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ON FREDHOLM COMPLETIONS OF PARTIAL OPERATOR MATRICES

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Abstract. The aim of this article is to study the Fredholm completion problem of two-by-two partial operator matrices in which the lower-left entry is unspecified and others are specified. By using the methods of operator matrix representation and operator equation, we obtain necessity and sufficiency conditions for the partial operator matrices to have a Fredholm completion with the property that the lower-right entry of its Fredholm inverses is specified.

1. Introduction and preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be separable infinite-dimensional Hilbert spaces, and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$, and write $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, we use $\mathcal{R}(T)$, $\mathcal{N}(T)$, and T^* to denote the range, the null space, and the adjoint of T. For a subspace $\mathcal{M} \subset \mathcal{H}$, its orthogonal complement and closure are denoted by $\mathcal{M}^\perp$ and $\overline{\mathcal{M}}$. Write $P_{\overline{M}}$ for the orthogonal projection onto $\overline{\mathcal{M}}$. For $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, if there exists a unique operator $T^\dagger \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that

$$T^\dagger T T^\dagger = T^\dagger, \qquad T T^\dagger T = T, \qquad (TT^\dagger)^* = TT^\dagger, \qquad (T^\dagger T)^* = T^\dagger T,$$

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then $T^\dagger$ is called the *Moore–Penrose inverse of* T. It is well known that T has the Moore–Penrose inverse if and only if $\mathcal{R}(T)$ is closed. Furthermore,

$$TT^\dagger = P_{\mathcal{R}(T)}, \qquad T^\dagger T = I - P_{\mathcal{N}(T)}.$$

Recall that an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be *left semi-Fredholm* if $\mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T) < \infty$ (see [1, p. 125]). Analogously, T is right semi-Fredholm if $\mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T^*) < \infty$. Note also that T is a Fredholm operator if it is both left and right semi-Fredholm. By Atkinson's theorem (see [1, Theorem 1.51]), an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Fredholm if and only if there exist $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and finite-rank operators $F_1 \in \mathcal{B}(\mathcal{H}), F_2 \in \mathcal{B}(\mathcal{K})$ such that $ST = I + F_1$ and $TS = I + F_2$, and any such operator S will be called a *Fredholm inverse of* T, written T^Φ. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a Fredholm operator, it is well known that the Moore–Penrose inverse $T^\dagger$ is a Fredholm inverse of T. For $S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, we write $S \overset{e}{=} T$ if there exists a finite-rank operator $F \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S = T + F$. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Fredholm, then any two Fredholm inverses S_1 and S_2 of T differ by a finite-rank operator, and so $S_1 \overset{e}{=} S_2$ (see [11, pp. 126–130]).

A partial operator matrix is an operator matrix with some entries specified and the other, unspecified, entries free to be chosen from an arbitrary set. A completion of a partial operator matrix is the conventional operator matrix resulting from a particular choice of values for the unspecified entries. In general, a completion problem is to find conditions on the specified entries so that the partial operator matrix has completions satisfying some nice properties. The completion problem was considered by many mathematicians (see [2]–[4], [6]–[10], [12]–[14], among others). This paper is concerned with the completion problem of partial operator matrices.

In [6], Fiedler and Markham considered the following completion problem:

$$\begin{pmatrix} A & B \\ C & ? \end{pmatrix}^{-1} = \begin{pmatrix} ? & ? \\ ? & D \end{pmatrix};$$

that is, under what conditions on A, B, C and D does there exist X such that

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}$$

is invertible and its inverse has the form

$$\begin{pmatrix} * & * \\ * & D \end{pmatrix},$$

where A, B, C, D are given matrices? Motivated by these results, the completion problem for many types of partial matrices were considered (see [3], [14, pp. 1299–1302], and the references therein). In particular, Tian and Takane in [14] have mentioned that it would be of interest to consider the corresponding completion problems for generalized inverses of partial matrices.

Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $C \in \mathcal{B}(\mathcal{K})$, and $D \in \mathcal{B}(\mathcal{K})$ be given operators. The completion problems of the partial operator matrix

$$\begin{pmatrix} A & B \\ ? & C \end{pmatrix}$$

has been studied by many authors. For example, Takahashi [13] and Hai and Chen [7], [8] considered the completion problem of under what conditions on A, B, and C does there exist $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that

$$\begin{pmatrix} A & B \\ X & C \end{pmatrix}$$

is invertible, right- (left-)invertible, and right- (left-)Fredholm, respectively. In [12], Li considered the following completion problem:

$$\begin{pmatrix} A & B \\ ? & C \end{pmatrix}^{-1} = \begin{pmatrix} ? & ? \\ ? & D \end{pmatrix}.$$

Inspired by the results of [6], [12], and [14], we are interested in the following Fredholm completion problem:

$$\begin{pmatrix} A & B \\ ? & C \end{pmatrix}^{\Phi} \stackrel{e}{=} \begin{pmatrix} ? & ? \\ ? & D \end{pmatrix}.$$

In this paper a systematic use is made of operator matrix representations and of operator equations.

