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MINIMAL REDUCING SUBSPACES OF AN OPERATOR-WEIGHTED SHIFT

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ABSTRACT. We introduce a family $\mathcal{T}$ consisting of invertible matrices with exactly one nonzero entry in each row and each column. The elements of $\mathcal{T}$ are possibly mutually noncommuting, and they need not be normal or self-adjoint. We consider an operator-valued unilateral weighted shift W with a uniformly bounded sequence of weights belonging to $\mathcal{T}$, and we describe its minimal reducing subspaces.

1. Introduction

If K is a separable complex Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$, and $\{\alpha_n\}_{n=0}^{\infty}$ is a bounded sequence of scalars, then the operator T defined by $Te_n = \alpha_n e_{n+1}$ is called a *scalar shift* with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$. Now let $l^2(K) = \bigoplus_0^{\infty} K$ be the orthogonal sum of $\aleph_o$ copies of the Hilbert space K with a scalar product defined by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \langle f_n, g_n\rangle, \quad f = (f_0, f_1, \dots) \in l^2(K), g = (g_0, g_1, \dots) \in l^2(K).$$

Let $\{A_n\}_{n=0}^{\infty}$ be a uniformly bounded sequence of linear operators on K. The operator W on $l^2(K)$ defined by

$$W(x_0, x_1, \dots) = (0, A_0x_0, A_1x_1, \dots)$$

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is called an *operator-valued unilateral weighted shift* with weights $\{A_n\}_{n=0}^{\infty}$. Clearly, W is bounded, and $\|W\| = \sup_n \|A_n\|$. If each A_n is invertible, then W is an *invertibly weighted operator shift* (see [9]).

Operator-weighted shifts were introduced by N. K. Nikol'skii [11] in 1967. These are generalizations of the scalar-weighted shifts; however, this generalization is not just formal. For example, by means of an operator-weighted shift, Pearcy and Petrovic [12] proved that an n-normal operator is power bounded if and only if it is similar to a contraction. Since their introduction, operator-weighted shifts have been widely studied. A general understanding of their various properties can be found in a number of sources (see [1], [3], [6], [8]–[11]).

Our interest is to determine the minimal reducing subspaces of an invertibly weighted operator shift W on $l^2(K)$. A subspace M of $l^2(K)$ is *invariant* under W if $W(M) \subseteq M$. If M is invariant under both W and W^*, then M is said to be *reducing* for W. A reducing subspace M is said to be *minimal-reducing* if it does not contain any proper nonzero reducing subspace.

The invariant and reducing subspaces of specific types of invertibly weighted operator shifts are known from [4], [5], [9], [11], [13], [15]. However, we observe that, in all these cases, W is an operator-weighted shift with weight sequence $\{A_n\}_{n=0}^{\infty}$ where it is either assumed that the A_n's are commuting normal operators or it is assumed that each A_n is positive diagonal. In this paper we consider an operator-weighted shift W with weights $\{A_n\}_{n=0}^{\infty}$ such that the A_n's are neither normal nor commuting.

For this, let $\mathcal{B}(K)$ denote the set of all bounded linear operators on the separable complex Hilbert space K with orthonormal basis $\{e_n\}_{n=0}^{\infty}$, and let $\mathcal{T}$ be the subset of $\mathcal{B}(K)$ defined as follows: $\mathcal{T} := T \in \mathcal{B}(K) \mid T$ is invertible in $\mathcal{B}(K)$, and the matrix of T with respect to $\{e_n\}_0^{\infty}$ has exactly one nonzero entry in each row and each column.

We observe the following:

(i) If $T_1, T_2 \in \mathcal{T}$, then $T_1 T_2 \in \mathcal{T}$; however, T_1 and T_2 need not commute. Hence elements of $\mathcal{T}$ are not simultaneously diagonalizable with respect to $\{e_n\}_0^{\infty}$.

(ii) If $T \in \mathcal{T}$, then its Hilbert adjoint T^* and inverse T^{-1} are also in $\mathcal{T}$.

(iii) Elements of $\mathcal{T}$ may not be self-adjoint or normal.

In this paper, we determine the minimal reducing subspaces of the unilateral operator-weighted shift W with weight sequence $\{A_n\}_{n=0}^{\infty}$, where $A_n \in \mathcal{T}\ \forall n \geq 0$.

2. Unitary equivalence

Let K be a separable complex Hilbert space, and let $\mathcal{B}(K)$ denote the space of all bounded linear operators on K with norm defined as $\|T\| = \sup_{\|x\|=1} \|Tx\|$ for $T \in \mathcal{B}(K)$.

Let $\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for K. Also, for $i, j = 0, 1, 2, \ldots$, let $g_{i,j} := (0, \ldots, e_i, 0, \ldots)$ where e_i occurs at the jth position. If $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, then $\{g_{i,j}\}_{i,j \in \mathbb{N}_0}$ is an orthonormal basis for $l^2(K)$.

Let us now consider the *operator-weighted sequence space* $l_B^2(K)$. To define $l_B^2(K)$, let $B = \{B_n\}_{n=0}^{\infty}$ be a sequence of invertible bounded linear operators on

K, let $l_B^2(K) := \{(f_0, f_1, \ldots) : f_i \in K$, and $\sum_{i=0}^{\infty} \|B_i f_i\|^2 < \infty\}$. For $f = (f_i)$ and $g = (g_i)$ in $l_B^2(K)$, we have

$$\langle f, g\rangle_B := \sum_{i=0}^{\infty} \langle B_i f_i, B_i g_i\rangle \qquad \text{and} \qquad \|f\|_B^2 = \sum_{i=0}^{\infty} \|B_i f_i\|^2.$$

Since $\|g_{i,j}\|_B = \|B_j e_i\|$, therefore if $f_{i,j} := \frac{g_{i,j}}{\|B_j e_i\|}$, then $\{f_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for the Hilbert space $l_B^2(K)$. If $\dim K = 1$, then each B_n is a nonzero scalar β_n, and $l_B^2(K)$ is the scalar-weighted sequence space l_β^2 defined in [14, Section 3]. The unilateral shift S on $l_B^2(K)$ is defined as $S(f_0, f_1, \ldots) = (0, f_0, f_1, \ldots)$, and S is bounded if and only if $\sup_{i,j} \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} < \infty$. The meaning of the above terms and notation will remain unchanged throughout this article unless specifically stated otherwise.

