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SUPPORTING VECTORS OF CONTINUOUS LINEAR OPERATORS

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Abstract. The set of supporting vectors of a continuous linear operator, that is, the normalized vectors at which the operator attains its norm, is decomposed into its convex components. In the complex case, the set of supporting vectors of a nonzero functional is proved to be path-connected. We also introduce the concept of generalized supporting vectors for a sequence of operators as the normalized vectors that maximize the summation of the squared norm of those operators. We determine the set of generalized supporting vectors for the particular case of a finite sequence of real matrices. Finally, we unveil the relation between the supporting vectors of a real matrix A and the Tikhonov regularization $\min_{x\in\mathbb{R}^n} \|Ax-b\|+\alpha\|x\|$ reaching the conclusion that, by an appropriate choice of b and α, the supporting vectors of A can be obtained via solving the Tikhonov regularization $\min_{x\in\mathbb{R}^n} \|Ax-b\|+\alpha\|x\|$.

1. Introduction

Recall that the set of supporting vectors of a continuous linear operator between normed spaces X and Y is defined as

$$\operatorname{suppv}(T) := \arg\max_{\|x\|=1}\big\|T(x)\big\| = \big\{x\in \mathsf{S}_X : \big\|T(x)\big\| = \|T\|\big\}.$$

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The supporting vectors play a fundamental role in the geometry of Banach spaces due to famous classical results such as the Bishop–Phelps theorem and the Hahn–Banach theorem. We refer the reader to [1], [2] for a broad perspective on those theorems and their generalizations.

In the second section of this manuscript we study the topological structure of the set $\mathrm{suppv}(T)$ by identifying its convex components, which turn out to be closed. Needless to say, $\mathrm{suppv}(T)$ is in general not even connected except in extremal cases (such as $T = 0$) or in an isometry. Special attention will be paid to the closed convex set $\mathrm{suppv}_1(x^*) := \{x \in \mathsf{S}_X : x^*(x) = \|x^*\|\}$, where $x^* \in X^*$. These sets are also known by the Banach space geometers as the *exposed faces of the unit ball* of X, B_X. We will prove (see Theorem 2.1) that $\mathrm{suppv}_1(x^*)$ is path-connected if X is complex.

In the third section we introduce the new concept of generalized supporting vectors for a sequence of continuous and linear operators, and we give a complete description of them in the particular case of an eventually null sequence of finite-rank operators. The generalized supporting vectors are crucial to solving several optimization problems that involve the design of transcranial magnetic stimulation coils (see [4]). The fourth and final section is devoted to obtaining the supporting vectors of a real matrix A via the Tikhonov regularization $\min_{x\in\mathbb{R}^n} \|Ax - b\| + \alpha\|x\|$ (see [6]).

2. Topological structure of the set of supporting vectors

It is trivially verified that $\mathrm{suppv}(x^*) = \bigcup_{\lambda\in\mathsf{S}_\mathbb{K}} \lambda\,\mathrm{suppv}_1(x^*)$ for every $x^* \in X^*$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In fact, if $\mathbb{K} = \mathbb{R}$, and $x^* \neq 0$, then $\{\mathrm{suppv}_1(x^*), -\mathrm{suppv}_1(x^*)\}$ are the two connected components of $\mathrm{suppv}(x^*)$. We remind the reader that a convex component is a maximal convex subset (see [3]). Therefore, $\{\mathrm{suppv}_1(x^*), -\mathrm{suppv}_1(x^*)\}$ also constitute the two convex components of $\mathrm{suppv}(x^*)$. The complex case is completely different.

Theorem 2.1. *Let X be a complex normed space, and let $x^* \in X^* \setminus \{0\}$. Then*

(1) $\mathrm{suppv}(x^*)$ *is path-connected,*
(2) *the convex components of* $\mathrm{suppv}(x^*)$ *are* $\{\lambda\,\mathrm{suppv}_1(x^*) : \lambda \in \mathsf{S}_\mathbb{C}\}$.