2. Main results

Our main results are the following theorems.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $C \in \mathcal{B}(\mathcal{K})$, and $D \in \mathcal{B}(\mathcal{K})$ be given operators, and let A and D not be finite-rank operators. Assume that $\mathcal{R}(A)$ is closed. Then there exist $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $Y \in \mathcal{B}(\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{K},\mathcal{H})$, and $W \in \mathcal{B}(\mathcal{H},\mathcal{K})$ such that*

$$M_X = \begin{pmatrix} A & B \\ X & C \end{pmatrix}$$

is a Fredholm operator with

$$M_X^{\Phi} \stackrel{e}{=} \begin{pmatrix} Y & Z \\ W & D \end{pmatrix}$$

if and only if the following statements hold:

(i) *$\mathcal{R}(D)$ is closed,*
(ii) *both $\mathcal{N}(D)$ and $\mathcal{N}(A)$ are finite-dimensional or infinite-dimensional,*
(iii) *$B_3 = P_{\mathcal{R}(A)^\perp} B P_{\mathcal{R}(D)} : \mathcal{R}(D) \longrightarrow \mathcal{R}(A)^\perp$ is a finite-rank operator,*
(iv) *$B_4 = P_{\mathcal{R}(A)^\perp} B P_{\mathcal{R}(D)^\perp} : \mathcal{R}(D)^\perp \longrightarrow \mathcal{R}(A)^\perp$ is Fredholm,*

(v) $\mathcal{R}((D_1^{-1})^* - C_1^* + F) \subset \mathcal{R}(B_1^*(A_1^{-1})^*)$ *for some finite-rank operator* $F \in \mathcal{B}(\mathcal{N}(D)^\perp, \mathcal{R}(D))$, *where*

$$\begin{aligned} A_1 &= P_{\mathcal{R}(A)} A P_{\mathcal{N}(A)^\perp} : \mathcal{N}(A)^\perp \to \mathcal{R}(A), \\ B_1 &= P_{\mathcal{R}(A)} B P_{\mathcal{R}(D)} : \mathcal{R}(D) \to \mathcal{R}(A), \\ C_1 &= P_{\mathcal{N}(D)^\perp} C P_{\mathcal{R}(D)} : \mathcal{R}(D) \to \mathcal{N}(D)^\perp, \\ D_1 &= P_{\mathcal{R}(D)} D P_{\mathcal{N}(D)^\perp} : \mathcal{N}(D)^\perp \to \mathcal{R}(D). \end{aligned}$$

Proof. Necessity: Suppose that

$$M_X = \begin{pmatrix} A & B \\ X & C \end{pmatrix}$$

is a Fredholm operator for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and suppose that

$$M_X^\Phi \overset{e}{=} \begin{pmatrix} Y & Z \\ W & D \end{pmatrix}.$$

Note that $\mathcal{R}(A)$ is closed, and hence M_X as an operator from $\mathcal{N}(A)^\perp \oplus \mathcal{N}(A) \oplus \overline{\mathcal{R}(D)} \oplus \mathcal{R}(D)^\perp$ to $\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{N}(D)^\perp \oplus \mathcal{N}(D)$ has a matrix representation

$$M_X = \begin{pmatrix} A_1 & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \\ X_1 & X_2 & C_1 & C_2 \\ X_3 & X_4 & C_3 & C_4 \end{pmatrix},$$

and M_X^Φ as an operator from $\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \oplus \mathcal{N}(D)^\perp \oplus \mathcal{N}(D)$ to $\mathcal{N}(A)^\perp \oplus \mathcal{N}(A) \oplus \overline{\mathcal{R}(D)} \oplus \mathcal{R}(D)^\perp$ has a matrix representation

$$M_X^\Phi = \begin{pmatrix} Y_1 & Y_2 & Z_1 & Z_2 \\ Y_3 & Y_4 & Z_3 & Z_4 \\ W_1 & W_2 & D_1 & 0 \\ W_3 & W_4 & 0 & 0 \end{pmatrix}.$$

