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ORTHOGONAL-PRESERVING AND SURJECTIVE CUBIC STOCHASTIC OPERATORS

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Abstract. In the present paper, we consider cubic stochastic operators, and prove that the surjectivity of such operators is equivalent to their orthogonal-preserving property. In the last section we provide a full description of orthogonal-preserving (respectively, surjective) cubic stochastic operators on the 2–dimensional simplex.

1. Introduction

It is known that a discrete Markov chain is described by transition probabilities which depend only on the current state of the process. Recently, nonlinear Markov chains are intensively studied by many scientists (see [8] for a recent review). A process described by a nonlinear Markov chain is a discrete-time stochastic process whose transitions may depend on both the current state and the present distribution of the process. The simplest nonlinear Markov chain is described by a *quadratic stochastic operator* (QSO) which is associated with a cubic stochastic matrix. This kind of operator arises in the problem of describing the evolution of biological populations (see [9]). The notion of QSO was first introduced by Bernstein [2], and the theory of QSOs was developed in many works (see for example [6], [9], [15], [16]). In [4] and [10], it is given via a self-contained exposition of the recent achievements and open problems in the theory of the QSOs.

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In general, the surjectivity of a quadratic operator is strongly tied up with nonlinear optimization problems (see [1]). The criteria for the surjectivity of QSOs associated with cubic stochastic matrices was given in [15]. Based on the results of [15] and the results of [12], we conclude that a QSO is surjective if and only if it is an orthogonal-preserving QSO. So it is natural to study another class of nonlinear Markov operators described by *cubic stochastic operators* (CSO). (It should be noted that cubic stochastic operators were introduced and developed in [3], [7], [13].)

One can comprehend CSOs as equivalent to the time evolution of genetics in biology by the following situation. Let $I = \{1, 2, \ldots, m\}$ be equivalent to n different genetic types of a species in a population. We denote by $\mathbf{x}^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)})$ the initial probability distribution of the species in the present generation. Here $\mathbf{x}^{(0)}$ is an element of the simplex S^{m-1} (i.e., the set of probability distributions on I). By $P_{ijk,l}$, we mean the probability of the species with ith, jth, and kth genotypes to crosslink each other and produce an individual with lth genotype. For the given current distribution, we can find the probability distribution of the first generation, $\mathbf{x}' = (x_1', \ldots, x_m')$, by mean of the total probability, that is,

$$x_l' = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i^{(0)} x_j^{(0)} x_k^{(0)}.$$

Hence, the correspondence $\mathbf{x}^{(0)} \to \mathbf{x}'$ defines a mapping called an *evolutionary operator*. Therefore, the CSO is a mapping $V : S^{m-1} \to S^{m-1}$ of the form

$$V(\mathbf{x})_l = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k, \quad \mathbf{x} \in S^{m-1}, \tag{1.1}$$

where $P_{ijk,l}$ are heredity coefficients such that

$$P_{ijk,\ell} \geq 0, \qquad \sum_{\ell=1}^{m} P_{ijk,\ell} = 1, \quad i, j, k, \ell \in I,$$

and the coefficients $P_{ijk,l}$ do not change for any permutation of i, j, and k if the types are not connected with the gender.

In the following, we are going to study the surjectivity of CSOs in terms of the orthogonal-preserving property. Namely, we will prove that surjectivity of CSOs is equivalent to its orthogonal-preserving property. This allows us to fully describe all surjective CSOs. As an application, we provide in the last section a full description of the orthogonal-preserving (or surjective) CSO on a 2-dimensional simplex.

2. Preliminaries

Let us recall some necessary notation. Let $I = \{1, \ldots, m\}$. Throughout this paper, we consider the simplex as

$$S^{m-1} = \Big\{ \mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, i \in I, \sum_{i=1}^{m} x_i = 1 \Big\}. \tag{2.1}$$

The complement of set $A \subset I$ is denoted by $A^c = I \backslash A$. By *support* of $\mathbf{x} \in S^{m-1}$, we mean a set $\mathrm{supp}(\mathbf{x}) = \{i \in I : x_i \neq 0\}$. Put the set $\mathrm{null}(\mathbf{x}) = \{i \in \mathbb{N} : x_i = 0\}$.