Proof. We will only prove the first item since the second one will be generalized in an upcoming theorem. Let $x, y \in \mathrm{suppv}(x^*)$, and let $\lambda, \gamma \in \mathsf{S}_\mathbb{C}$ such that $x \in \lambda\,\mathrm{suppv}_1(x^*)$, and $y \in \gamma\,\mathrm{suppv}_1(x^*)$. Note that $\mathsf{S}_\mathbb{C} x \subseteq \mathrm{suppv}(x^*)$, and note that $\mathsf{S}_\mathbb{C} x$ is path-connected since it is homeomorphic to $\mathsf{S}_\mathbb{C}$. Thus we can construct a continuous path inside $\mathrm{suppv}(x^*)$ joining x with $\frac{\gamma}{\lambda} x \in \gamma\,\mathrm{suppv}_1(x^*)$. Finally, since $\gamma\,\mathrm{suppv}_1(x^*)$ is convex, we can consider the segment joining $\frac{\gamma}{\lambda} x$ with y, which is entirely contained in $\gamma\,\mathrm{suppv}_1(x^*) \subseteq \mathrm{suppv}(x^*)$. By adding together these two paths we obtain a continuous path inside $\mathrm{suppv}(x^*)$ joining x and y. □

We proceed to obtain a similar decomposition of $\mathrm{suppv}(T)$ for T a continuous linear operator between normed spaces.

Lemma 2.2. *Let X, Y, and Z be normed spaces, and consider nonzero continuous linear operators $S : X \to Y$ and $T : Y \to Z$. If $\|T \circ S\| = \|T\|\|S\|$, then*

$$\mathrm{suppv}(T \circ S) \subseteq \mathrm{suppv}(S) \quad \text{and} \quad \frac{S(\mathrm{suppv}(T \circ S))}{\|S\|} \subseteq \mathrm{suppv}(T).$$

Proof. Fix an arbitrary $x \in \mathrm{suppv}(T \circ S)$. By hypothesis, $\|T\|\|S\| = \|T \circ S\| = \|T(S(x))\|$, which implies that $\|T(\frac{S(x)}{\|S\|})\| = \|T\|$. This means that $\|\frac{S(x)}{\|S\|}\| = 1$, and thus that $\|S(x)\| = \|S\|$. □

Note that Theorem 2.1(2) is a direct consequence of the following result.

Theorem 2.3. *Let X and Y be normed spaces, and let $T : X \to Y$ be a continuous linear operator. Then*

(1) $\mathrm{suppv}(T) = \bigcup_{y^* \in \mathrm{suppv}(T^*)} \mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$*;*
(2) *if C is a convex component of* $\mathrm{suppv}(T)$*, then* $C = \mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$ *for some* $y^* \in \mathrm{suppv}(T^*)$*;*
(3) *if Y is smooth, then every nonempty* $\mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$ *with* $y^* \in \mathrm{suppv}(T^*)$ *is a convex component of* $\mathrm{suppv}(T)$*.*

Proof. (1) We will prove only that

$$\mathrm{suppv}(T) \supseteq \bigcup_{y^* \in \mathrm{suppv}(T^*)} \mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$$

since the other inclusion will be shown in the next item. If $y^* \in \mathrm{suppv}(T^*)$, then $\|y^* \circ T\| = \|T^*(y^*)\| = \|T^*\| = \|T\| = \|y^*\|\|T\|$; therefore we can apply Lemma 2.2 to deduce that $\mathrm{suppv}_{\mathbf{1}}(y^* \circ T) \subseteq \mathrm{suppv}(T)$.

(2) Let C be a convex component of $\mathrm{suppv}(T)$. Note that $T(C)$ is a convex set contained in $\mathsf{S}_Y(0, \|T\|)$; thus, by the Hahn–Banach theorem, there exists $y^* \in \mathsf{S}_{Y^*}$ such that $y^*(T(C)) = \{\|T\|\}$. As a consequence $\|T^*(y^*)\| = \|y^* \circ T\| = \|T\| = \|T^*\|$; therefore $y^* \in \mathrm{suppv}(T^*)$, and $C \subseteq \mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$. Since $\mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$ is convex, and, by virtue of the previous item contained in $\mathrm{suppv}(T)$, we have that the maximality of C allows that $C = \mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$.

