathcal{R}(T)}$ and $T^M\pi_{\mathcal{R}(T)} = T^M$. Hence

$$\begin{aligned} &\bar{T}^M - T^M \\ &\quad = (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M\delta T)^{-1}T^M\pi_{\mathcal{R}(\bar{T})} \\ &\qquad - (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M\delta T)^{-1}T^M\pi_{\mathcal{R}(T)} \\ &\qquad + (I_X - \pi_{\mathcal{N}(\bar{T})})T^M\pi_{\mathcal{R}(T)} - (I_X - \pi_{\mathcal{N}(T)})T^M\pi_{\mathcal{R}(T)} \\ &\qquad + (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M\delta T)^{-1}T^M\pi_{\mathcal{R}(T)} - (I_X - \pi_{\mathcal{N}(\bar{T})})T^M\pi_{\mathcal{R}(T)} \\ &\quad = (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M\delta T)^{-1}T^M(\pi_{\mathcal{R}(\bar{T})} - \pi_{\mathcal{R}(T)}) \quad \text{(using quasiadditivity)} \\ &\qquad + (\pi_{\mathcal{N}(T)} - \pi_{\mathcal{N}(\bar{T})})T^M + (I_X - \pi_{\mathcal{N}(\bar{T})})\big((I_X + T^M\delta T)^{-1} - I_X\big)T^M. \end{aligned}$$

Since $\|T^M\|\|\delta T\| < 1$, then we can check easily that

$$\big\|(I_X + T^M\delta T)^{-1}\big\| \leq \frac{1}{1 - \|T^M\delta T\|},$$
$$\big\|(I_X + T^M\delta T)^{-1} - I_X\big\| \leq \frac{\|T^M\delta T\|}{1 - \|T^M\delta T\|}.$$

From Lemma 2.7, we also have $\|I_X - \pi_{\mathcal{N}(\bar{T})}\| \leq 1$. Therefore, we obtain

$$\begin{aligned} \|\bar{T}^M - T^M\| \leq \frac{\|T^M\|}{1 - \|T^M\delta T\|}\|\pi_{\mathcal{R}(\bar{T})} - \pi_{\mathcal{R}(T)}\| \\ + \|T^M\|\|\pi_{\mathcal{N}(T)} - \pi_{\mathcal{N}(\bar{T})}\| + \frac{\|T^M\|\|T^M\delta T\|}{1 - \|T^M\delta T\|}. \end{aligned} \tag{3.2}$$

Finally, by using Lemma 3.3 and the inequality (3.2), we can obtain

$$\begin{aligned} &\|\bar{T}^M - T^M\| \\ &\quad \leq \frac{\|T^M\|}{1 - \|T^M\delta T\|}\big(\|\pi_{\mathcal{R}(\bar{T})} - \pi_{\mathcal{R}(T)}\| \\ &\qquad + \big(1 - \|T^M\delta T\|\big)\big\|\pi_{\mathcal{N}(T)} - \pi_{\mathcal{N}(\bar{T})}\| + \|T^M\delta T\|\big) \end{aligned}$$

$$\leq \begin{cases} \frac{\|T^M\|}{1-\|T^M\delta T\|}\{10(p-1)^{-\frac{1}{2}}\hat{\delta}(\mathcal{R}(T),\mathcal{R}(\bar{T}))^{\frac{1}{2}} \\ \quad +10(1-\|T^M\delta T\|)(p-1)^{-\frac{1}{2}}\hat{\delta}(\mathcal{N}(\bar{T}),\mathcal{N}(T))^{\frac{1}{2}} \\ \quad +\|T^M\delta T\|\} & (1<p\leq 2), \\ \frac{\|T^M\|}{1-\|T^M\delta T\|}\{10C_p^{-\frac{1}{p}}\hat{\delta}(\mathcal{R}(T),\mathcal{R}(\bar{T}))^{\frac{1}{p}} \\ \quad +10(1-\|T^M\delta T\|)C_p^{-\frac{1}{p}}\hat{\delta}(\mathcal{N}(\bar{T}),\mathcal{N}(T))^{\frac{1}{p}} \\ \quad +\|T^M\delta T\|\} & (2\leq p<+\infty). \end{cases}$$

This completes the proof. □

When the perturbed operator $\delta T$ satisfies $\mathcal{R}(\bar{T})=\mathcal{R}(T)$ and $\mathcal{N}(\bar{T})=\mathcal{N}(T)$, then it is easy to get the following result from Theorem 3.4.