In what follows, by $\mathbf{e}_i$ we denote the standard basis in $\mathbb{R}^m$ (i.e., $\mathbf{e}_i = (\delta_{i1}, \ldots, \delta_{im})$ $(i \in I)$, where δ_{ij} is the Kronecker delta). We define the face Γ_A of the simplex S^{m-1} by setting $\Gamma_A = \mathrm{conv}\{\mathbf{e}_i\}_{i \in A}$, here $\mathrm{conv}(B)$ stands for the convex hull of a set B. The set $\mathrm{int}\,\Gamma_A = \{\mathbf{x} \in \Gamma_A : x_i > 0, \forall i \in A\}$ is the interior of Γ_A. We recall that a vector $\mathbf{x} \in S^{m-1}$ is orthogonal or singular to $\mathbf{y} \in S^{m-1}$ $(\mathbf{x} \perp \mathbf{y})$ if and only if $\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y}) = \emptyset$ for any $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_m) \in S^{m-1}$. It is clear that $\mathbf{x} \perp \mathbf{y}$ if and only if $x_k \cdot y_k = 0$ for all $k \in I_m$ whenever $\mathbf{x}, \mathbf{y} \in S^{m-1}$. Let V be a CSO given by (1.1) associated with heredity coefficients $\{P_{ijk,\ell}\}$. We define a vector

$$P_{ijk,\bullet} = (P_{ijk,1}, \ldots, P_{ijk,m}), \quad i, j, k \in I.$$

Definition 2.1. A CSO V is called *orthogonal-preserving* (OP) if, for any $\mathbf{x}, \mathbf{y} \in S^{m-1}$ with $\mathbf{x} \perp \mathbf{y}$, we have $V(\mathbf{x}) \perp V(\mathbf{y})$.

An *absorbing state* plays an important role in the theory of the classical (linear) Markov chains. Analogously, in [14] the concept of absorbing sets for nonlinear Markov chains was introduced.

Definition 2.2. A subset $A \subset I$ is called *absorbing* if $A^c = \bigcap_{i,j,k \in A} \mathrm{null}(P_{i,j,k,\bullet})$.

The following results have been proved in [14, Propositions 5.2, 5.5].

Proposition 2.3. *The following statements hold:*

(i) $\mathrm{supp}(V(\mathbf{x})) = \bigcup_{i,j,k \in \mathrm{supp}(\mathbf{x})} \mathrm{supp}(P_{i,j,k,\bullet})$,
(ii) $\mathrm{null}(V(\mathbf{x})) = \bigcap_{i,j,k \in \mathrm{supp}(\mathbf{x})} \mathrm{null}(P_{i,j,k,\bullet})$,
(iii) $V(\mathrm{int}\,\Gamma_A) \subset \mathrm{int}\,\Gamma_B$ *where* $B = \bigcup_{i,j,k \in A} \mathrm{supp}(P_{i,j,k,\bullet})$,
(iv) $V(\mathrm{int}\,\Gamma_A) \subset \mathrm{int}\,\Gamma_B$ *if and only if* $V(\mathbf{x}^{(0)}) \in \mathrm{int}\,\Gamma_B$ *for some* $\mathbf{x}^{(0)} \in \mathrm{int}\,\Gamma_A$.

Proposition 2.4. *Let $A \subset I$ be a subset. The following statements are equivalent:*

(i) *the set A is absorbing,*
(ii) $V(\mathrm{int}\,\Gamma_A) \subset \mathrm{int}\,\Gamma_A$,
(iii) $V(\mathbf{x}^{(0)}) \in \mathrm{int}\,\Gamma_A$ *for some* $\mathbf{x}^{(0)} \in \mathrm{int}\,\Gamma_A$.

3. Surjective and orthogonal-preserving CSOs

Let V be a CSO defined on S^{m-1} which is given by the following form:

$$V(\mathbf{x})_\bullet = \sum_{i,j,k=1}^{m} x_i x_j x_k P_{ijk,\bullet},$$

where the coefficients $P_{ijk,\ell}$ satisfy

$$P_{ijk,\ell} \geq 0, \qquad P_{ijk,\ell} = P_{jki,\ell} = P_{kij,\ell} = P_{kji,\ell} = P_{jik,\ell} = P_{ikj,\ell}, \qquad \sum_{\ell=1}^{m} P_{ijk,\ell} = 1.$$

In this section, we will show that the surjectivity of CSOs is equivalent to the process of orthogonal preserving. First, we need the following auxiliary result.