Clearly, A_1 is an invertible operator, and D_1 is an injective operator with dense range. By the definition of the Fredholm inverse, we can see that

$$M_X M_X^\Phi = \begin{pmatrix} I + F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & I + F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & I + F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & I + F_{44} \end{pmatrix}, \tag{2.1}$$

$$M_X^\Phi M_X = \begin{pmatrix} I + F'_{11} & F'_{12} & F'_{13} & F'_{14} \\ F'_{21} & I + F'_{22} & F'_{23} & F'_{24} \\ F'_{31} & F'_{32} & I + F'_{33} & F'_{34} \\ F'_{41} & F'_{42} & F'_{43} & I + F'_{44} \end{pmatrix}, \tag{2.2}$$

where F_{ij} and F'_{ij} are finite-rank operators, $i, j = 1, 2, 3, 4$.

First, from (2.1) and (2.2) it follows that

$$B_3D_1 = F_{23},$$
$$A_1Z_2 = F_{14},$$
$$D_1X_2 = F'_{32},$$
$$W_3A_1 = F'_{41}$$

are finite-rank operators. Note that A_1 and D_1 are not finite-rank operators, and hence B_3, W_3, Z_2, and X_2 are finite-rank operators.

Next, by (2.1) and (2.2) we can see that

$$B_4W_4 = I + F_{22} - B_3W_2,$$
$$W_4B_4 = I + F'_{44} - W_3B_2,$$
$$X_4Z_4 = I + F_{44} - X_3Z_2,$$
$$Z_4X_4 = I + F'_{22} - Z_3X_2.$$

This, together with the fact that B_3, W_3, Z_2, and X_2 are finite-rank operators, shows that B_4 and X_4 are Fredholm operators. Since X_4 is an operator from $\mathcal{N}(A)$ to $\mathcal{N}(D)$, it follows from the Fredholmness of X_4 that both $\mathcal{N}(A)$ and $\mathcal{N}(D)$ are finite-dimensional or infinite-dimensional.

Now, from (2.1) we can find that

$$A_1Z_1 + B_1D_1 = F_{13},$$
$$X_1Z_1 + X_2Z_3 + C_1D_1 = I + F_{33},$$

and so

$$(C_1 - X_1A_1^{-1}B_1)D_1 = I + F_{33} - X_2Z_3 - X_1A_1^{-1}F_{13}. \tag{2.3}$$

Similarly, from (2.2) we can also find that

$$W_1A_1 + D_1X_1 = F'_{31},$$
$$W_1B_1 + W_2B_3 + D_1C_1 = I + F'_{33},$$

and hence

$$D_1(C_1 - X_1A_1^{-1}B_1) = I + F'_{33} - W_2B_3 - F'_{31}A_1^{-1}B_1. \tag{2.4}$$

Since B_3, X_2, F_{13}, F_{33}, F'_{31}, and F'_{33} are finite rank operators, it follows that $F_{33} - X_2Z_3 - X_1A_1^{-1}F_{13}$ and $F'_{33} - W_2B_3 - F'_{31}A_1^{-1}B_1$ are finite-rank operators, which together with (2.3) and (2.4) shows that D_1 is a Fredholm operator with

$$D_1^{\Phi} \stackrel{e}{=} C_1 - X_1A_1^{-1}B_1.$$

It is not hard to find that $\mathcal{R}(D) = \mathcal{R}(D_1)$ is closed. Thus D_1 is invertible, and so $D_1^{\Phi} \stackrel{e}{=} D_1^{-1}$. Therefore,

$$C_1 - D_1^{-1} \stackrel{e}{=} X_1A_1^{-1}B_1.$$

By the range inclusion theorem [5, Theorem 1], $C_1 - D_1^{-1} \stackrel{e}{=} X_1A_1^{-1}B_1$ if and only if

$$\mathcal{R}\big(C_1^* - (D_1^{-1})^* + F\big) \subset \mathcal{R}\big(B_1^*(A_1^{-1})^*\big)$$

for some finite-rank operator $F \in \mathcal{B}(\mathcal{N}(D)^{\perp}, \mathcal{R}(D))$.