**Corollary 3.5** ([3, Corollary 4.7])**.** *Let $X=L^p$ $(1<p<+\infty)$. Let $T\in B(X)$ with $\mathcal{R}(T)$ closed. Assume that $T^M$ is quasiadditive on $\mathcal{R}(T)$. Let $\delta T\in B(X)$ such that $T^M$ is quasiadditive on $\mathcal{R}(\delta T)$ and $\|T^M\|\|\delta T\|<1$. Put $\bar{T}=T+\delta T$. If $\mathcal{R}(\bar{T})=\mathcal{R}(T)$ and $\mathcal{N}(\bar{T})=\mathcal{N}(T)$, then $\bar{T}^M$ exists and*

$$\|\bar{T}^M-T^M\|\leq\frac{\|T^M\delta T\|\|T^M\|}{1-\|T^M\delta T\|}.$$

*Remark* 3.6. Let $X=L^p$ $(1<p<+\infty)$. Let $T$, $\delta T\in B(X)$ such that $T^M$ exists. Put $\bar{T}=T+\delta T$. From [6, Lemma 2.14], we know that $\|T^M\|\geq\gamma(T)^{-1}$ (here, $\gamma(T)$ is the reduced minimum module of $T$). Then, similar to [17, Lemma 1.3.5], we can check that

$$\delta\big(\mathcal{R}(T),\mathcal{R}(\bar{T})\big)\leq\|T^M\|\|\delta T\|,\qquad \delta\big(\mathcal{N}(\bar{T}),\mathcal{N}(T)\big)\leq\|T^M\|\|\delta T\|. \tag{3.3}$$

So from Theorem 3.4 and Lemma 3.2, we also have the following corollary.

**Corollary 3.7.** *Let $X=L^p$ $(1<p<+\infty)$. Let $T,\delta T\in B(X)$ with $\mathcal{R}(T)$ closed. Assume that $T^M$ is quasiadditive on $\mathcal{R}(T)$ and $\mathcal{R}(\delta T)$. Put $\bar{T}=T+\delta T$. If $\|T^M\|\|\delta T\|<1$ and $\mathcal{R}(\bar{T})\cap\mathcal{N}(T^M)=\{0\}$, then $\bar{T}^M$ exists and*

$$\|\bar{T}^M-T^M\| \leq \begin{cases} \frac{\|T^M\|}{1-\|T^M\delta T\|}\{20(p-1)^{-\frac{1}{2}}\eta \\ \quad +\|T^M\delta T\|(1-10(p-1)^{-\frac{1}{2}}\eta)\} & (1<p\leq 2), \\ \frac{\|T^M\|}{1-\|T^M\delta T\|}\{20C_p^{-\frac{1}{p}}\eta \\ \quad +\|T^M\delta T\|(1-10C_p^{-\frac{1}{p}}\eta)\} & (2\leq p<+\infty), \end{cases}$$

*where $\eta=\max\{\|T^M\|\|\delta T\|,\frac{2\|T^M\|^2\|\delta T\|}{1-\|T^M\delta T\|}\}$.*

*Proof.* From Lemma 3.2(3), we know that $\|\bar{T}^M\|\leq\frac{2\|T^M\|}{1-\|T^M\delta T\|}$. Then, similar to inequalities (3.3), for $\bar{T}^M$, we also get

$$\delta\big(\mathcal{R}(\bar{T}),\mathcal{R}(T)\big)\leq\|\bar{T}^M\|\|\delta T\|\leq\frac{2\|T^M\|^2\|\delta T\|}{1-\|T^M\delta T\|},$$

$$\delta\big(\mathcal{N}(T), \mathcal{N}(\bar{T})\big) \leq \|\bar{T}^M\| \|\delta T\| \leq \frac{2\|T^M\|^2 \|\delta T\|}{1 - \|T^M \delta T\|}.$$