(3) Let $\mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$ be nonempty with $y^* \in \mathrm{suppv}(T^*)$, and consider C to be a convex subset $\mathrm{suppv}(T)$ containing $\mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$. We will show that $C = \mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$. We may assume without loss of generality that C is a convex component of $\mathrm{suppv}(T)$. In this case, we already know that $C = \mathrm{suppv}_{\mathbf{1}}(z^* \circ T)$ for some $z^* \in \mathrm{supp}(T^*)$. Hence we have that $\mathrm{suppv}_{\mathbf{1}}(y^* \circ T) \subseteq \mathrm{suppv}_{\mathbf{1}}(z^* \circ T)$. Choose any $x \in \mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$. Observe that $y^*(\frac{T(x)}{\|T\|}) = 1 = z^*(\frac{T(x)}{\|T\|})$. Now the smoothness of Y allows us to deduce that $y^* = z^*$. □

Note that if we drop the hypothesis of smoothness from the third item of Theorem 2.3, then we cannot conclude that all the nonempty exposed faces of the form $\mathrm{suppv}_{\mathbf{1}}(y^* \circ T)$ with $y^* \in \mathrm{suppv}(T^*)$ are convex components of $\mathrm{suppv}(T)$. Indeed, consider $X = Y = \ell_\infty$, $T = I$, and $y^* := \frac{e_1+e_2}{2}$. Note that $C_1 := \{(x_n)_{n\in\mathbb{N}} \in \mathsf{S}_{\ell_\infty} : x_1 = 1\}$ is a convex component of $\mathsf{S}_{\ell_\infty} = \mathrm{suppv}(T)$ and that

$$\mathrm{suppv}_{\mathbf{1}}(y^* \circ T) = \left\{(x_n)_{n\in\mathbb{N}} \in \mathsf{S}_{\ell_\infty} : x_1 = x_2 = 1\right\} \subsetneq C_1.$$

As an immediate consequence of Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *Let X and Y be normed spaces, and let $T : X \to Y$ be a continuous linear operator. Then*

(1) *if* $\operatorname{suppv}(T) \neq \varnothing$, *then* $\operatorname{suppv}(T^*) \neq \varnothing$;
(2) *if X is reflexive and* $\operatorname{suppv}(T^*) \neq \varnothing$, *then* $\operatorname{suppv}(T) \neq \varnothing$.

Proof. The first item is a direct consequence of Theorem 2.3. Therefore, assume that X is reflexive and that $\operatorname{suppv}(T^*) \neq \varnothing$. If $y^* \in \operatorname{suppv}(T^*)$, then $\|y^* \circ T\| = \|T^*(y^*)\| = \|T^*\| = \|T\| = \|y^*\|\|T\|$; thus Lemma 2.2 shows that $\operatorname{suppv}(y^* \circ T) \subseteq \operatorname{suppv}(T)$. Finally, the reflexivity of X allows us to conclude that $\operatorname{suppv}(y^* \circ T) \neq \varnothing$. $\square$

We conclude this section by pointing out that the converse to the previous corollary does not hold true. Indeed, consider any nonreflexive Banach space and a non-norm-attaining functional $f \in \mathsf{S}_{X^*}$. Then $\operatorname{suppv}(f) = \varnothing$; however, $f^* : \mathbb{K} \to X^*$ verifies that $\operatorname{suppv}(f^*) = \mathsf{S}_{\mathbb{K}}$.

3. Generalized supporting vectors

The generalized supporting vectors appear in an implicit way in many optimization problems in physics and engineering (see [4]). Here we will properly define them in a more general and abstract setting.

Definition 3.1. Let X and Y be normed spaces, and consider a sequence $(T_n)_{n\in\mathbb{N}}$ of continuous linear operators between them. The generalized supporting vectors of $(T_n)_{n\in\mathbb{N}}$ are defined as the elements of

$$\operatorname{gsuppv}\big((T_n)_{n\in\mathbb{N}}\big) := \arg\max_{\|x\|_2=1} \sum_{n=1}^{\infty} \big\|T_i(x)\big\|^2.$$

Notice that it can easily happen that $\operatorname{gsuppv}((T_n)_{n\in\mathbb{N}})$ is empty. In order to avoid this, some conditions are required on the sequence of operators and on the normed spaces, such as reflexivity for X and that $(T_n)_{n\in\mathbb{N}} \in \ell_2(\mathcal{B}(X,Y))$.

We will focus now on the generalized supporting vectors of an eventually null sequence of finite-rank operators $A_1, \dots, A_k \in \mathbb{R}^{m\times n}$. For this we need to recall several basic concepts of linear algebra along with other basic concepts from the spectral theory of normed algebras, such as the point spectrum of a continuous linear operator. (Throughout this section, the 2-norm of a matrix $A \in \mathbb{R}^{m\times n}$ is considered to be the operator norm of A between $\mathbb{R}^n$ and $\mathbb{R}^m$ when both are endowed with the Euclidean norm.)