Theorem 2.1. *Let S be the unilateral shift on $l_B^2(K)$, and for each $n \in \mathbb{N}_0$ we define operator A_n on K as $A_n e_i = (\frac{\|B_{n+1}e_i\|}{\|B_n e_i\|})e_i$. Then S is unitarily equivalent to the operator-weighted shift W on $l^2(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$.*

Proof. Let $V : l_B^2(K) \to l^2(K)$ be defined as $V f_{i,j} = g_{i,j}$ for all $i, j \in \mathbb{N}_0$, and let it extend linearly. Then V is unitary, and $V^* g_{i,j} = f_{i,j}$. We claim that $S = V^*WV$. To establish our claim, choose $i, j \in \mathbb{N}_0$. Then

$$Sf_{i,j} = \frac{1}{\|B_j e_i\|} S g_{i,j} = \frac{g_{i,j+1}}{\|B_j e_i\|} = \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} f_{i,j+1}.$$

Also,

$$\begin{aligned} V^*WV f_{i,j} &= V^*W g_{i,j} \\ &= V^*W(0, 0, \ldots, e_i, 0, \ldots) \\ &= V^*(0, 0, \ldots, A_j e_i, 0, \ldots) \\ &= \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} V^* g_{i,j+1} \\ &= \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} f_{i,j+1}. \end{aligned}$$

Hence $V^*WV = S$. □

For the converse, we consider a sequence $\{A_n\}_{n=0}^{\infty}$ of bounded linear operators on K such that $\sup_n \|A_n\| < \infty$. We first consider the case where A_n's are simultaneously diagonalizable with respect to $\{e_i\}_{i=0}^{\infty}$.

Theorem 2.2. *For $n \in \mathbb{N}_0$, let A_n be an invertible bounded linear operator on K such that the matrix of A_n with respect to $\{e_i\}_{i=0}^{\infty}$ is $\operatorname{diag}(\delta_0^{(n)}, \delta_1^{(n)}, \delta_2^{(n)}, \ldots)$. Also let $\sup_n \|A_n\| < \infty$. If W is the operator-weighted shift on $l^2(K)$ with weight sequence $\{A_n\}_{n=0}^{\infty}$, then W is unitarily equivalent to the unilateral shift S on $l_B^2(K)$, where B denotes the sequence $\{B_n\}_{n=0}^{\infty}$ with $B_0 := I$ and $B_{n+1} := A_n A_{n-1} A_{n-2} \ldots A_0$ for $n \in \mathbb{N}_0$.*

Proof. By [9, Theorem 3.4] we may assume that each A_n is positive. If $V : l^2_B(K) \to l^2(K)$ is defined linearly such that $Vf_{i,j} = g_{i,j}$ for all $i, j \in \mathbb{N}_0$, then V is unitary. Let $B_0 := I$, and let $B_{n+1} := A_n A_{n-1} A_{n-2} \dots A_0$ for $n \in \mathbb{N}_0$. Then $\|B_{n+1}e_i\| = \delta_i^{(n)}\delta_i^{(n-1)} \dots \delta_i^{(0)}$ for all $i, n \in \mathbb{N}_0$ so that $\frac{\|B_{n+1}e_i\|}{\|B_n e_i\|} = \delta_i^{(n)}$. Then as in Theorem 2.1, it can be shown that $V^*WV = S$. □

Next we consider the case where each A_n is in $\mathcal{T}$. Now elements of $\mathcal{T}$ have a specific type of matrix representation with respect to $\{e_i\}_{i\in\mathbb{N}_0}$. Let $T \in \mathcal{T}$, and for $j \in \mathbb{N}_0$ let γ_j denote the nonzero entry occurring in the jth column of the matrix of T with respect to $\{e_i\}_{i=0}^\infty$. Then there exists a unique bijective map $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ such that γ_j occurs at the $\psi(j)$th row. Thus if $[a_{i,j}]$ $(i, j \in \mathbb{N}_0)$ denotes the matrix of T with respect to $\{e_i\}_{i=0}^\infty$, then

$$a_{i,j} := \begin{cases} \gamma_j & \text{if i} = \psi(\text{j}), \\ 0 & \text{otherwise.} \end{cases}$$

Thus for each $j \in \mathbb{N}_0$, $Te_j = \gamma_j e_{\psi(j)}$, and $\|T\| = \sup_j |\gamma_j|$.

Since T is invertible in $\mathcal{B}(K)$, $\gamma_j \neq 0$ for each j, and $T^{-1}e_{\psi(j)} = \frac{1}{\gamma_j}e_j$. Hence if $\varphi := \psi^{-1}$, then for each $i \in \mathbb{N}_0$,

$$T^{-1}e_i = \frac{1}{\gamma_{\varphi(i)}}e_{\varphi(i)}, \qquad \text{and} \qquad \|T^{-1}\| = \sup_i \frac{1}{|\gamma_{\varphi(i)}|} = \frac{1}{\inf_i |\gamma_{\varphi(i)}|} = \frac{1}{\inf_j |\gamma_j|}.$$

If β_i denotes the nonzero entry in the ith row of $[a_{i,j}]$, then for $x = \sum_i x_i e_i \in K$,

$$T(x_0, x_1, x_2, \dots) = (\beta_0 x_{\varphi(0)}, \beta_1 x_{\varphi(1)}, \dots). \tag{2.1}$$

Theorem 2.3. *Let $\{A_n\}_{n=0}^\infty$ be a sequence in $\mathcal{T}$, and let $\sup_n \|A_n\| < \infty$. Then there exists a sequence $B = \{B_n\}_{n=0}^\infty$ of positive invertible diagonal bounded linear operators on K such that the operator-weighted shift W on $l^2(K)$ with weight sequence $\{A_n\}_{n=0}^\infty$ is unitarily equivalent to the unilateral shift S on $l^2_B(K)$.*

To prove the above theorem, we first prove the following lemma.

Lemma 2.4. *Let $\{A_n\}_{n=0}^\infty$ be a sequence in $\mathcal{T}$ with $\sup_n \|A_n\| < \infty$, and let W be an operator-weighted shift on $l^2(K)$ with weight sequence $\{A_n\}_{n=0}^\infty$. Then there exists a sequence $\{D_n\}_{n=0}^\infty$ of positive invertible diagonal operators on K such that W is unitarily equivalent to the operator-weighted shift T on $l^2(K)$ with weight sequence $\{D_n\}_{n=0}^\infty$.*

Proof. For each $n \in \mathbb{N}_0$ there exists a bijective map $\psi_n : \mathbb{N}_0 \to \mathbb{N}_0$ such that $A_n e_i = \gamma_i^{(n)} e_{\psi_n(i)}$ for nonzero scalars $\gamma_i^{(n)}$ and $i \in \mathbb{N}_0$.

Let $A_n = U_n P_n$ be the polar decomposition of A_n. Then $P_n \geq 0$ is invertible diagonal, and $P_n e_i = |\gamma_i^{(n)}| e_i$ for all $i \in \mathbb{N}_0$. Also, U_n is unitary with $U_n e_i = \frac{\gamma_i^{(n)}}{|\gamma_i^{(n)}|} e_{\psi_n(i)}$ for all $i \in \mathbb{N}_0$. Define $P, U, U_+ : l^2(K) \to l^2(K)$ as follows:

$$\begin{aligned} P(x_0, x_1, \dots) &= (P_0 x_0, P_1 x_1, \dots), \\ U(x_0, x_1, \dots) &= (U_0 x_0, U_1 x_1, \dots), \\ U_+(x_0, x_1, \dots) &= (0, x_0, x_1, \dots). \end{aligned}$$

Then $W = (U_+U)P$, which is in fact the polar decomposition of W. Let $V_0 = I$, and let $V_{n+1} = U_nV_n$ for all $n \in \mathbb{N}_0$. Then each V_n is unitary on K. Let $V : l^2(K) \to l^2(K)$ be defined as $V(x_0, x_1, \dots) = (V_0x_0, V_1x_1, \dots)$. Then V is unitary, and $U_+U = VU_+V^*$. Thus $W = U_+UP = VU_+V^*P = V(U_+V^*PV)V^*$. Moreover, V is unitary, and hence W is unitarily equivalent to U_+V^*PV. Let $D_n := V_n^*P_nV_n$ for all $n \in \mathbb{N}_0$. For each $x \in K$, $\langle D_nx, x\rangle = \langle V_n^*P_nV_nx, x\rangle = \langle P_nV_nx, V_nx\rangle \geq 0$. This implies that $D_n \geq 0$. Also, the fact that P_n is diagonal and V_n is unitary implies that D_n is diagonal.