Now, set $\eta = \max\{\|T^M\|\|\delta T\|, \frac{2\|T^M\|^2\|\delta T\|}{1-\|T^M\delta T\|}\}$. Then from the definition of gap function and Theorem 3.4, we can get our estimations easily by simple computation. □

In order to give an example in $L^p$ $(1 < p < +\infty)$, we need the definition of dual mapping for Banach spaces. Recall that (see [2]) the set-valued mapping $F_X : X \to X^*$ defined by

$$F_X(x) = \big\{ f \in X^* \;\big|\; f(x) = \|x\|^2 = \|f\|^2 \big\}, \quad \forall x \in X,$$

is called the *dual mapping* of $X$, where $X^*$ is the dual space of $X$. It is well known that the dual mapping $F_X$ is a homogeneous set-valued mapping (we refer the reader to [2] for more information about the mapping $F_X$ and the geometric properties of Banach spaces, such as strict convexity, reflexivity, and complemented subspaces). In particular, if $X$ is a reflexive and strictly convex Banach space, and $L \subset X$ is a hyperplane, then we have the following useful result.

**Lemma 3.8** ([16, Theorem 1.2.17])**.** *Let $X$ be a reflexive and strictly convex Banach space. Let $f \in X^*$. Suppose that $L = \{x \in X \mid f(x) = 0\}$ (i.e., $L \subset X$ is a hyperplane). Then $\pi_L$ is a bounded linear operator on $X$ and*

$$\pi_L(x) = x - \frac{f(x)}{\|f\|^2} F_X^{-1}(f), \quad \forall x \in X.$$

We now consider the following example in $L^p$ $(1 < p < +\infty)$.

*Example* 3.9. Let $X$ be the Banach space $L^p$ $(1 < p < +\infty)$. Fix an element $s \in X \setminus \{0\}$ and a continuous linear functional $f : X \to \mathbb{C}$ with $f(s) = 1$. We consider the following operator $T : X \to X$ defined by

$$T(x) = x - f(x)s.$$

Then, we can check easily that $T$ is a bounded linear operator on $X$ and that $\mathcal{N}(T) = \{\lambda s \mid \lambda \in \mathbb{C}\}$ and $\mathcal{R}(T) = \mathcal{N}(f)$. So, in general, the operator $T$ is not invertible in $B(X)$. But by Remark 2.6 and Lemma 2.9, we can obtain that the Moore–Penrose metric generalized inverse $T^M$ uniquely exists.

Noting that $\mathcal{N}(f) \subset X$ is a maximum linear subspace and that $s \notin \mathcal{N}(f) = \mathcal{R}(T)$, we have $X = \mathcal{N}(T) \dotplus \mathcal{R}(T)$. Now we define

$$T_0 : \mathcal{R}(T) \to \mathcal{R}(T) \quad \text{ by } T_0(x) = T(x), \forall x \in \mathcal{R}(T).$$

Clearly, $T_0$ is invertible with $T_0^{-1} = I_{\mathcal{R}(T)} \in B(\mathcal{R}(T))$. Put

$$By = \begin{cases} T_0^{-1} y, & y \in \mathcal{R}(T), \\ 0, & y \in \mathcal{N}(T). \end{cases}$$

Then, it is easy to check that the operator $B$ is a bounded homogeneous generalized inverse of $T$ with $B\pi_{\mathcal{R}(T)} = \pi_{\mathcal{R}(T)}$. Thus, we obtain that $T^M = (I_X -$

$\pi_{\mathcal{N}(T)})\pi_{\mathcal{R}(T)}$ by Lemma 3.1. From Lemma 2.7, we get $\|\pi_{\mathcal{R}(T)}\| \le 2$. Note that $\|I_X - \pi_{\mathcal{N}(T)}\| \le 1$, so we obtain $\|T^M\| \le 2$.