Proposition 3.1. *If any subset $A \subset I$ with $|A| \leq 3$ is absorbing, then all subsets of I are absorbing.*

Proof. Using the assumption, one concludes that for any $i, j, k \in I$, we have

$$\begin{aligned} &P_{iii,\bullet} = \mathbf{e}_i, \qquad P_{iij,\bullet}, P_{iji,\bullet}, P_{jii,\bullet} \in \operatorname{conv}\{\mathbf{e}_i, \mathbf{e}_j\}, \\ &P_{ijk,\bullet} \in \operatorname{conv}\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\}, \end{aligned} \tag{3.1}$$

where $i \neq j \neq k$. Keeping in mind the symmetricity of $P_{ijk,l}$ and due to the fact (3.1), we obtain $\operatorname{null}(P_{ijk,\bullet}) \supset I\backslash\{i,j,k\}$, therefore for any $B \subset I$, one gets

$$\begin{aligned} \bigcap_{i,j,k\in B} \operatorname{null}(P_{ijk,\bullet}) &= \bigcap_{i\in B}\operatorname{null}(P_{iii,\bullet}) \cap \bigcap_{\substack{i\neq j\\ i,j\in B}} \operatorname{null}(P_{iij,\bullet}) \cap \bigcap_{\substack{i\neq j\neq k\\ i,j,k\in B}} \operatorname{null}(P_{ijk,\bullet}) \\ &= I\backslash B = B^c. \end{aligned}$$

This means that B is absorbing. This completes the proof. □

Proposition 3.2. *If any subset $A \subset I$ with $|A| \leq 3$ is absorbing, then the associated CSO $V : S^{m-1} \to S^{m-1}$ is surjective.*

Proof. Clearly, from Propositions 2.4 and 3.1 we find that the associated CSO $V : S^{m-1} \to S^{m-1}$ maps each face of the simplex into itself. To show that the operator V is surjective, we use mathematical induction by means of the dimension of the simplex. In the case of $m = 2$, we can write V (see (3.1)) in the following form:

$$\begin{aligned} V(\mathbf{x})_1 &= x_1^3 + 3P_{112,1}x_1^2x_2 + 3P_{122,1}x_1x_2^2, \\ V(\mathbf{x})_2 &= x_2^3 + 3P_{112,2}x_1^2x_2 + 3P_{122,2}x_1x_2^2, \end{aligned}$$

where $\mathbf{x} = (x_1, x_2) \in S^1$. It is enough for us to study $V(\mathbf{x})_1$ because of $V(\mathbf{x})_1 + V(\mathbf{x})_2 = 1$. Let

$$f(x) = x^3 + 3P_{112,1}x^2(1-x) + 3P_{122,1}(1-x)^2.$$

Clearly, $f(x_1) \leq 1$ and continuous on interval $[0,1]$. Due to $f(0) = 0$ and $f(1) = 1$, we can infer that $f(x_1)$ is surjective over interval $[0,1]$, hence implying the surjectivity of $V(\mathbf{x})$. Thus, the statement is true for $m = 2$. Further, we assume that the statement holds for $m \leq n-1$, and we will prove it for $m = n$. From the assumption, if we restrict the mapping of V to the face, then the mapping is surjective (i.e., $V : \partial S^{n-1} \to \partial S^{n-1}$ is surjective). Now, we consider $\mathbf{y} \in \operatorname{int} S^{n-1}$. Here, the surjectivity means that the set $V^{-1}(\mathbf{y})$ is nonempty. To prove this statement, we suppose that the set $V^{-1}(\mathbf{y})$ is empty. Then we define a mapping $g : S^{n-1}\backslash\{\mathbf{y}\} \to \partial S^{n-1}$, which maps every point $\mathbf{z} \in S^{n-1}\backslash\{\mathbf{y}\}$ to the intersection point of the ray starting from $\mathbf{z}$ in the direction of $\mathbf{y}$ with the boundary of the simplex. It is easy to check that the mapping $\mathcal{F} : S^{n-1} \to S^{n-1}, \mathcal{F} = g \circ V$, does not have any fixed point. However, this contradicts the Brouwer fixed-point theorem. This completes the proof. □

Now we are ready to prove our main result here.

Theorem 3.3. *Let V be a CSO on S^{m-1} such that $V(\mathbf{e}_i) = \mathbf{e}_i$ for all $i \in I$. Then the following statements are equivalent:*

(i) V *is orthogonal-preserving;*
(ii) V *is surjective;*
(iii) V *satisfies the following conditions:*
  (a) $V^{-1}(\mathbf{e}_i) = \mathbf{e}_i$ *for any* $i \in I$*,*
  (b) $V^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_i\mathbf{e}_j}) = \operatorname{int}\Gamma_{\mathbf{e}_i\mathbf{e}_j}$ *for any* $i, j \in I$*,*
  (c) $V^{-1}(\operatorname{int}\Gamma_{\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k}) = \operatorname{int}\Gamma_{\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k}$ *for any* $i, j, k \in I$*, where* $\Gamma_{\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_l}} = \operatorname{conv}\{\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_l}\}$.