A matrix $P \in \mathbb{R}^{n\times n}$ is said to be orthogonal provided that $P^T = P^{-1}$. Orthogonal matrices induce isometries on $\ell_2^n := (\mathbb{R}^2, \|\cdot\|_2)$; that is,

$$\|Px\|_2^2 = (Px)^T(Px) = x^T P^T P x = x^T x = \|x\|_2^2$$

for all $x \in \mathbb{R}^n$. In particular, the 2-norm of an orthogonal matrix is 1.

It is well known that the eigenvalues of a symmetric real matrix are real. This fact remains true in more general settings, for instance in operator theory. If $A \in \mathbb{R}^{m\times n}$ is symmetric, then $\lambda_{\max}(A)$ stands for the largest eigenvalue of A, and

$V(\lambda_{\max}(A)) := \ker(A - \lambda_{\max}(A)\mathrm{I})$ is the vector subspace of eigenvectors associated to $\lambda_{\max}(A)$. Before stating and proving the following crucial lemma, we note that if $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix, then $\|D\|_2 = |\lambda_{\max(D)}|$. From the spectral theory of C^*-algebras, we can easily deduce the following lemma. We include its proof for the sake of completeness.

Lemma 3.2. *If $A \in \mathbb{R}^{n\times n}$ is positive semidefinite and symmetric, then*
(1) $\|A\|_2 = \lambda_{\max}(A)$,
(2) $x^T Ax \leq \lambda_{\max}(A)\|x\|_2^2$ *for all* $x \in \mathbb{R}^n$.

Proof. Since A is symmetric, we have that A is orthogonally diagonalizable. In other words, there exist an orthogonal matrix P and a diagonal matrix D such that $A = P^T DP$ and the eigenvalues of A are the elements of the main diagonal of D. On the other hand, since A is also positive semidefinite, then the eigenvalues of A are positive.

(1) Since P and P^T are both isometries, it holds that
$$\|A\|_2 = \|P^T DP\|_2 = \|D\|_2 = \lambda_{\max}(D) = \lambda_{\max}(A).$$
(2) By restating that P is an isometry and by relying on the above item, we see that
$$x^T Ax = |x^T Ax| \leq \|x^T\|_2\|A\|_2\|x\|_2 = \lambda_{\max}(A)\|x\|_2^2. \qquad \square$$

Since every element x of a normed space X can be regarded as an element of X^{**}, it makes sense to consider $\operatorname{suppv}_1(x)$, which consists of all $x^* \in \mathsf{S}_{X^*}$ such that $x^*(x) = \|x\|$. On the other hand, we note that the set of smooth points of the unit ball of a normed space X is defined as
$$\operatorname{smo}(\mathsf{B}_X) := \left\{x \in \mathsf{S}_X : \exists x^* \in \mathsf{S}_{X^*} \text{ with } \operatorname{suppv}_1(x) = \{x^*\}\right\}.$$
Whenever $\operatorname{smo}(\mathsf{B}_X) = \mathsf{S}_X$, we consider X a smooth normed space. It is well known that all Hilbert spaces are smooth.

Theorem 3.3. *Let $A_1, \ldots, A_k \in \mathbb{R}^{m\times n}$. Then*
$$\max_{\|x\|_2=1} \sum_{i=1}^k \|A_i x\|_2^2 = \lambda_{\max}\Big(\sum_{i=1}^k A_i^T A_i\Big),$$
and
$$V\Big(\lambda_{\max}\Big(\sum_{i=1}^k A_i^T A_i\Big)\Big) \cap \mathsf{S}_{\ell_2^n} = \arg\max_{\|x\|_2=1} \sum_{i=1}^k \|A_i x\|_2^2.$$

Proof. First, for any $x \in \mathbb{R}^n$ and by virtue of Lemma 3.2(2),
$$\sum_{i=1}^k \|A_i x\|_2^2 = \sum_{i=1}^k x^T A_i^T A_i x = x^T\Big(\sum_{i=1}^k A_i^T A_i\Big)x \leq \lambda_{\max}\Big(\sum_{i=1}^k A_i^T A_i\Big)\|x\|_2.$$
Therefore,
$$\max_{\|x\|_2=1} \sum_{i=1}^k \|A_i x\|_2^2 \leq \lambda_{\max}\Big(\sum_{i=1}^k A_i^T A_i\Big).$$