If $T = U_+V^*PV$, then $T(x_0, x_1, \dots) = (0, D_0x_0, D_1x_1, \dots)$; that is, T is an operator-weighted shift on $l^2(K)$ with weight sequence $\{D_n\}_{n=0}^{\infty}$ of positive invertible diagonal operators on K. □

Proof of Theorem 2.3. By Lemma 2.4, there exists a sequence $\{D_n\}_{n=0}^{\infty}$ of positive invertible diagonal operators on K and an operator-weighted shift T on $l^2(K)$ with weight sequence $\{D_n\}_{n=0}^{\infty}$ such that W is unitarily equivalent to T. By Theorem 2.2, T is unitarily equivalent to the unilateral shift S on $l^2_B(K)$ with $B = \{B_n\}_{n=0}^{\infty}$, where $B_0 := I$, and $B_n := D_nD_{n-1}\dots D_0$ for $n \in \mathbb{N}_0$. Thus W is also unitarily equivalent to S on $l^2_B(K)$. □

Remark 2.5. The D_n's, as given in Lemma 2.4, are defined as follows: if for each $n \in \mathbb{N}_0, \psi_n : \mathbb{N}_0 \to \mathbb{N}_0$ is the bijective map such that $A_ne_i = \gamma_i^{(n)}e_{\psi_n(i)}$, then each D_n is given as

$$D_0 = \operatorname{diag}\big(|\gamma_0^{(0)}|, |\gamma_1^{(0)}|, |\gamma_2^{(0)}|, \dots\big) \quad \text{for } n = 0,$$
$$D_n = \operatorname{diag}\big(|\gamma^{(n)}_{\psi_{n-1}\psi_{n-2}\dots\psi_0(0)}|, |\gamma^{(n)}_{\psi_{n-1}\psi_{n-2}\dots\psi_0(1)}|, |\gamma^{(n)}_{\psi_{n-1}\psi_{n-2}\dots\psi_0(2)}|, \dots\big) \quad \text{for } n > 0.$$

The minimal reducing subspaces of S on $l^2_B(K)$ are determined in [5], where it is assumed that B represents a uniformly bounded sequence of invertible diagonal operators on K. So in view of Theorem 2.3 and [5], we should be able to determine the minimal reducing subspaces of the operator-weighted shift W on $l^2(K)$ with weights $\{A_n\}$ in $\mathcal{T}$. However, because of the complex transformations involved in the process, it is quite difficult to easily appreciate the end result. Hence in the present article, we adopt a different approach.

For operator-valued weighted shift W with nondiagonal operator weights, we first try representing W as a direct sum of scalar-weighted shift operators, as suggested in [13]. In this respect we have a theorem from [9].

Theorem 2.6 ([9, Theorem 3.9])**.** *The operator-weighted shift W on $l^2(K)$ with operator weights $\{A_n\}_{n=0}^{\infty}$ is a direct sum of scalar-weighted shifts if and only if the weakly closed $*$ algebra generated by $\{I, A_0, A_1, \dots\}$ is diagonalizable.*

Note that an algebra $\mathcal{B}$ of operators is regarded as diagonalizable if there is an orthonormal basis for the underlying space such that each operator in $\mathcal{B}$ is diagonal with respect to this basis. We consider the operator-weighted shift W on $l^2(K)$ with weights A_n in $\mathcal{T}$. In view of Lemma 2.4 and Theorem 3.1, it is possible to express W as a direct sum of scalar-weighted shift operators. Based on these scalar-weighted shifts, we then proceed to determine the minimal reducing subspaces of W.

3. Direct sum of scalar shifts

Since K is assumed to be a separable complex Hilbert space, $K \cong l^2$, where $l^2 = \{x = (x_0, x_1, \dots) : x_i \in \mathbb{C} \text{ and } \sum_i |x_i|^2 < \infty\}$. Let $\{\xi_i\}_{i\in\mathbb{N}_0}$ denote the standard orthonormal basis for l^2. If $\mu_{i,j} := (0, 0, \dots, \xi_j, 0, \dots)$ where ξ_j occurs at the ith place, then $\{\mu_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for $l^2 \oplus l^2 \oplus \dots$.

Theorem 3.1. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$, where each A_n is positive invertible diagonal with respect to the orthonormal basis $\{e_i\}_{i\in\mathbb{N}_0}$ of K. Then there exists scalar-weighted shift operators $S_0, S_1, \dots$ on l^2 such that W on $l^2(K)$ is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$ on $l^2 \oplus l^2 \oplus \dots$.*

Proof. For $n \in \mathbb{N}_0$, let A_n with respect to $\{e_i\}_{i\in\mathbb{N}_0}$ be the diagonal matrix $\operatorname{diag}(\delta_0^{(n)}, \delta_1^{(n)}, \dots)$. Define S_n to be the scalar-weighted shift on l^2 with weight sequence $\{\delta_n^{(j)}\}_{j\in\mathbb{N}_0}$. Then $S_n\xi_j = \delta_n^{(j)}\xi_{j+1}$ for all $j \in \mathbb{N}_0$. Therefore,

$$(S_0 \oplus S_1 \oplus \dots)\mu_{i,j} = \delta_i^{(j)}\mu_{i,j+1}.$$

Also, $Wg_{i,j} = W(0, 0, \dots, e_i, 0, \dots) = (0, 0, \dots, 0, A_je_i, 0, \dots) = \delta_i^{(j)}g_{i,j+1}$. If $V : l^2(K) \to l^2 \oplus l^2 \oplus \dots$ is defined by $Vg_{i,j} = \mu_{i,j}$, then V is unitary, and $VWV^*\mu_{i,j} = VWg_{i,j} = \delta_i^{(j)}Vg_{i,j+1} = \delta_i^{(j)}\mu_{i,j+1} = (S_0 \oplus S_1 \oplus \dots)\mu_{i,j}$.

Thus W on $l^2(K)$ is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$ on $l^2 \oplus l^2 \oplus \dots$. □

Remark 3.2. If $\dim K < \infty$, then the above result can also be deduced using Lemma 2.1 from [10].

Theorem 3.3. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded operator weights $\{A_n\}_{n\in\mathbb{N}_0}$, where each $A_n \in \mathcal{T}$. Then there exist scalar-weighted shift operators S_0, S_1, … on l^2 such that W on $l^2(K)$ is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$ on $l^2 \oplus l^2 \oplus \dots$.*

The proof follows immediately from Lemma 2.4 and Theorem 3.1; however, we include an independent proof so that the structure of S_n, which is often used in later sections, is explicitly given.