Proof. Let us prove, consecutively, the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let V be an orthogonal-preserving CSO. Due to assumption (i.e., $V(\mathbf{e}_i) = \mathbf{e}_i$), we then have

$$P_{iii,\bullet} = \mathbf{e}_i.$$

Now, choose

$$\mathbf{x}^{(\ell)} = \Big(\frac{1}{m-1}, \ldots, \frac{1}{m-1}, \underbrace{0}_{\ell\text{th term}}, \frac{1}{m-1}, \ldots, \frac{1}{m-1}\Big).$$

Using the definition of CSO, we have

$$\begin{aligned} V(\mathbf{x}^{(\ell)})_\ell &= \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k \\ &= \frac{3}{(m-1)^3} \sum_{\substack{i,j=1 \\ i\neq j\neq \ell}}^{m} P_{iij,\ell} + \frac{6}{(m-1)^3} \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq \ell}}^{m} P_{ijk,\ell}. \end{aligned}$$

Clearly, $\mathbf{x}^{(\ell)}$ is orthogonal to $\mathbf{e}_\ell$, so the orthogonality of $V(\mathbf{x}^{(\ell)})$ to $V(\mathbf{e}_\ell)$ yields

$$P_{iij,\ell} = 0 \quad \text{if } i, j \neq \ell \qquad \text{and} \qquad P_{ijk,\ell} = \quad \text{if } i, j, k \neq \ell.$$

This gives us (3.1), and therefore any subset $A \subset I$ with $|A| \leq 3$ is absorbing. Due to Proposition 3.2, we cab infer that V is surjective.

(ii) $\Rightarrow$ (iii). Assume that V is surjective and let $V^{-1}(\mathbf{e}_i)$ be the preimage of $\mathbf{e}_i$. We set

$$\operatorname{supp}\big(V^{-1}(\mathbf{e}_i)\big) = \bigcup_{\mathbf{x}\in V^{-1}(\mathbf{e}_i)} \operatorname{supp}(\mathbf{x}), \qquad \Gamma_{\operatorname{supp}(V^{-1}(\mathbf{e}_i))} = \operatorname{conv}\{\mathbf{e}_j\}_{j\in\operatorname{supp}(V^{-1}(\mathbf{e}_i))}.$$

Due to Proposition 2.3, we have $V(\Gamma_{\operatorname{supp}(V^{-1}(\mathbf{e}_i))}) = \mathbf{e}_i$. Consequently,

$$\{\mathbf{e}_j\}_{j\in\operatorname{supp}(V^{-1}(\mathbf{e}_i))} \subset V^{-1}(\mathbf{e}_i) \quad \text{for any } i \in I.$$

This implies that $|\operatorname{supp}(V^{-1}(\mathbf{e}_i))| = 1$. This means that only $\mathbf{e}_i$ then maps to $\mathbf{e}_i$, and hence (iii)(a). Further, let $\mathbf{y} \in \operatorname{int}\Gamma_{\mathbf{e}_i\mathbf{e}_j}$ and $\mathbf{x} \in V^{-1}(\mathbf{y})$. Using Proposition 2.3, we have

$$V(\operatorname{int}\Gamma_{\operatorname{supp}(\mathbf{x})}) \subset \operatorname{int}\Gamma_{\mathbf{e}_i\mathbf{e}_j}.$$

In fact, we have

$$\operatorname{supp}(\mathbf{x}) = \{i, j\} \quad \text{for any } \mathbf{x} \in V^{-1}(\mathbf{y}).$$

If not, then $k \in \mathrm{supp}(\mathbf{x})\backslash\{i,j\} \neq \emptyset$. Then $V(\mathbf{e}_k) \in V(\mathrm{int}\,\Gamma_{\mathrm{supp}(\mathbf{x})}) \subset \Gamma_{\mathbf{e}_i\mathbf{e}_j}$, which is a contradiction. Therefore, we obtain (iii)(b). The case (iii)(c) can be done similarly.

(iii) $\Rightarrow$ (ii). This implication follows from Proposition 3.2.