Now let $w \in V(\lambda_{\max}(\sum_{i=1}^k A_i^T A_i)) \cap \mathsf{S}_{\ell_2^n}$. Then

$$\sum_{i=1}^k \|A_i w\|_2^2 = w^T \Big(\sum_{i=1}^k A_i^T A_i\Big) w = \lambda_{\max}\Big(\sum_{i=1}^k A_i^T A_i\Big).$$

This shows that

$$\max_{\|x\|_2=1} \sum_{i=1}^k \|A_i x\|_2^2 = \lambda_{\max}\Big(\sum_{i=1}^k A_i^T A_i\Big)$$

and that

$$V\Big(\lambda_{\max}\Big(\sum_{i=1}^k A_i^T A_i\Big)\Big) \cap \mathsf{S}_{\ell_2^n} \subseteq \arg\max_{\|x\|_2=1} \sum_{i=1}^k \|A_i x\|_2^2.$$

Finally, let $v \in \arg\max_{\|x\|_2=1} \sum_{i=1}^k \|A_i x\|_2^2$. On the one hand, in accordance with Lemma 3.2(1), we deduce that

$$\Big\|\frac{(\sum_{i=1}^k A_i^T A_i)v}{\lambda_{\max}(\sum_{i=1}^k A_i^T A_i)}\Big\|_2 \leq \frac{\|\sum_{i=1}^k A_i^T A_i\|_2}{\lambda_{\max}(\sum_{i=1}^k A_i^T A_i)} = 1.$$

On the other hand,

$$v^T \frac{(\sum_{i=1}^k A_i^T A_i)v}{\lambda_{\max}(\sum_{i=1}^k A_i^T A_i)} = \frac{\sum_{i=1}^k \|A_i v\|_2^2}{\lambda_{\max}(\sum_{i=1}^k A_i^T A_i)} = 1,$$

which implies that

$$\Big\|\frac{(\sum_{i=1}^k A_i^T A_i)v}{\lambda_{\max}(\sum_{i=1}^k A_i^T A_i)}\Big\|_2 = 1.$$

The smoothness of ℓ_2^n allows us to deduce that

$$\frac{(\sum_{i=1}^k A_i^T A_i)v}{\lambda_{\max}(\sum_{i=1}^k A_i^T A_i)} = v;$$

that is,

$$\Big(\sum_{i=1}^k A_i^T A_i\Big)v = \lambda_{\max}\Big(\sum_{i=1}^k A_i^T A_i\Big)v,$$

and so $v \in V(\lambda_{\max}(\sum_{i=1}^k A_i^T A_i)) \cap \mathsf{S}_{\ell_2^n}$. □

As a consequence, we easily obtain the well-known formula of the 2-norm of a matrix.

Corollary 3.4. *If $A \in \mathbb{R}^{m\times n}$, then $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$, and $V(\lambda_{\max}(A^T A)) \cap \mathsf{S}_{\ell_2^n} = \operatorname{suppv}(A)$.*

4. Supporting vectors and the Tikhonov regularization

Many times in physics and engineering, the following problem arises:

$$\begin{cases} \min \|Ax - b\| + \alpha\|x\| \\ x \in \mathbb{R}^n, \end{cases}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, and $\alpha > 0$ are constant. This problem is often related to identifying the supporting vectors of A, and it is known as the *Tikhonov regularization* due to the addition of the term $\alpha\|x\|$ (see [6] for a broad perspective on the Tikhonov regularization). In this section we will prove that an appropriate choice of b and α allows the computation of the supporting vectors of A via the Tikhonov regularization $\min_{x\in\mathbb{R}^n} \|Ax - b\| + \alpha\|x\|$. We will consider both $\mathbb{R}^n$ and $\mathbb{R}^m$ endowed with arbitrary norms and $\mathbb{R}^{m\times n}$ endowed with the corresponding operator norm. Likewise, X and Y will stand for $\mathbb{R}^n$ and $\mathbb{R}^m$ endowed with the previous norms, respectively.

Lemma 4.1. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, and let $\alpha > 0$. If*

$$\operatorname{suppv}(A) \cap \arg\min_{x\in\mathbb{R}^n} \big(\|Ax - b\| + \alpha\|x\|\big) \neq \varnothing,$$

then $\alpha \leq \min\{\|A\|, \|b\|\}$.