When the perturbed operator $\delta T \in B(X)$ satisfies $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$ with $\|\delta T\| < \frac{1}{2}$, for example, we can take $\delta T(x) = \frac{1}{3(1+\|f\|\|s\|_p)} T(x)$. Put $\bar{T} = T + \delta T$. Then, it follows from $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$ that $\mathcal{R}(\bar{T}) \subset \mathcal{R}(T)$ and $TT^M \delta T = \delta T$. Consequently, we get

$$\bar{T} = T + \delta T = T(I_X + T^M \delta T). \tag{3.4}$$

If $\|\delta T\| < 1/2$, then $\|T^M\|\|\delta T\| < 1$, so $(I_X + T^M \delta T)$ is invertible in $B(X)$. Thus, from (3.4), we have $\mathcal{R}(\bar{T}) = \mathcal{R}(T)$, which means that $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \mathcal{R}(T) \cap \mathcal{N}(T^M) = \{0\}$. Now, from the error estimates in Theorem 3.4, we can get the following:

$$\begin{aligned} &\|\bar{T}^M - T^M\| \\ &\le \begin{cases} \frac{2}{1-\|T^M \delta T\|}\{10(1 - \|T^M \delta T\|)(p-1)^{-\frac{1}{2}} \hat{\delta}(\mathcal{N}(\bar{T}), \mathcal{N}(T))^{\frac{1}{2}} \\ \quad + \|T^M \delta T\|\} & (1 < p \le 2), \\ \frac{2}{1-\|T^M \delta T\|}\{10(1 - \|T^M \delta T\|)C_p^{-\frac{1}{p}} \hat{\delta}(\mathcal{N}(\bar{T}), \mathcal{N}(T))^{\frac{1}{p}} \\ \quad + \|T^M \delta T\|\} & (2 \le p < +\infty). \end{cases} \end{aligned}$$

## 4. Perturbation for $T^M$ in uniformly convex Banach spaces

Finally, in this short section, we consider the perturbation problems in uniformly convex Banach spaces based on strong uniqueness-type results. It is well known that uniformly convex Banach spaces are reflexive and strictly convex Banach spaces [2].

Let $X$ be a Banach space, and let $S_X = \{x \in X : \|x\| = 1\}$ be its unit sphere. The moduli of convexity and smoothness of $X$ are the functions defined, respectively, by the formulas

$$\delta_X(\epsilon) = \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : x, y \in S_X, \|x - y\| = \epsilon\Big\}, \quad 0 < \epsilon \le 2.$$

The space $X$ is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for every $\epsilon > 0$. If, in addition, $\delta_X(\epsilon) \ge C\epsilon^q$ for some constant $C > 0$ and $q \ge 2$, we say that $X$ has *modulus of convexity* of power type $q$.

*Definition* 4.1 ([9, Definition 2.1]). The metric projection $\pi_M$ is said to be *strongly unique* of order $p > 0$ at $M$ if for each $x \in X$ and every $m \in M$ we have

$$\gamma_M(x)\|\pi_M x - m\|^\alpha \le \|x - m\|^\alpha - \|x - \pi_M x\|^\alpha \tag{4.1}$$

with some constant $\gamma_M(x) > 0$ depending only on $x$ and $M$.

**Lemma 4.2** ([10, Theorem 1])**.** *Let $X$ be a uniformly convex Banach space of power type $p$, and let $M$ be a closed subspace of $X$. Then the best approximation to $x \in X$ from $M$ is strongly unique of order $p$; that is, the metric projection $\pi_M$ satisfies the strong uniqueness condition (4.1).*

**Lemma 4.3** ([9, Theorem 2.1]). *Let $M$ be a closed linear subspace of the normed linear space $X$ such that the metric projection $\pi_M$ satisfies the strong uniqueness condition (4.1). Then for any $x \in X$ and every other closed linear subspace $N \subset X$ we have*

$$\|\pi_M - \pi_N\| \leq 10\gamma_M(x)^{-\frac{1}{\alpha}}\hat{\delta}(M,N)^{\frac{1}{\alpha}}. \tag{4.2}$$

We now present the main result of this section.