Proof. For each $A_n \in \mathcal{T}$, there exists a unique bijective map ψ_n on $\mathbb{N}_0$ such that $A_ne_j = \gamma_j^{(n)}e_{\psi_n(j)}$ for all $j \in \mathbb{N}_0$.

Let $U : l^2(K) \to l^2 \oplus l^2 \oplus \dots$ be linearly defined such that

$$Ug_{i,j} := \begin{cases} \mu_{i,0} & \text{if } j = 0, \\ \mu_{\psi_0^{-1}\psi_1^{-1}\dots\psi_{j-1}^{-1}(i),j} & \text{if } j > 0. \end{cases}$$

Then U is unitary. For $n \in \mathbb{N}_0$, let S_n be a scalar-weighted shift on l^2 with weight sequence $\{\gamma_n^{(0)}, \gamma_{\psi_0(n)}^{(1)}, \gamma_{\psi_1\psi_0(n)}^{(2)}, \dots\}$; that is,

$$S_n\xi_j := \begin{cases} \gamma_n^{(0)}\xi_1 & \text{if } j = 0, \\ \gamma_{\psi_{j-1}\psi_{j-2}\dots\psi_0(n)}^{(j)}\xi_{j+1} & \text{if } j > 0. \end{cases}$$

Hence

$$(S_0 \oplus S_1 \oplus \dots)\mu_{i,j} = \begin{cases} \gamma_i^{(0)}\mu_{i,1} & \text{if } j = 0, \\ \gamma^{(j)}_{\psi_{j-1}\psi_{j-2}\dots\psi_0(i)}\mu_{i,j+1} & \text{if } j > 0, \end{cases}$$
$$= UWU^*\mu_{i,j}. \qquad \square$$

In view of Theorem 3.3, we now propose the following definitions.

Definition 3.4. Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. Let $S_0, S_1, \dots$ be scalar-weighted shifts on l^2 such that W is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$. For $n, m \in \mathbb{N}_0$, we say 'n is related to m with respect to W' denoted by $n \sim^W m$ if S_n and S_m are identical. Clearly $\sim^W$ is an equivalence relation on $\mathbb{N}_0$.

Definition 3.5. Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Let $S_0, S_1, \dots$ be scalar-weighted shifts on l^2 such that W is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$. Note that W is considered to be of Type I if no two S_n's are identical. Otherwise W is said to be of Type II. Thus W is of Type II if and only if there exist distinct nonnegative integers n and m such that S_n and S_m are identical. An operator-weighted shift W of Type II is said to be of Type III if $\sim^W$ partitions $\mathbb{N}_0$ into a finite number of equivalence classes.

The above definition is motivated by similar definitions given in [15, Section 1]. In fact for $\dim K = N < \infty$, the two definitions refer to the same idea, as can be seen from the following. In [15] the minimal reducing subspaces of M_z^N $(N > 1)$ on the space $H^2_w := \{f(z) = \sum_{k=0}^{\infty} a_k z^k : \|f\|^2_w = \sum w_k |a_k|^2 < \infty\}$ are determined, where $w = \{w_0, w_1, \dots\}$ is a sequence of positive numbers.

If in the present study we consider $\dim K = N$, and for each $n \in \mathbb{N}_0$, we define $B_n = \operatorname{diag}(\sqrt{w_{nN}}, \sqrt{w_{nN+1}}, \dots, \sqrt{w_{(n+1)N-1}})$, then M_z^N on H^2_w is unitarily equivalent to the unilateral shift S on $l^2_B(K)$.

Again, if for each $n \in \mathbb{N}_0$ we define

$$A_n = \operatorname{diag}\Big(\sqrt{\frac{w_{(n+1)N}}{w_{nN}}}, \sqrt{\frac{w_{(n+1)N+1}}{w_{nN+1}}}, \dots, \sqrt{\frac{w_{(n+2)N-1}}{w_{(n+1)N-1}}}\Big),$$

and we consider W to be the operator-weighted shift on $l^2(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$, then S is unitarily equivalent to W, as in Theorem 2.1. Thus M_z^N on H^2_w is unitarily equivalent to operator-weighted shift W on $l^2(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$.

For $0 \le n \le N-1$, let S_n be the scalar-weighted shift on l^2 with weight sequence $\{\sqrt{\frac{w_{n+N}}{w_n}}, \sqrt{\frac{w_{n+2N}}{w_{n+N}}}, \sqrt{\frac{w_{n+3N}}{w_{n+2N}}}, \dots\}$. Then the operator-weighted shift W on $l^2(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$ is unitarily equivalent to $S_0 \oplus \dots \oplus S_{N-1}$ on $l^2 \oplus \dots \oplus l^2$ (N copies), as in Theorem 3.1.

By Definition 3.5, W is of Type I if no two S_n's are identical. This means that for each $0 \le n \le N-1$ and $0 \le m \le N-1$ with $n \ne m$, there exists $l > 0$ such that $\sqrt{\frac{w_{n+lN}}{w_{n+(l-1)N}}} \ne \sqrt{\frac{w_{m+lN}}{w_{m+(l-1)N}}}$. If k is the smallest positive integer for which

$\sqrt{\frac{w_{n+kN}}{w_{n+(k-1)N}}} \neq \sqrt{\frac{w_{m+kN}}{w_{m+(k-1)N}}}$, then $\frac{w_{n+kN}}{w_n} \neq \frac{w_{m+kN}}{w_m}$. So W is of Type I if, for each $0 \leq n \leq N-1$ and $0 \leq m \leq N-1$ with $n \neq m$, there exists $k > 0$ such that $\frac{w_{n+kN}}{w_n} \neq \frac{w_{m+kN}}{w_m}$, and this implies that the sequence w is of Type I (see [15]).

4. Extremal functions of reducing subspaces

We begin the section by introducing a few definitions and notation which are to be used in subsequent results.

Definition 4.1. Let $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ be a nonzero vector in $l^2(K)$. The order of F, denoted as $o(F)$, is defined as the smallest nonnegative integer m such that $\alpha_m \neq 0$.

Definition 4.2. If $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$ is a nonzero vector in K, then the order of f, denoted as $o(f)$, is defined to be the smallest nonnegative integer m such that $\alpha_m \neq 0$.

Definition 4.3. If $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i \in K$, then we define F_f in $l^2(K)$ as $F_f = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$. It follows that if $f \neq 0$, then $o(f) = o(F_f)$.

Definition 4.4. Let Y be a nonzero nonempty subset of K. Then the order of Y, denoted as $o(Y)$, is defined to be the nonnegative integer m satisfying the following conditions:

(i) $o(f) \geq m$ for all $f \in Y$,
(ii) there exists $\tilde{f} \in Y$ such that $o(\tilde{f}) = m$.

Definition 4.5. Let X be a subset of $l^2(K)$, and let $\mathcal{L}_X := \{f_0 : (f_0, f_1, \dots) \in X\}$. If $\mathcal{L}_X$ is a nonzero subset of K, then the order of X, denoted as $o(X)$, is defined as $o(\mathcal{L}_X)$.