Proof. Let $y \in \operatorname{suppv}(A) \cap \arg\min_{x\in\mathbb{R}^n}(\|Ax - b\| + \alpha\|x\|)$. Notice that $\|y\| = 1$ and that $\|Ay\| = \|A\|$. We will distinguish between the two following cases:

- $\|b\| \leq \|A\|$. On the one hand,

$$\begin{aligned} \|b\| &= \|A0 - b\| + \alpha\|0\| \\ &\geq \|Ay - b\| + \alpha\|y\| \\ &\geq \big|\|Ay\| - \|b\|\big| + \alpha\|y\| \\ &= \big|\|A\| - \|b\|\big| + \alpha \\ &= \|A\| - \|b\| + \alpha \\ &\geq \alpha. \end{aligned}$$

- $\|A\| < \|b\|$. On the other hand, following the same change of inequalities as above, we conclude that

$$\|b\| \geq \big|\|A\| - \|b\|\big| + \alpha = \|b\| - \|A\| + \alpha,$$

  from which we immediately deduce that $\alpha \leq \|A\|$. □

The preceding lemma suggests the choice $\alpha = \|A\| = \|b\|$. Before stating our next result on the Tikhonov regularization, we need the following technical lemma.

Lemma 4.2. *Let Z be a strictly convex normed space. If $z \in Z$ with $0 < \|z\| < 1$, then $\mathsf{S}_Z \cap \mathsf{S}_Z(0, 1 - \|z\|) = \{\frac{z}{\|z\|}\}$.*

Proof. Let $y \in \mathsf{S}_Z \cap \mathsf{S}_Z(0, 1 - \|z\|)$. Notice that $\|y - z\| = 1 - \|z\| = \|y\| - \|z\|$; thus $\|z + (y - z)\| = \|z\| + \|y - z\|$. According to [5, Proposition 5.1.11], there exists $\alpha > 0$ such that $y - z = \alpha z$; that is, $y = (1+\alpha)z$. Since $\|y\| = 1$, and $1 + \alpha > 0$, then the only possibility is that $y = \frac{z}{\|z\|}$. ☐

Theorem 4.3. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, and $\alpha > 0$ satisfy the condition $\alpha = \|A\| = \|b\|$. Then:*

(1) $\min_{x\in\mathbb{R}^n}(\|Ax - b\| + \alpha\|x\|) = \min_{\|x\|\leq 1}(\|Ax - b\| + \alpha\|x\|) = \|A\|$,
(2) $0 \in \arg\min_{x\in\mathbb{R}^n}(\|Ax - b\| + \alpha\|x\|) \subseteq \mathsf{B}_X$,
(3) $(\arg\min_{\|x\|\leq 1}(\|Ax - b\| + \alpha\|x\|)) \setminus \{0\} \subseteq (0, 1] \operatorname{suppv}(A)$,
(4) *if Y is strictly convex, and* $(\arg\min_{\|x\|\leq 1}(\|Ax - b\| + \alpha\|x\|)) \setminus \{0\} \neq \varnothing$, *then $b \in A \operatorname{suppv}(A)$.*

Proof. (1)–(2) Let $x \in \mathbb{R}^n$. We will distinguish between the two following cases:

- $\|Ax\| \geq \|A\|$. In this case, $\|A\|\|x\| \geq \|Ax\| \geq \|A\|$, so $\|x\| \geq 1$; hence

$$\|Ax - b\| + \alpha\|x\| \geq \alpha = \|A\|.$$

- $\|Ax\| < \|A\|$. In this case,

$$\begin{aligned}\|Ax - b\| + \alpha\|x\| &\geq \big|\|Ax\| - \|b\|\big| + \alpha\|x\| \\ &= \|A\| - \|Ax\| + \alpha\|x\| \\ &\geq \|A\| - \|A\|\|x\| + \alpha\|x\| \\ &= \|A\|.\end{aligned}$$

This shows that

$$\min\big\{\|Ax - b\| + \alpha\|x\| : x \in \mathbb{R}^n\big\} \geq \|A\|.$$

By taking $x = 0$, we obtain $\|A0 - b\| + \alpha\|0\| = \|b\| = \|A\|$. Now, if $\|x\| > 1$, then $\|Ax - b\| + \alpha\|x\| > \alpha = \|A\|$.