**Theorem 4.4.** *Let $X$ be a uniformly convex Banach space of power type $p$. Let $T$, $\delta T \in B(X)$ with $\mathcal{R}(T)$ closed. Assume that $T^M$ is quasiadditive on $\mathcal{R}(T)$ and $\mathcal{R}(\delta T)$. Put $\bar{T} = T + \delta T$. If $\|T^M\|\|\delta T\| < 1$ and $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, then $\bar{T}^M$ exists and*

$$\begin{aligned}
&\|\bar{T}^M - T^M\| \\
&\quad\leq \frac{\|T^M\|}{1-\|T^M\delta T\|}\Big\{10\gamma_{\mathcal{R}(\bar{T})}(x)^{-\frac{1}{\alpha}}\hat{\delta}\big(\mathcal{R}(\bar{T}),\mathcal{R}(T)\big)^{\frac{1}{\alpha}} \\
&\qquad + 10\big(1-\|T^M\delta T\|\big)\gamma_{\mathcal{N}(\bar{T}}(x)^{-\frac{1}{\alpha}}\hat{\delta}\big(\mathcal{N}(\bar{T}),\mathcal{N}(T)\big)^{\frac{1}{\alpha}} + \|T^M\delta T\|\Big\}, \quad \forall x \in X.
\end{aligned}$$

*Proof.* First, from Lemma 3.2, we see that $\bar{T}^M = (I_X - \pi_{\mathcal{N}(\bar{T})})(I_X + T^M\delta T)^{-1}T^M \times \pi_{\mathcal{R}(\bar{T})}$. Then, similar to the proof of Theorem 3.4, but using Lemma 4.3, we get our desired estimate

$$\begin{aligned}
&\|\bar{T}^M - T^M\| \\
&\quad\leq \frac{\|T^M\|}{1-\|T^M\delta T\|}\big(\|\pi_{\mathcal{R}(\bar{T})} - \pi_{\mathcal{R}(T)}\| \\
&\qquad + \big(1-\|T^M\delta T\|\big)\|\pi_{\mathcal{N}(T)} - \pi_{\mathcal{N}(\bar{T})}\| + \|T^M\delta T\|\big) \\
&\quad\leq \frac{\|T^M\|}{1-\|T^M\delta T\|}\Big\{10\gamma_{\mathcal{R}(\bar{T})}(x)^{-\frac{1}{\alpha}}\hat{\delta}\big(\mathcal{R}(\bar{T}),\mathcal{R}(T)\big)^{\frac{1}{\alpha}} \\
&\qquad + 10\big(1-\|T^M\delta T\|\big)\gamma_{\mathcal{N}(\bar{T}}(x)^{-\frac{1}{\alpha}}\hat{\delta}\big(\mathcal{N}(\bar{T}),\mathcal{N}(T)\big)^{\frac{1}{\alpha}} + \|T^M\delta T\|\Big\}, \quad \forall x \in X.
\end{aligned}$$

This completes the proof. □

## 5. Concluding remark

In this article, based on some error estimates of metric projections and the tool of stable perturbation, we obtained some new perturbation bounds for the Moore–Penrose metric generalized inverses in some Banach spaces, particularly in $L^p$ $(1 < p < +\infty)$ space and uniformly convex Banach space of power type $p$. We should note that, in a recent paper [6], using the gap between homogeneous subsets, under range-preserving, kernel-preserving, and other more general cases, respectively, the author also considered some perturbation problems for the Moore–Penrose metric generalized inverse in reflexive strictly convex Banach space. However, we have proved different results. It is well known that the theory of the Moore–Penrose metric generalized inverses has its genesis in the context of the so-called *ill-posed* problems. It is our hope that such new perturbation bounds obtained in these papers can find applications in the solution of the least problem (1.2) and its perturbation analysis in some Banach spaces. We will investigate these applications in a future research topic. It would also be interesting to extend our results from bounded linear operators to more general closed operators (see [7]). Since various differential operators in mathematical physics are unbounded closed operators, we would similarly like to propose this issue as an interesting project for further research.

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[1]Postdoctoral Research Station of Physics, Henan Normal University, Xinxiang, Henan, 453007, People's Republic of China, and Postdoctoral Research Base, Henan Institute of Science and Technology, Xinxiang, Henan, 45003, People's Republic of China.

*E-mail address*: caocjb@gmail.com

[2]Department of Mathematics, Henan Institute of Science and Technology, Xinxiang, Henan, 453003, People's Republic of China.

*E-mail address*: wqzhang@hist.edu.cn