(ii) $\Rightarrow$ (i). Due to the surjectivity of V and condition $V(\mathbf{e}_i) = \mathbf{e}_i$, we get that any subsets $A \subset I$ with $|A| \leq 3$ are absorbing. From (3.1) we obtain

$$\begin{aligned} V(\mathbf{x})_\ell &= \sum_{i,j,k=1}^{m} P_{ijk,\ell} x_i x_j x_k \\ &= P_{\ell\ell\ell,\ell} x_\ell^3 + 3\sum_{\substack{j=1\\ j\neq\ell}}^{m} P_{\ell\ell j,l} x_\ell^2 x_j + 3\sum_{\substack{i=1\\ i\neq\ell}}^{m} P_{ii\ell,\ell} x_i^2 x_\ell + 6\sum_{\substack{j=1\\ i\neq j\neq\ell}}^{m} P_{ij\ell,\ell} x_i x_j x_\ell \\ &= x_\ell\Big(x_\ell^2 + 3\sum_{\substack{j=1\\ j\neq\ell}}^{m} P_{\ell\ell j,\ell} x_\ell x_j + 3\sum_{\substack{i=1\\ i\neq\ell}}^{m} P_{ii\ell,\ell} x_i^2 + 6\sum_{\substack{j=1\\ i\neq j\neq\ell}}^{m} P_{ij\ell,\ell} x_i x_j\Big) \end{aligned} \tag{3.2}$$

for any $\mathbf{x} \in S^{m-1}$ and $\ell \in I$. Next, take any two vectors in the simplex S^{m-1} such that $\mathbf{x} = (x_1, \ldots, x_m) \perp \mathbf{y} = (y_1, \ldots, y_m)$. This means that for any fixed $\ell \in I$, we have either $x_\ell = 0$ or $y_\ell = 0$. Therefore, $V(\mathbf{x})_\ell \cdot V(\mathbf{y})_\ell = 0$ for any $\ell \in I$, which gives the orthogonality of $V(\mathbf{x})$ and $V(\mathbf{y})$. This completes the proof. □

Immediately from Theorem 3.3, we conclude with the following.

Corollary 3.4. *Let V be a CSO on S^{m-1}. Then the following statements are equivalent:*

(i) *V is orthogonal-preserving;*
(ii) *V is surjective;*
(iii) *V satisfies the following conditions:*
  (a) $V^{-1}(\mathbf{e}_{\pi(i)}) = \mathbf{e}_{\pi(i)}$ *for any $i \in I$,*
  (b) $V^{-1}(\mathrm{int}\,\Gamma_{\mathbf{e}_i\mathbf{e}_j}) = \mathrm{int}\,\Gamma_{\mathbf{e}_{\pi(i)}\mathbf{e}_{\pi(j)}}$ *for any $i, j \in I$,*
  (c) $V^{-1}(\mathrm{int}\,\Gamma_{\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k}) = \mathrm{int}\,\Gamma_{\mathbf{e}_{\pi(i)}\mathbf{e}_{\pi(k)}\mathbf{e}_{\pi(k)}}$ *for any $i, j, k \in I$,*
  where as before $\Gamma_{\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_l}} = \mathrm{conv}\{\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_l}\}$.

Remark 3.5. We notice that if V is a surjective CSO with $V(\mathbf{e}_i) = \mathbf{e}_i$ $(i \in I)$, then, from the proof above (see (3.2)), we conclude that V is a Lotka–Volterra operator (see [5] for the definitions). It is known (see [5, Theorem 6]) that if a Lotka–Volterra operator is $\mathbf{f}$-monotone, then it is a bijection, but in [11, Example 5.5] it was shown that a bijective cubic Lotka–Volterra operator has no need to be $\mathbf{f}$-monotone. On the other hand, it is known (see [4]) that quadratic stochastic operators are bijective if and only if they are surjective. Therefore, we formulate the following conjecture.

Conjecture 3.6. Any surjective CSO is bijective.

4. Description of orthogonal-preserving CSOs on 2-dimensional simplex

In general, the description of surjective nonlinear Markov operators is a tricky job. Therefore, in this section, we are going to describe surjective CSOs by means of OP CSOs instead of applying surjectivity directly. In this section, we restrict ourselves to CSOs defined on a 2-dimensional simplex.

Now let us assume that $V : S^2 \to S^2$ is an orthogonal-preserving CSO. This means that

$$\mathbf{e}_1 \perp \mathbf{e}_2 \perp \mathbf{e}_3 \perp\Rightarrow V(\mathbf{e}_1) \perp V(\mathbf{e}_2) \perp V(\mathbf{e}_3).$$

Now from the definition of CSO, we immediately get

$$P_{111,\bullet} \perp P_{222,\bullet} \perp P_{333,\bullet}$$

since in the simplex S^2 there is a unique orthogonal system which is $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$. We conclude that the possible vectors $\{P_{111,\bullet}, P_{222,\bullet}, P_{333,\bullet}\}$ must be permutations of the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Therefore, we have six possibilities and we consider each of these possibilities one by one.