(3) Let $z \in \arg\min_{\|x\|\leq 1}(\|Ax - b\| + \alpha\|x\|)$ with $z \neq 0$. By taking into account that $\|A\| = \|Az - b\| + \alpha\|z\|$ and that $\|Az\| \leq \|A\|\|z\| \leq \|A\| = \|b\|$, we obtain

$$\begin{aligned}\|b\| - \|Az\| &= \big|\|b\| - \|Az\|\big| \\ &\leq \|b - Az\| \\ &= \|A\| - \alpha\|z\| \\ &= \|A\| - \|A\|\|z\| \\ &= \|b\| - \|A\|\|z\|.\end{aligned}$$

Therefore, $\|A\|\|z\| \leq \|Az\| \leq \|A\|\|z\|$, which means that

$$\left\|A\frac{z}{\|z\|}\right\| = \|A\|;$$

hence $z \in \|z\| \operatorname{suppv}(A)$.

(4) Let $z \in \arg\min_{\|x\|\leq 1}(\|Ax - b\| + \alpha\|x\|)$ with $z \neq 0$. From the proof of the previous item, we know that $\frac{z}{\|z\|} \in \operatorname{suppv}(A)$ and that

$$\|Az - b\| = \|A\|\big(1 - \|z\|\big) = \left\|Az - A\frac{z}{\|z\|}\right\|.$$

Therefore, $b \in \mathsf{S}_Y(0, \|A\|) \cap \mathsf{S}_Y(Az, \|A\|(1 - \|z\|))$. By making use of the triangular inequality, it can be seen that $\mathsf{B}_Y(Az, \|A\|(1 - \|z\|)) \subseteq \mathsf{B}_Y(0, \|A\|)$. Now Lemma 4.2 comes into play to assure that

$$\mathsf{S}_Y\big(Az, \|A\|\big(1 - \|z\|\big)\big) \cap \mathsf{S}_Y\big(0, \|A\|\big) = \Big\{A\frac{z}{\|z\|}\Big\}.$$

Thus we conclude that $b = A\frac{z}{\|z\|} \in A\,\mathrm{suppv}(A)$. $\square$

The preceding theorem indicates that under the choice of $\alpha = \|A\| = \|b\|$, the Tikhonov regularization can be restricted to $\|x\| \leq 1$. On the other hand, the hypothesis of strict convexity for Y in Theorem 4.3(4) cannot be removed, as we will show in the next example.

Example 4.4. Take $X = Y := \ell_\infty^2$, $A := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $b := \left(\begin{smallmatrix} 1 \\ 1/2 \end{smallmatrix}\right)$, $\alpha := 1$, and $z := \left(\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right)$. Then

- $\alpha = \|A\|_\infty = \|b\|_\infty = 1$,
- $\mathrm{suppv}(A) = \left\{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : |x| = 1, |y| \leq 1\right\}$,
- $A\,\mathrm{suppv}(A) = \left\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)\right\}$, so $b \notin A\,\mathrm{suppv}(A)$, and
- $\|Az - b\|_\infty + \alpha\|z\|_\infty = \frac{1}{2} + \frac{1}{2} = 1 = \|A\|_\infty$.

The next result shows a necessary and sufficient condition to force that $b \in A\,\mathrm{suppv}(A)$.

Theorem 4.5. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, and $\alpha > 0$ satisfy the condition $\alpha = \|A\| = \|b\|$. Then* $\min\{\|Ax - b\| + \alpha\|x\| : \|x\| = 1\} = \|A\|$ *if and only if* $b \in A\,\mathrm{suppv}(A)$. *In this situation,*

$$\arg\min_{\|x\|=1}\big(\|Ax - b\| + \alpha\|x\|\big) \subseteq \mathrm{suppv}(A).$$

Proof. Assume first that there exists $y \in \mathbb{R}^n$ with $\|y\| = 1$, $\|Ay\| = \|A\|$, and $b = Ay$. Then

$$\begin{aligned} \|A\| &= \min\big\{\|Ax - b\| + \alpha\|x\| : x \in \mathbb{R}^n\big\} \\ &\leq \min\big\{\|Ax - b\| + \alpha\|x\| : \|x\| = 1\big\} \\ &\leq \|Ay - b\| + \alpha\|y\| \\ &= \|A\|. \end{aligned}$$

Conversely, assume that $\min\{\|Ax - b\| + \alpha\|x\| : \|x\| = 1\} = \|A\|$. Due to the compacity of S_X we can find $y \in \arg\min_{\|x\|=1}(\|Ax - b\| + \alpha\|x\|)$. Notice that

$$\|A\| = \|Ay - b\| + \alpha\|y\| = \|Ay - b\| + \|A\|,$$

which means that $Ay = b$. Finally, $\|Ay\| = \|b\| = \|A\|$, and so $y \in \mathrm{suppv}(A)$. $\square$

A first consequence of Theorem 4.5 follows. We note that U_X stands for the open unit ball of X, that is, the set of vectors with norm strictly less than 1.