Definition 4.6. Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. A linear expression $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ is said to be W-transparent if, for every pair of nonzero scalars α_i and α_j, we have $i \sim^W j$.

Definition 4.7. Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$, and let $\mathcal{S}$ be the vector space of all finite linear combinations of finite products of W and W^*. For nonzero $F \in l^2(K)$, let $\mathcal{S}F := \{TF : T \in \mathcal{S}\}$. Then the closure of $\mathcal{S}F$ in $l^2(K)$ is a reducing subspace of W, denoted by X_F. Clearly, X_F is the smallest reducing subspace of $l^2(K)$ containing F.

Lemma 4.8. *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and let W be the operator-weighted shift on $l^2(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$. The following will hold:*

(i) *for each $n \in \mathbb{N}_0$, $A_n^* e_i = \gamma_{\psi_n^{-1}(i)}^{(n)} e_{\psi_n^{-1}(i)}$ for all $i \in \mathbb{N}_0$,*
(ii) *$W^*(f_0, f_1, \dots) = (A_0^* f_1, A_1^* f_2, \dots)$ for $(f_0, f_1, \dots) \in l^2(K)$.*

(iii) *For* $i, j \in \mathbb{N}_0$, $Wg_{i,j} = \gamma_i^{(j)} g_{\psi_j(i),j+1}$, *and*

$$W^* g_{i,j} = \begin{cases} 0 & \text{if } j = 0, \\ \gamma^{(j-1)}_{\psi^{-1}_{j-1}(i)} g_{\psi^{-1}_{j-1}(i),j-1} & \text{if } j > 0. \end{cases}$$

(iv) *For* $i, j \in \mathbb{N}_0$,

$$(W^*)^k W^k g_{i,j} = \begin{cases} [\gamma_i^{(j)}]^2 g_{i,j} & \text{if } k = 1, \\ [\gamma_i^{(j)} \gamma^{(j+1)}_{\psi_j(i)} \dots \gamma^{(j+k-1)}_{\psi_{j+k-2}\dots\psi_j(i)}]^2 g_{i,j} & \text{if } k > 1. \end{cases}$$

(v) *For distinct nonnegative integers* n *and* m, *if* $n \sim^W m$, *then it holds that* $\|(W^*)^k W^k g_{n,0}\| = \|(W^*)^k W^k g_{m,0}\|$ *for each* $k \in \mathbb{N}$.

Proof. (i) For $f = \sum_{j\in\mathbb{N}_0} \alpha_j e_j \in K$, and $n \in \mathbb{N}_0$, $\langle A_n f, e_i\rangle = \sum_j \alpha_j \langle \gamma_j^{(n)} e_{\psi_n(j)}, e_i\rangle = \alpha_{\psi_n^{-1}(i)} \gamma^{(n)}_{\psi_n^{-1}(i)} = \langle f, \gamma^{(n)}_{\psi_n^{-1}(i)} e_{\psi_n^{-1}(i)}\rangle$. Hence $A_n^* e_i = \gamma^{(n)}_{\psi_n^{-1}(i)} e_{\psi_n^{-1}(i)}$ for all $i \in \mathbb{N}_0$.

(ii) For $g = (g_0, g_1, \dots) \in l^2(K)$, $\langle Wg, f\rangle = \sum_{i=0}^{\infty} \langle A_i g_i, f_{i+1}\rangle = \sum_{i=0}^{\infty} \langle g_i, A_i^* f_{i+1}\rangle = \langle g, (A_0^* f_1, A_1^* f_2, \dots)\rangle$, and so $W^*(f_0, f_1, \dots) = (A_0^* f_1, A_1^* f_2, \dots)$ for $f = (f_0, f_1, \dots) \in l^2(K)$.

(iii) This follows from (i) and (ii), and (iv) follows from (iii).

(v) For $n \in \mathbb{N}_0$, let S_n be the scalar-weighted shift on l^2 with weight sequence $\{\gamma_n^{(0)}, \gamma^{(1)}_{\psi_0(n)}, \gamma^{(2)}_{\psi_1\psi_0(n)}, \dots\}$. Then by Theorem 3.3, W is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$. As $n \sim^W m$, S_n and S_m are identical according to Definition 3.4. Therefore, $\gamma_n^{(0)} = \gamma_m^{(0)}$, and $\gamma^{(k+1)}_{\psi_k\psi_{k-1}\dots\psi_0(n)} = \gamma^{(k+1)}_{\psi_k\psi_{k-1}\dots\psi_0(m)}$ $\forall k \geq 0$. The result now follows immediately from (iv). □

Lemma 4.9. *Let* $\{A_n\}_{n\in\mathbb{N}_0}$ *be a uniformly bounded sequence of operators in* $\mathcal{T}$ *and let* W *be the operator-weighted shift on* $l^2(K)$ *with weight sequence* $\{A_n\}_{n\in\mathbb{N}_0}$. *Let* ψ_n *denote the unique bijective map on* $\mathbb{N}_0$ *such that* $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ *with* $\gamma_j^{(n)} > 0$. *Let* $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ *be* W*-transparent in* $l^2(K)$ *with* $o(F) = m$. *If*

$$\tilde{F}_k := \begin{cases} F & \text{if } k = 0, \\ \sum_{i\in\mathbb{N}_0} \alpha_i g_{\psi_{k-1}\psi_{k-2}\dots\psi_0(i),k} & \text{if } k > 1, \end{cases}$$

then the following will hold:

(i)

$$(W^*)^k W^k F = \begin{cases} [\gamma_m^{(0)}]^2 F & \text{if } k = 1, \\ [\gamma_m^{(0)} \gamma^{(1)}_{\psi_0(m)} \dots \gamma^{(k-1)}_{\psi_{k-2}\dots\psi_0(m)}]^2 F & \text{if } k > 1. \end{cases}$$

(ii)

$$W\tilde{F}_k = \begin{cases} \gamma_m^{(0)} \tilde{F}_1 & \text{if } k = 0, \\ \gamma^{(k)}_{\psi_{k-1}\dots\psi_0(m)} \tilde{F}_{k+1} & \text{if } k > 0. \end{cases}$$

(iii)

$$W^*\tilde{F}_k = \begin{cases} 0 & \text{for } k = 0, \\ \gamma_m^{(0)}\tilde{F}_0 & \text{for } k = 1, \\ \gamma_{\psi_{k-2}\ldots\psi_0(m)}^{(k-1)}\tilde{F}_{k-1} & \text{for } k > 1, \end{cases}$$

(iv) *X_F is the closed linear span of $\{\tilde{F}_k : k \in \mathbb{N}_0\}$.*

Proof. As $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ is W-transparent in $l^2(K)$ with $o(F) = m$, the following must hold:

(a) $\alpha_m \neq 0$, and $\alpha_i = 0$ for $0 \leq i < m$;
(b) if $\alpha_i \neq 0$, and $\alpha_j \neq 0$, then $i \sim^W j$.