Sufficiency: Since $\mathcal{R}(A)$ and $\mathcal{R}(D)$ are closed, it follows that A, B, C, and D have the following operator matrix representation:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A) \longrightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp},$$
$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} : \mathcal{R}(D) \oplus \mathcal{R}(D)^{\perp} \longrightarrow \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp},$$
$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \mathcal{R}(D) \oplus \mathcal{R}(D)^{\perp} \longrightarrow \mathcal{N}(D)^{\perp} \oplus \mathcal{N}(D),$$
$$D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{N}(D)^{\perp} \oplus \mathcal{N}(D) \longrightarrow \mathcal{R}(D) \oplus \mathcal{R}(D)^{\perp},$$

respectively. It is obvious that A_1 and D_1 are invertible. Note that both $\mathcal{N}(A)$ and $\mathcal{N}(D)$ are infinite-dimensional or finite-dimensional, and thus there exists a bounded linear operator $S : \mathcal{N}(A) \longrightarrow \mathcal{N}(D)$ such that S is Fredholm. Also, by (iii) and the range inclusion theorem [5, Theorem 1], there exists $T \in \mathcal{B}(\mathcal{N}(A)^{\perp}, \mathcal{N}(D)^{\perp})$ such that

$$D_1^{-1} + F = C_1 - TA_1^{-1}B_1,$$

where $F \in \mathcal{B}(\mathcal{R}(D), \mathcal{N}(D)^{\perp})$ is a finite-rank operator. Let

$$X = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} : \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A) \longrightarrow \mathcal{N}(D)^{\perp} \oplus \mathcal{N}(D),$$
$$Y = \begin{pmatrix} A_1^{-1}(I + B_1D_1TA_1^{-1}) & A_1^{-1}(B_1D_1C_2 - B_2 - B_1D_1TA_1^{-1}B_2)B_4^{\dagger} \\ S^{\dagger}C_3D_1TA_1^{-1} & -S^{\dagger}(C_3D_1TA_1^{-1}B_2 - C_3D_1C_2 + C_4)B_4^{\dagger} \end{pmatrix}$$
$$: \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp} \longrightarrow \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A),$$
$$Z = \begin{pmatrix} -A_1^{-1}B_1D_1 & 0 \\ -S^{\dagger}C_3D_1 & S^{\dagger} \end{pmatrix} : \mathcal{N}(D)^{\perp} \oplus \mathcal{N}(D) \longrightarrow \mathcal{N}(A)^{\perp} \oplus \mathcal{N}(A),$$
$$W = \begin{pmatrix} -D_1TA_1^{-1} & D_1(TA_1^{-1}B_2 - C_2)B_4^{\dagger} \\ 0 & B_4^{\dagger} \end{pmatrix}$$
$$: \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp} \longrightarrow \mathcal{R}(D) \oplus \mathcal{R}(D)^{\perp}.$$

Then $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $Y \in \mathcal{B}(\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Now, a direct calculation shows that

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} A & B \\ X & C \end{pmatrix}\begin{pmatrix} Y & Z \\ W & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} Y & Z \\ W & D \end{pmatrix}\begin{pmatrix} A & B \\ X & C \end{pmatrix}$$

are finite-rank operators. This complete the proof. □

Theorem 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $C \in \mathcal{B}(\mathcal{K})$, and $D \in \mathcal{B}(\mathcal{K})$ be given operators. Assume that A and D are finite-rank operators. Then there exist $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $Y \in \mathcal{B}(\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$\begin{pmatrix} A & B \\ X & C \end{pmatrix}$$

is a Fredholm operator with

$$\begin{pmatrix} A & B \\ X & C \end{pmatrix}^{\Phi} \overset{e}{=} \begin{pmatrix} Y & Z \\ W & D \end{pmatrix}$$

if and only if B is a Fredholm operator.

Proof. Since A and D are finite-rank operators, then we only need to show that there exist $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $Y \in \mathcal{B}(\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}$$

is a Fredholm operator with

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}^{\Phi} \overset{e}{=} \begin{pmatrix} Y & Z \\ W & 0 \end{pmatrix}$$

if and only if B is a Fredholm operator.

Suppose that there are $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $Y \in \mathcal{B}(\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}$$

is a Fredholm operator with

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}^{\Phi} \overset{e}{=} \begin{pmatrix} Y & Z \\ W & 0 \end{pmatrix}.$$

Thus there exist finite-rank operators F_{ij} and F'_{ij}, $i, j = 1, 2$ such that

$$\begin{aligned}
\begin{pmatrix} 0 & B \\ X & C \end{pmatrix} \begin{pmatrix} Y & Z \\ W & 0 \end{pmatrix} &= \begin{pmatrix} BW & 0 \\ XY + CW & XZ \end{pmatrix} \\
&= \begin{pmatrix} I + F_{11} & F_{12} \\ F_{21} & I + F_{22} \end{pmatrix}, \\
\begin{pmatrix} Y & Z \\ W & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ X & C \end{pmatrix} &= \begin{pmatrix} ZX & YB + ZC \\ 0 & WB \end{pmatrix} \\
&= \begin{pmatrix} I + F'_{11} & F'_{12} \\ F'_{21} & I + F'_{22} \end{pmatrix},
\end{aligned}$$

and hence

$$\begin{aligned} BW &= I + F_{11}, \\ WB &= I + F'_{22}, \end{aligned}$$

which shows that B is a Fredholm operator.