Corollary 4.6. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, and $\alpha > 0$ satisfy the condition $\alpha = \|A\| = \|b\|$. If $b \notin A\,\mathrm{suppv}(A)$, then*

$$\arg\min_{\|x\|\leq 1}\big(\|Ax - b\| + \alpha\|x\|\big) \subseteq \mathsf{U}_X.$$

Proof. If $\mathsf{S}_X \cap \arg\min_{\|x\|\leq 1}(\|Ax-b\|+\alpha\|x\|) \neq \varnothing$, then $\min\{\|Ax-b\|+\alpha\|x\| : \|x\|=1\} = \|A\|$; thus by virtue of Theorem 4.5 we obtain that $b \in A\,\mathrm{suppv}(A)$. $\square$

Another consequence of Theorem 4.5 is the next corollary.

Corollary 4.7. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$, and $\alpha > 0$ satisfy the condition $\alpha = \|A\|$ and $b = Ay$ for some $y \in \mathrm{suppv}(A)$. Then*

(1) $[0,1]y \subseteq \arg\min_{\|x\|\leq 1}(\|Ax-b\|+\alpha\|x\|)$;

(2) *if* $\ker(A) = \{0\}$, *then*

$$\arg\min_{\|x\|=1}\big(\|Ax-b\|+\alpha\|x\|\big) = \{y\}.$$

Proof. (1) Let $t \in [0,1]$. Simply observe that

$$\begin{aligned}\big\|A(ty)-b\big\|+\alpha\|ty\| &= \big\|A(ty)-Ay\big\|+\alpha\|ty\| \\ &= (1-t)\|Ay\|+t\alpha\|y\| \\ &= (1-t)\|A\|+t\|A\| \\ &= \|A\|.\end{aligned}$$

(2) Let $z \in \arg\min_{\|x\|=1}(\|Ax-b\|+\alpha\|x\|)$. According to Theorem 4.5,

$$\begin{aligned}\|A\| &= \min\big\{\|Ax-b\|+\alpha\|x\| : \|x\|=1\big\} \\ &= \|Az-b\|+\alpha\|z\| \\ &= \|Az-b\|+\|A\|,\end{aligned}$$

which results in $\|Az-b\|=0$; hence $Az=b=Ay$, which by hypothesis implies that $z=y$. $\square$

We now attempt to find an example of a Tikhonov regularization of the form $\min_{x\in\mathbb{R}^n}\|Ax-b\|+\alpha\|x\|$ and also in which $\alpha=\|A\|=\|b\|$ in such a way that $\arg\min_{x\in\mathbb{R}^n}(\|Ax-b\|+\alpha\|x\|)=\{0\}$. In accordance with Corollary 4.7(1) we must choose $b \notin A\,\mathrm{suppv}(A)$. This hint leads us to rely strongly on the following proposition.

Proposition 4.8. *Let Z be a normed space, and consider a linear projection $P : Z \to Z$ such that $\|P\| = \|I-P\| = 1$. Then $\|Pz-y\|+\|z\| > 1$ for each $z \neq 0$, and every $y \in \mathsf{S}_Z \cap \ker(P)$.*

Proof. Note that $(I-P)(Pz-y) = y$, so $\|Pz-y\| \geq 1$. Therefore, $\|Pz-y\|+\|z\| \geq 1+\|z\| > 1$. $\square$

The preceding proposition is the key to our example.

Example 4.9. Take $X = Y := \ell_\infty^2$, $A := \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $b := \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, and $\alpha := 1$. Then $\alpha = \|A\|_\infty = \|b\|_\infty = 1$, and Proposition 4.8 allows

$$\arg\min_{x\in\mathbb{R}^n}\big(\|Ax-b\|_\infty+\alpha\|x\|_\infty\big) = \{0\}.$$

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