Thus we must have $i \sim^W m$ for all $i \in \mathbb{N}_0$ with $\alpha_i \neq 0$. Hence

$$\gamma_i^{(0)} = \gamma_m^{(0)} \qquad \text{and} \qquad \gamma_{\psi_k\psi_{k-1}\ldots\psi_0(i)}^{(k+1)} = \gamma_{\psi_k\psi_{k-1}\ldots\psi_0(m)}^{(k+1)} \quad \forall k \geq 0. \tag{4.1}$$

(i) This follows from 4.1 and Lemma 4.8(iv).
(ii) Here $W\tilde{F}_0 = WF = \sum_i \alpha_i W g_{i,0} = \sum_i \alpha_i \gamma_i^{(0)} g_{\psi_0(i),1} = \gamma_m^{(0)}\tilde{F}_1$.
For $k > 0$,

$$\begin{aligned} W\tilde{F}_k &= \sum_i \alpha_i W g_{\psi_{k-1}\ldots\psi_0(i),k} \\ &= \sum_i \alpha_i \gamma_{\psi_{k-1}\ldots\psi_0(i)}^{(k)} g_{\psi_k\ldots\psi_0(i),k+1} \\ &= \gamma_{\psi_{k-1}\ldots\psi_0(m)}^{(k)}\tilde{F}_{k+1}. \end{aligned}$$

(iii) This can be shown similarly using 4.1 and Lemma 4.8(iii).
(iv) By (ii) and (iii), each $\tilde{F}_k \in X_F$, and the closed linear span$\{\tilde{F}_k : k \in \mathbb{N}_0\}$ is a nonzero reducing subspace of W contained in X_F. Thus by minimality of X_F, we have $X_F =$ closed linear span$\{\tilde{F}_k : k \in \mathbb{N}_0\}$.

□

Definition 4.10. Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. Let $\Omega_1, \Omega_2, \ldots$ be the disjoint equivalence classes of $\mathbb{N}_0$ under the relation $\sim^W$. Consider $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0} \in l^2(K)$. For each k, let $q_k := \sum_{i\in\Omega_k} \alpha_i g_{i,0}$. Dropping those q_k's which are zero, the remaining q_k's are arranged as $f_1, f_2, \ldots$ in such a way that for $i < j$ we have $o(f_i) < o(f_j)$. The resulting decomposition $F = f_1 + f_2 + \cdots$ is called the *canonical decomposition* of F with respect to W. Clearly, each f_i is W-transparent in $l^2(K)$.

If there exists a finite positive integer n such that $F = f_1 + f_2 + \cdots + f_n$, then F is said to have a finite canonical decomposition.

Lemma 4.11. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. Let X be a reducing subspace of W, and let $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ be in X. If F has a finite canonical decomposition $F = f_1 + f_2 + \cdots + f_n$, then each $f_i \in X_F$.*

Proof. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$.

Let $o(f_i) = m_i$ so that $m_1 < m_2 < \cdots < m_n$. Clearly, $m_i \nsim^W m_j$ for $i \neq j$.

Step I: Since $m_1 \nsim^W m_n$, either $\gamma_{m_1}^{(0)} \neq \gamma_{m_n}^{(0)}$ or there exists $k > 0$ such that $\gamma_{\psi_{k-1}\ldots\psi_0(m_1)}^{(k)} \neq \gamma_{\psi_{k-1}\ldots\psi_0(m_n)}^{(k)}$. In case $\gamma_{m_1}^{(0)} = \gamma_{m_n}^{(0)}$, let k_1 be the smallest positive integer such that $\gamma_{\psi_{k_1-1}\ldots\psi_0(m_1)}^{(k_1)} \neq \gamma_{\psi_{k_1-1}\ldots\psi_0(m_n)}^{(k_1)}$.

Let

$$Q_1 := \begin{cases} [(\gamma_{m_n}^{(0)})^2 - W^*W]F & \text{if } \gamma_{m_1}^{(0)} \neq \gamma_{m_n}^{(0)}, \\ [(\gamma_{m_n}^{(0)}\gamma_{\psi_0(m_n)}^{(1)} \cdots \gamma_{\psi_{k_1-1}\ldots\psi_0(m_n)}^{(k_1)})^2 - (W^*)^{k_1+1}W^{k_1+1}]F & \text{otherwise.} \end{cases}$$

For $1 \leq i \leq n-1$, let $\beta_i^{(1)} := (\gamma_{m_n}^{(0)})^2 - (\gamma_{m_i}^{(0)})^2$ if $\gamma_{m_1}^{(0)} \neq \gamma_{m_n}^{(0)}$; otherwise, let

$$\beta_i^{(1)} := (\gamma_{m_n}^{(0)}\gamma_{\psi_0(m_n)}^{(1)} \cdots \gamma_{\psi_{k_1-1}\ldots\psi_0(m_n)}^{(k_1)})^2 - (\gamma_{m_i}^{(0)}\gamma_{\psi_0(m_i)}^{(1)} \cdots \gamma_{\psi_{k_1-1}\ldots\psi_0(m_i)}^{(k_1)})^2.$$

Then $\beta_1^{(1)} \neq 0$. Also, since each f_i is W-transparent, by applying Lemma 4.9(i), we get $Q_1 = \sum_{i=1}^{n-1} \beta_i^{(1)} f_i \in X_F$.

Step II: As $m_1 \nsim^W m_{n-1}$, either $\gamma_{m_1}^{(0)} \neq \gamma_{m_{n-1}}^{(0)}$ or k_2 is the smallest positive integer such that $\gamma_{\psi_{k_2-1}\ldots\psi_0(m_1)}^{(k_2)} \neq \gamma_{\psi_{k_2-1}\ldots\psi_0(m_{n-1})}^{(k_2)}$.

Let

$$Q_2 := \begin{cases} [(\gamma_{m_{n-1}}^{(0)})^2 - W^*W]Q_1 & \text{if } \gamma_{m_1}^{(0)} \neq \gamma_{m_{n-1}}^{(0)}, \\ [(\gamma_{m_{n-1}}^{(0)}\gamma_{\psi_0(m_{n-1})}^{(1)} \cdots \gamma_{\psi_{k_2-1}\ldots\psi_0(m_{n-1})}^{(k_2)})^2 - (W^*)^{k_2+1}W^{k_2+1}]Q_1. \end{cases}$$

For $1 \leq i \leq n-2$, let $\beta_i^{(2)} := (\gamma_{m_{n-1}}^{(0)})^2 - (\gamma_{m_i}^{(0)})^2$ if $\gamma_{m_1}^{(0)} \neq \gamma_{m_{n-1}}^{(0)}$; otherwise, let

$$\beta_i^{(2)} := (\gamma_{m_{n-1}}^{(0)}\gamma_{\psi_0(m_{n-1})}^{(1)} \cdots \gamma_{\psi_{k_2-1}\ldots\psi_0(m_{n-1})}^{(k_2)})^2 - (\gamma_{m_i}^{(0)}\gamma_{\psi_0(m_i)}^{(1)} \cdots \gamma_{\psi_{k_2-1}\ldots\psi_0(m_i)}^{(k_2)})^2.$$

Then $\beta_1^{(2)} \neq 0$, and $Q_2 = \sum_{i=1}^{n-2} \beta_i^{(1)}\beta_i^{(2)} f_i \in X_F$.