Conversely, assume that B is a Fredholm operator. Let $G \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a Fredholm operator. A direct calculation shows that

$$\begin{pmatrix} 0 & B \\ G & C \end{pmatrix} \begin{pmatrix} -G^{\dagger}CB^{\dagger} & G^{\dagger} \\ B^{\dagger} & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} -P_{\mathcal{R}(B)^{\perp}} & 0 \\ P_{\mathcal{R}(G)^{\perp}}CB^{\dagger} & -P_{\mathcal{R}(G)^{\perp}} \end{pmatrix}$$

and

$$\begin{pmatrix} -G^\dagger C B^\dagger & G^\dagger \\ B^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ G & C \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} -P_{\mathcal{N}(G)} & G^\dagger C P_{\mathcal{N}(B)} \\ 0 & -P_{\mathcal{N}(B)} \end{pmatrix}.$$

Note that the Fredholmness of B and G implies that $\mathcal{N}(B)$, $\mathcal{N}(G)$, $\mathcal{R}(B)^\perp$, and $\mathcal{R}(G)^\perp$ are finite-dimensional, and so

$$\begin{pmatrix} 0 & B \\ G & C \end{pmatrix}$$

is a Fredholm operator with

$$\begin{pmatrix} 0 & B \\ G & C \end{pmatrix}^\Phi \overset{e}{=} \begin{pmatrix} -G^\dagger C B^\dagger & G^\dagger \\ B^\dagger & 0 \end{pmatrix}.$$

This complete the proof. □

Theorem 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $C \in \mathcal{B}(\mathcal{K})$, and $D \in \mathcal{B}(\mathcal{K})$ be given operators. Assume that A is a finite-rank operator and D is not a finite-rank operator. Then there exist $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $Y \in \mathcal{B}(\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$\begin{pmatrix} A & B \\ X & C \end{pmatrix}$$

is a Fredholm operator with

$$\begin{pmatrix} A & B \\ X & C \end{pmatrix}^\Phi \overset{e}{=} \begin{pmatrix} Y & Z \\ W & D \end{pmatrix}$$

if and only if the following statements hold:

(i) *B is a right semi-Fredholm operator with $\dim \mathcal{N}(B) = \infty$;*
(ii) *there exists a closed subspace $\mathcal{M} \subset \mathcal{K}$ with $\dim \mathcal{M} = \dim \mathcal{M}^\perp = \infty$ such that $C_1 = P_{\mathcal{M}} C P_{\mathcal{N}(B)} : \mathcal{N}(B) \longrightarrow \mathcal{M}$ is Fredholm and*

$$D \overset{e}{=} \begin{pmatrix} C_1^\dagger & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{M} \oplus \mathcal{M}^\perp \longrightarrow \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp.$$

For the proof of Theorem 2.3, we need a lemma.

Lemma 2.4 ([1, p. 28])**.** *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, and let $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If $ST \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator, then S is right-semi-Fredholm and T is left-semi-Fredholm.*

Proof of Theorem 2.3. Suppose that there exist $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $Y \in \mathcal{B}(\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$\begin{pmatrix} A & B \\ X & C \end{pmatrix}$$

is a Fredholm operator with

$$\begin{pmatrix} A & B \\ X & C \end{pmatrix}^\Phi \overset{e}{=} \begin{pmatrix} Y & Z \\ W & D \end{pmatrix}.$$

Note that A is a finite-rank operator. It follows that

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}$$

is a Fredholm operator and

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}^{\Phi} \overset{e}{=} \begin{pmatrix} Y & Z \\ W & D \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & C \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix}$$

is Fredholm, it follows from Lemma 2.4 that B is right-semi-Freholm and X is left-semi-Fredholm. Thus

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}$$

has the following matrix representation:

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix} = \begin{pmatrix} 0 & 0 & B_1 \\ 0 & C_1 & C_2 \\ X_1 & C_3 & C_4 \end{pmatrix} : \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{H} \oplus \mathcal{R}(X)^{\perp} \oplus \mathcal{R}(X),$$

where B_1 is left-invertible and X_1 is right-invertible. Therefore, there exist invertible operators U and V on $\mathcal{H} \oplus \mathcal{K}$ such that

$$U \begin{pmatrix} 0 & B \\ X & C \end{pmatrix} V = \begin{pmatrix} 0 & 0 & B_1 \\ 0 & C_1 & 0 \\ X_1 & 0 & 0 \end{pmatrix}.$$

This, together with the Fredholmness of

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}$$

and the invertibility of U and V, shows that C_1 is Fredholm. Note that B_1 is left-invertible, X_1 is right-invertible, and C_1 is Fredholm, and hence one can infer that a Fredholm inverse of

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}$$

has the following matrix representation form:

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}^{\Phi} \overset{e}{=} \left(\begin{array}{c|cc} * & * & X_1^{\dagger} \\ \hline * & C_1^{\dagger} & 0 \\ B_1^{\dagger} & 0 & 0 \end{array}\right) : \mathcal{H} \oplus \mathcal{R}(X)^{\perp} \oplus \mathcal{R}(X) \longrightarrow \mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp}.$$

On the other hand,

$$\begin{pmatrix} 0 & B \\ X & C \end{pmatrix}^{\Phi} \overset{e}{=} \begin{pmatrix} Y & Z \\ W & D \end{pmatrix},$$

and hence

$$D \overset{e}{=} \begin{pmatrix} C_1^{\dagger} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{R}(X)^{\perp} \oplus \mathcal{R}(X) \longrightarrow \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp}.$$

Since D is not finite-rank, then C_1 is not a finite-rank operator, which implies $\dim \mathcal{N}(B) = \dim \mathcal{R}(X)^{\perp} = \infty$. Let $\mathcal{M} = \mathcal{R}(X)^{\perp}$. Then clearly (i) and (ii) hold.

Conversely, assume that (i) and (ii) hold. Since $\dim \mathcal{M} = \dim \mathcal{M}^{\perp} = \infty$, there exists a Fredholm operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{M}^{\perp})$ such that $\mathcal{R}(T) = \mathcal{M}^{\perp}$. Let

$$\begin{aligned} B_1 &= BP_{\mathcal{N}(B)^{\perp}} : \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{H}, \\ C_2 &= P_{\mathcal{M}} C P_{\mathcal{N}(B)^{\perp}} : \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{M}, \\ C_3 &= P_{\mathcal{M}^{\perp}} C P_{\mathcal{N}(B)} : \mathcal{N}(B) \longrightarrow \mathcal{M}^{\perp}, \\ C_4 &= P_{\mathcal{M}^{\perp}} C P_{\mathcal{N}(B)^{\perp}} : \mathcal{N}(B)^{\perp} \longrightarrow \mathcal{M}^{\perp}, \end{aligned}$$

and let

$$\begin{aligned} X &= \begin{pmatrix} 0 \\ T \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{M} \oplus \mathcal{M}^{\perp}, \\ Y &= T^{\dagger}(C_3 C_1^{\dagger} C_2 - C_4) B_1^{\dagger} : \mathcal{H} \longrightarrow \mathcal{H}, \\ Z &= \begin{pmatrix} -T^{\dagger} C_3 C_1^{\dagger} & T^{\dagger} \end{pmatrix} : \mathcal{M} \oplus \mathcal{M}^{\perp} \longrightarrow \mathcal{H}, \\ W &= \begin{pmatrix} -C_1^{\dagger} C_2 B_1^{\dagger} \\ B_1^{\dagger} \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp}. \end{aligned}$$

It is easy to check that

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} A & B \\ X & C \end{pmatrix} \begin{pmatrix} Y & Z \\ W & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} Y & Z \\ W & D \end{pmatrix} \begin{pmatrix} A & B \\ X & C \end{pmatrix}$$

are finite-rank operators. This complete the proof. □

Remark 2.5. When (i) A is not a finite-rank operator and D is a finite-rank operator, or (ii) A and D are not finite-rank operators and $\mathcal{R}(A)$ is not closed, the completion problem

$$\begin{pmatrix} A & B \\ ? & C \end{pmatrix}^{\Phi} \overset{e}{=} \begin{pmatrix} ? & ? \\ ? & D \end{pmatrix}$$

is more complicated, and is unsolved in this paper.

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