Consider the first possibility by assuming that

$$P_{111,\bullet} = \mathbf{e}_1, \qquad P_{222,\bullet} = \mathbf{e}_2, \qquad P_{333,\bullet} = \mathbf{e}_3.$$

Now our aim is to find conditions for the other coefficients of the given CSO. Let us consider the following orthogonal vectors:

$$\mathbf{x} = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \qquad \mathbf{y} = (0, 0, 1).$$

Then from

$$\begin{aligned} V(\mathbf{x}) &= \frac{1}{8}(3P_{112,1} + 3P_{122,1} + 1, 3P_{112,2} + 3P_{122,2} + 1, 3P_{112,3} + 3P_{122,3}),\\ V(\mathbf{y}) &= (0, 0, 1), \end{aligned}$$

and the orthogonal preservation of V, we get $P_{112,3} = P_{122,3} = 0$. From

$$\sum_{i=1}^{3} P_{112,i} = 1 \qquad \text{and} \qquad \sum_{i=1}^{3} P_{122,i} = 1,$$

we get

$$P_{112,1} + P_{112,2} = 1 \qquad \text{and} \qquad P_{122,1} + P_{122,2} = 1.$$

Now consider

$$\mathbf{x} = \left(\frac{1}{2}, 0, \frac{1}{2}\right), \qquad \mathbf{y} = (0, 1, 0).$$

Then one finds

$$\begin{aligned} V(\mathbf{x}) &= \frac{1}{8}(1 + 3P_{113,1} + 3P_{133,1}, 3P_{113,2} + 3P_{133,2}, 3P_{113,3} + 3P_{133,3} + 1),\\ V(\mathbf{y}) &= (0, 1, 0). \end{aligned}$$

Again the orthogonal preservation of V yields $P_{113,2} = P_{133,2} = 0$. Since

$$\sum_{i=1}^{3} P_{113,i} = 1 \qquad \text{and} \qquad \sum_{i=1}^{3} P_{133,i} = 1,$$

we have

$$P_{113,1} + P_{113,3} = 1 \qquad \text{and} \qquad P_{133,1} + P_{133,3} = 1.$$

Further, let us consider

$$\mathbf{x} = \Big(0, \frac{1}{2}, \frac{1}{2}\Big), \qquad \mathbf{y} = (1, 0, 0).$$

Hence,

$$\begin{aligned} V(\mathbf{x}) &= \frac{1}{8}(3P_{223,1} + 3P_{233,1}, 1 + 3P_{223,2} + 3P_{233,2}, 3P_{223,3} + 3P_{233,3} + 1), \\ V(\mathbf{y}) &= (1, 0, 0). \end{aligned}$$

By the same argument as before, we infer that $P_{223,1} = P_{233,1} = 0$, which implies that

$$P_{223,2} + P_{223,3} = 1 \qquad \text{and} \qquad P_{233,1} + P_{233,3} = 1.$$

Taking into account the obtained equations, we denote

$$\begin{array}{lll} P_{112,1} = \alpha, & P_{122,1} = \beta, & P_{113,1} = \gamma, \\ P_{133,1} = \theta, & P_{223,2} = \lambda, & P_{233,2} = \varphi, \\ P_{123,1} = \xi, & P_{123,2} = \eta. & \end{array}$$

Correspondingly, we get

$$\begin{array}{lll} P_{112,2} = 1 - \alpha, & P_{122,2} = 1 - \beta, & P_{113,3} = 1 - \gamma, \\ P_{133,3} = 1 - \theta, & P_{223,3} = 1 - \lambda, & P_{233,3} = 1 - \varphi, \\ P_{123,3} = 1 - \xi - \eta. & & \end{array}$$

By $V^{(1)}$, we denote the obtained OP CSO. Then we can see that $V^{(1)}$ has the following form:

$$V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta} : \begin{cases} x' = x(x^2 + 3\alpha xy + 3\beta y^2 + 3\gamma xz + 3\theta z^2 + 6\xi yz), \\ y' = y(y^2 + 3(1-\alpha)x^2 + 3(1-\beta)xy + 3\lambda yz + 3\varphi z^2 + 6\eta xz), \\ z' = z(z^2 + 3(1-\gamma)x^2 + 3(1-\theta)xz + 3(1-\lambda)y^2 \\ \qquad + 3(1-\varphi)yz + 6c_{\xi,\eta}xy), \end{cases}$$

where $c_{\xi,\eta} = 1 - \xi - \eta$. Similarly, by considering all other possibilities, we obtain the following operators:

$$V^{(2)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta} : \begin{cases} x' = z(z^2 + 3\alpha zx + 3\beta x^2 + 3\gamma zy + 3\theta y^2 + 6\xi xy), \\ y' = x(x^2 + 3(1-\alpha)z^2 + 3(1-\beta)zx + 3\lambda xy + 3\varphi y^2 + 6\eta yz), \\ z' = y(y^2 + 3(1-\gamma)z^2 + 3(1-\theta)zy + 3(1-\lambda)x^2 \\ \qquad + 3(1-\varphi)xy + 6c_{\xi,\eta}xz), \end{cases}$$

$$V^{(3)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}:\begin{cases} x' = y(y^2+3\alpha yx+3\beta x^2+3\gamma yz+3\theta z^2+6\xi xz),\\ y' = x(x^2+3(1-\alpha)y^2+3(1-\beta)yx+3\lambda xz+3\varphi z^2+6\eta yz),\\ z' = z(z^2+3(1-\gamma)y^2+3(1-\theta)yz+3(1-\lambda)x^2\\ \qquad +3(1-\varphi)xz+6c_{\xi,\eta}xy),\end{cases}$$

$$V^{(4)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}:\begin{cases} x' = x(x^2+3\alpha xz+3\beta z^2+3\gamma xy+3\theta y^2+6\xi yz),\\ y' = z(z^2+3(1-\alpha)x^2+3(1-\beta)xz+3\lambda zy+3\varphi y^2+6\eta xy),\\ z' = y(y^2+3(1-\gamma)x^2+3(1-\theta)xy+3(1-\lambda)z^2\\ \qquad +3(1-\varphi)zy+6c_{\xi,\eta}xz),\end{cases}$$

$$V^{(5)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}:\begin{cases} x' = y(y^2+3\alpha yz+3\beta z^2+3\gamma yx+3\theta x^2+6\xi xz),\\ y' = z(z^2+3(1-\alpha)y^2+3(1-\beta)yz+3\lambda zx+3\varphi x^2+6\eta xy),\\ z' = x(x^2+3(1-\gamma)y^2+3(1-\theta)yx+3(1-\lambda)z^2\\ \qquad +3(1-\varphi)zx+6c_{\xi,\eta}yz),\end{cases}$$

$$V^{(6)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}:\begin{cases} x' = z(z^2+3\alpha zy+3\beta y^2+3\gamma zx+3\theta x^2+6\xi xy),\\ y' = y(y^2+3(1-\alpha)z^2+3(1-\beta)zy+3\lambda yx+3\varphi x^2+6\eta xz),\\ z' = x(x^2+3(1-\gamma)z^2+3(1-\theta)zx+3(1-\lambda)y^2\\ \qquad +3(1-\varphi)yx+6c_{\xi,\eta}yz).\end{cases}$$

Hence, if V is an OP CSO, then it must be one of the above operators. The reverse of the statement is also true. The proof is given in the following theorem.

Theorem 4.1. *Let V be a CSO. Then V is OP if and only if it has one of the following forms:*

$$\begin{array}{lll} V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}, & V^{(2)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}, & V^{(3)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta},\\ V^{(4)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}, & V^{(5)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}, & V^{(6)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}.\end{array} \tag{4.1}$$

Proof. The "if" part comes from the previous calculations. Now let us prove the "only if" part—that is, by assuming a CSO V has the form as given in (4.1). Without loss of generality, we may consider operator $V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}$.

Assume that $\mathbf{x}\perp\mathbf{y}$. Then we obtain the following possibilities:

$$\mathbf{x}\perp\mathbf{y}\Leftrightarrow\begin{cases}\mathbf{x}=(x,y,0), & \mathbf{y}=(0,0.1),\\ \mathbf{x}=(x,0,z), & \mathbf{y}=(0,1,0),\\ \mathbf{x}=(0,y,z), & \mathbf{y}=(1,0,0).\end{cases}$$

First, consider $\mathbf{x}=(x,y,0)$ and $\mathbf{y}=(0,0,1)$. Consequently,

$$V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}(\mathbf{x}) = \big(x^3+3\alpha x^2y+3\beta xy^2, y^3+3(1-\alpha)x^2y+3(1-\beta)xy^2, 0\big),$$
$$V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}(\mathbf{y}) = (0,0,1).$$

It is clear that they are orthogonal. For the other two cases, we could establish the orthogonality of $V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}(\mathbf{x})$ and $V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}(\mathbf{y})$ by the same argument. This completes the proof. ∎

Remark 4.2. From Corollary 3.4, we conclude that all surjective CSOs of S^2 are given by (4.1). We point out that certain classes of operators of the form $V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}$ have been investigated in [7], [13].