Repeating the above argument $n-1$ times, we get $Q_{n-1} = \beta_1^{(1)}\beta_1^{(2)} \ldots \beta_1^{(n-1)} f_1 \in X_F$ with $\beta_1^{(i)} \neq 0$ for $1 \leq i \leq n-1$. This implies that $f_1 \in X_F$. By a similar procedure it can be shown that $f_i \in X_F$ for $1 < i \leq n$. □

Lemma 4.12. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. If X is a reducing subspace of W, then $\mathcal{L}_X = 0$ if and only if $X = 0$.*

Proof. Let $X = 0 \Rightarrow \mathcal{L}_X = 0$. Conversely, suppose that $X \neq 0$, and, if possible, let $\mathcal{L}_X = 0$. Since $X \neq 0$ we can choose $f = (0, f_1, f_2, \ldots) \in X$ with $f_n \neq 0$. Then by Lemma 4.8(ii), $(W^*)^n f = (g_1, g_2, \ldots)$ where $g_1 \neq 0$. As $(W^*)^n f \in X$, so $g_1 \in \mathcal{L}_X$, a contradiction. Thus $X \neq 0 \Rightarrow \mathcal{L}_X \neq 0$. □

Theorem 4.13. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. Let X be a nonzero reducing subspace of W with*

$o(X) = m$. Then the extremal problem

$$\sup\Big\{\operatorname{Re}\alpha_m : F = (f_0, f_1, \ldots) \in X, \|F\| \leq 1, f_0 = \sum_{i\in\mathbb{N}_0} \alpha_i e_i.\Big\}$$

has a unique solution $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0} \in X$ with $\|G\| = 1$ and $o(G) = m$.

Proof. Define $\varphi : X \to \mathbb{C}$ as $\varphi(F) = \alpha_m$, where $F = (f_0, f_1, \ldots)$, and $f_0 = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$. As $X \neq 0$, then $\mathcal{L}_X \neq 0$, by Lemma 4.12 and in view of Definition 4.5, $o(\mathcal{L}_X) = m = o(X)$. Therefore φ is a nonzero bounded linear functional on X. From [2] we know that there exists a unique $G \in X$ such that $\varphi(G) > 0$, $\|G\| = 1$, and that

$$\begin{aligned}\varphi(G) &= \sup\big\{\operatorname{Re}\varphi(F) : F \in X, \|F\| \leq 1\big\}\\ &= \sup\Big\{\operatorname{Re}\alpha_m : F = (f_0, f_1, \ldots) \in X, \|F\| \leq 1, f_0 = \sum_{i\in\mathbb{N}_0} \alpha_i e_i.\Big\}.\end{aligned}$$

We will show that $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ and that $o(G) = m$. For this we consider $G = (g_0, g_1, \ldots)$.

Claim I: If $F \in X$, and $\|F\| < 1$, then $\operatorname{Re}\varphi(F) < \varphi(G)$. If possible, let $\operatorname{Re}\varphi(F) = \varphi(G)$. Let $H := \frac{F}{\|F\|}$. Then $H \in X$, $\|H\| = 1$, and $\operatorname{Re}\varphi(H) > \varphi(G)$, contradicting the extremality of G. Hence claim I is established.

Now for each $F \in X$, $\operatorname{Re}\varphi(G + WF) = \varphi(G)$; hence by claim I, we must have $\|G + WF\| \geq 1$, which implies that $G \perp WF$. In particular,

$$\begin{aligned}&\langle G, WW^*G\rangle = 0\\ &\quad\Rightarrow A_i^* g_{i+1} = 0 \quad \forall i \geq 0, \text{by Lemma 4.8(ii)}\\ &\quad\Rightarrow g_{i+1} = 0 \quad \forall i \geq 0.\end{aligned}$$

Thus $G = (g_0, 0, 0, \ldots)$. Let $g_0 = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$. Since, $o(\mathcal{L}_X) = m$, then $\alpha_i = 0$ for all $0 \leq i < m$. Also, $\varphi(G) > 0$ implies that $\alpha_m \neq 0$. Thus $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$, and $o(G) = m$. □

Remark 4.14. The function G in Theorem 4.13 is called the *extremal function* of the nonzero reducing subspace X of W.

Theorem 4.15. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. If the extremal function of a nonzero reducing subspace X of W has a finite canonical decomposition, then it must be W-transparent.*

Proof. Let X be a nonzero reducing subspace of the order m, and let $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ be its extremal function. Also let $G = g_1 + g_2 + \cdots + g_n$ be the finite canonical decomposition of G. Then $g_1 = \sum_{i\in\mathbb{N}_0} \beta_i g_{i,0}$ such that $o(g_1) = m$, and $\beta_m = \alpha_m$. Also $\|g_1\| \leq \|G\| = 1$. Thus by extremality of G, we must have $G = g_1$. As g_1, by definition, is W-transparent, so G is also W-transparent. □

5. Minimal reducing subspaces

In this section we identify and study the minimal reducing subspaces of W in $H^2(K)$. It may be noted that in general there are many operators which have reducing subspaces that do not contain minimal reducing subspaces. One such operator is the operator of multiplication by z on the Bergman space $L^2(\mathbb{D}, dA)$, where $\mathbb{D}$ is the unit disk and dA is the area measure (see [7], [16]).

Lemma 5.1. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. Let F be W-transparent, and let $o(F) = m$. If $G \in X_F$ is such that G is nonzero and $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$, then $G = \lambda F$ for some nonzero scalar λ.*

Proof. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$. As $G = (g, 0, 0, \dots)$ with $g \neq 0$, and $F = (f, 0, 0, \dots)$ with $f \neq 0$, so by Definition 4.7, $G = \sum_k \lambda_k (W^*)^k W^k F$ for scalars λ_k, not all zero.

Let

$$\beta_k := \begin{cases} (\gamma_m^{(0)})^2 & \text{if } k = 1, \\ (\gamma_m^{(0)} \gamma_{\psi_0(m)}^{(1)} \cdots \gamma_{\psi_{k-2}\cdots\psi_0(m)}^{(k-1)})^2 & \text{if } k > 1. \end{cases}$$

Then by Lemma 4.9(i), $(W^*)^k W^k F = \beta_k F$, where $\beta_k \neq 0$ for all k. Therefore, $G = (\sum_k \lambda_k \beta_k) F = \lambda F$ for $\lambda = \sum_k \lambda_k \beta_k \neq 0$. □

Lemma 5.2. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. Let $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ with $o(F) = m_1$. If $G \in X_F$ such that G is nonzero, and $G = \sum_{i\in\mathbb{N}_0} \beta_i g_{i,0}$, then $o(G) \geq m_1$.*

Proof. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$. Let $F = f_1 + f_2 + \cdots$ be the canonical decomposition of F with $o(f_i) = m_i$. If for each $i \in \mathbb{N}_0$,

$$\beta_k^{(i)} := \begin{cases} (\gamma_{m_i}^{(0)})^2 & \text{if } k = 1, \\ (\gamma_{m_i}^{(0)} \gamma_{\psi_0(m_i)}^{(1)} \cdots \gamma_{\psi_{k-2}\cdots\psi_0(m_i)}^{(k-1)})^2 & \text{if } k > 1, \end{cases}$$

then $(W^*)^k W^k f_i = \beta_k^{(i)} f_i$ for all $k \in \mathbb{N}_0$ and $i \in \mathbb{N}$. Now $G \in X_F$ implies that $G = \sum_k \lambda_k (W^*)^k W^k F = \sum_k \lambda_k (\sum_i \beta_k^{(i)} f_i) = \sum_i (\sum_k \lambda_k \beta_k^{(i)}) f_i$. Therefore $o(G) = o(f_1)$ if $\sum_k \lambda_k \beta_k^{(1)} \neq 0$; otherwise, $o(G) > o(f_1)$. Hence $o(G) \geq m_1$. □

Theorem 5.3. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$, and let X be a minimal reducing subspace of W. If $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0} \in X$, then F must be W-transparent.*

Proof. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$. If possible, let F not be W-transparent. Then the canonical decomposition of $F = f_1 + f_2 + \cdots$ will have at least two components, f_1 and f_2.

If we let $o(f_i) = n_i$, then $n_1 \not\sim^W n_2$. Hence either $\gamma_{n_1}^{(0)} \neq \gamma_{n_2}^{(0)}$, or there exists a positive integer k such that $\gamma_{\psi_{k-1}\cdots\psi_0(n_1)}^{(k)} \neq \gamma_{\psi_{k-1}\cdots\psi_0(n_2)}^{(k)}$.

(i) If $\gamma_{n_1}^{(0)} \neq \gamma_{n_2}^{(0)}$, then define $G := W^*WF - (\gamma_{n_1}^{(0)})^2 F$ so that $G = [(\gamma_{n_2}^{(0)})^2 - (\gamma_{n_1}^{(0)})^2] f_2 + [(\gamma_{n_3}^{(0)})^2 - (\gamma_{n_1}^{(0)})^2] f_3 + \cdots$, which implies that $o(G) = o(f_2) = n_2$.

(ii) If $\gamma_{n_1}^{(0)} = \gamma_{n_2}^{(0)}$, then let k be the positive integer such that $\gamma_{\psi_{k-1}\ldots\psi_0(n_1)}^{(k)} \neq \gamma_{\psi_{k-1}\ldots\psi_0(n_2)}^{(k)}$, and $\gamma_{\psi_{i-1}\ldots\psi_0(n_1)}^{(i)} = \gamma_{\psi_{i-1}\ldots\psi_0(n_2)}^{(i)}$ for all $0 < i < k$. Then

$$\begin{aligned} G :&= (W^*)^{k+1} W^{k+1} F - (\gamma_{n_1}^{(0)} \gamma_{\psi_0(n_1)}^{(1)} \cdots \gamma_{\psi_{k-1}\ldots\psi_0(n_1)}^{(k)})^2 F \\ &= \big[(\gamma_{n_2}^{(0)} \gamma_{\psi_0(n_2)}^{(1)} \cdots \gamma_{\psi_{k-1}\ldots\psi_0(n_2)}^{(k)})^2 - (\gamma_{n_1}^{(0)} \gamma_{\psi_0(n_1)}^{(1)} \cdots \gamma_{\psi_{k-1}\ldots\psi_0(n_1)}^{(k)})^2\big] f_2 + \cdots , \end{aligned}$$

which implies that $o(G) = o(f_2) = n_2$.

Thus there exists $0 \neq G \in X$ such that $o(F) < o(G)$. Therefore X_G is a nonzero reducing subspace of W contained in X. By minimality of X, we must have $X_G = X$. But this implies that $F \in X_G$ so that, by Lemma 5.2, $o(F) \geq o(G)$, which is a contradiction. Thus, F must be W-transparent. □

Corollary 5.4. *Let W be an operator-weighted shift on $l^2(K)$ with weights $\{A_n\}$ in $\mathcal{T}$. The extremal function of a minimal reducing subspace of W is always W-transparent.*

Theorem 5.5. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. Let X be a nonzero reducing subspace of W. Then X is minimal if and only if $X = X_F$, where $F \in X$ is W-transparent.*

Proof. If X is minimal, then $X = X_G$ where G is the extremal function of X. Also, by Corollary 5.4, G must be W-transparent. Conversely, let $X = X_F$ where $F \in X$ is W-transparent. Then by Lemma 4.9, X_F is a reducing subspace of W. Thus we only need to show that X_F is minimal-reducing.

For this, let Y be a nonzero reducing subspace of W contained in X_F. If G is the extremal function of Y, then $G \in X_F$; thus by Lemma 5.1, $G = \lambda F$ for a nonzero scalar λ. This implies that $F \in Y$. Therefore $Y = X_F$, which shows that X_F is minimal. □

Corollary 5.6. *Let W be an operator-weighted shift on $l^2(K)$ with weights $\{A_n\}$ in $\mathcal{T}$. Every reducing subspace of W in $l^2(K)$ whose extremal function has a finite canonical decomposition must contain a minimal reducing subspace.*

The proof follows immediately from Lemma 4.11 and Theorem 5.5.

6. Conclusion

Theorem 6.1. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. If W is of Type I, then $X_{g_{n,0}}$ for $n \in \mathbb{N}_0$ are the only minimal reducing subspaces of W in $l^2(K)$.*

Proof. Let X be a minimal reducing subspace of W, and let G be the extremal function such that $X = X_G$. As W is of type I, the only W-transparent functions are $g_{n,0}$ and their scalar multiples. Hence $X = X_{g_{n,0}}$ for $n \in \mathbb{N}_0$. □

Theorem 6.2. *Let W be an operator-weighted shift on $l^2(K)$ with uniformly bounded weights $\{A_n\}$ in $\mathcal{T}$. If W is of Type II, then W has minimal reducing* *subspaces other than* $X_{g_{n,0}}$ $(n \in \mathbb{N}_0)$*. In fact, for every* W*-transparent* F*,* X_F *is a minimal reducing subspace. Hence* W *will have infinitely many minimal reducing subspaces in* $l^2(K)$*.*

Proof. Let Y be a nonzero reducing subspace of W such that $Y \subseteq X_F$. Let $Y = X_G$, where G is the extremal function. Then $G \in X_F$. By Lemma 5.1, $G = \lambda F$, $\lambda \neq 0$, which implies that $F \in Y$. Therefore $X_F = Y$. Hence X_F is minimal. □

Theorem 6.3. *Let* W *be an operator-weighted shift on* $l^2(K)$ *with uniformly bounded weights* $\{A_n\}$ *in* $\mathcal{T}$*. If* W *is of Type III, then every reducing subspace of* W *must contain a minimal reducing subspace.*

Proof. Let X be a nonzero reducing subspace of W. If $X = X_F$ for some transparent function F, then X is minimal. Otherwise, let $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0} \in X$, and let $G = f_1 + f_2 + \cdots + f_m$ be its canonical decomposition. Then by Lemma 4.11, each $f_i \in X$; hence X_{f_i} is a minimal reducing subspace in X. □

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