Moreover, we want to describe different classes nonconjugacy of OP CSOs on a 2-dimensional simplex.

Definition 4.3. Two stochastic operators $V^{(a)}$ and $V^{(b)}$ are conjugate if there exists a permutation π such that $\pi^{-1}V^{(a)}\pi = V^{(b)}$; the last one is denoted by $V^{(a)} \sim^{\pi} V^{(b)}$.

In our case, we need to consider only permutations of (x, y, z) given by

$$\pi_1 = \begin{bmatrix} x & y & z \\ y & z & x \end{bmatrix}, \qquad \pi_2 = \begin{bmatrix} x & y & z \\ x & z & y \end{bmatrix}.$$

Note that other permutations could be derived from those two.

Theorem 4.4. *Orthogonal-preserving CSOs can be divided into three nonconjugate classes which are*

$$\begin{aligned} K_1 &= \{V^{(3)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}, V^{(4)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}, V^{(6)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}\}, \\ K_2 &= \{V^{(2)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}, V^{(5)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}\}, \\ K_3 &= \{V^{(1)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}\}. \end{aligned}$$

Proof. Under the permutation π_1, let us first consider $V^{(3)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}\pi_1(x, y, z)$. Then we have

$$\begin{aligned} &\pi_1^{-1}V^{(3)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}\pi_1(x, y, z) \\ &\quad= \pi_1^{-1}V^{(3)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}(y, z, x) \\ &\quad= \big(x^3 + 3(1-\gamma)z^2x + 3(1-\theta)zx^2 + 3(1-\lambda)y^2x + 3(1-\varphi)yx^2 \\ &\qquad+ 6(1-\xi-\eta)yzx, z^3 + 3\alpha z^2y + 3\beta zy^2 + 3\gamma z^2x + 3\theta zx^2 \\ &\qquad+ 6(\xi yzx), y^3 + 3(1-\alpha)z^2y + 3(1-\beta)zy^2 + 3\lambda y^2x + 3\varphi yx^2 + 6(\eta yzx)\big) \\ &\quad= V^{(4)}_{1-\theta,1-\gamma,1-\varphi,1-\lambda,\alpha,\beta,1-\xi-\eta,\xi}, \\ &\pi_1^{-1}V^{(4)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}\pi_1(x, y, z) \\ &\quad= \pi_1^{-1}V^{(4)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}(y, z, x) \\ &\quad= \big(z^3 + 3(1-\gamma)y^2z + 3(1-\theta)yz^2 + 3(1-\lambda)x^2z + 3(1-\varphi)xz^2 \\ &\qquad+ 6(1-\xi-\eta)yzx, y^3 + 3\alpha y^2x + 3\beta yx^2 + 3\gamma y^2z + 3\theta yz^2 \\ &\qquad+ 6(\xi yzx), x^3 + 3(1-\alpha)y^2x + 3(1-\beta)yx^2 + 3\lambda x^2z + 3\varphi xz^2 + 6(\eta yzx)\big) \\ &\quad= V^{(6)}_{1-\theta,1-\gamma,1-\varphi,1-\lambda,\alpha,\beta,1-\xi-\eta,\xi}, \\ &\pi_1^{-1}V^{(6)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}\pi_1(x, y, z) \\ &\quad= \pi_1^{-1}V^{(6)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}(y, z, x) \\ &\quad= \big(y^3 + 3(1-\lambda)x^2y + 3(1-\theta)xy^2 + 3(1-\lambda)z^2y + 3(1-\varphi)zy^2 \end{aligned}$$

$$
\begin{aligned}
&+ 6(1-\xi-\eta)yzx, x^3 + 3\alpha x^2 z + 3\beta x z^2 + 3\gamma x^2 y + 3\theta x y^2 \\
&+ 6(\xi y z x), z^3 + 3(1-\alpha)x^2 z + 3(1-\beta)x z^2 + 3\lambda z^2 y + 3\varphi x y^2 + 6(\eta y z x)\big) \\
= {}& V^{(3)}_{1-\theta,1-\gamma,1-\varphi,1-\lambda,\alpha,\beta,1-\xi-\eta,\xi}.
\end{aligned}
$$

This implies that $V^{(3)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}$, $V^{(4)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}$, and $V^{(6)}_{\alpha,\beta,\gamma,\theta,\lambda,\varphi,\xi,\eta}$ are conjugate, and we put them into a class, namely, K_1. Taking into account π_1 and π_2, the other classes can be done similarly. This completes the proof. □

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