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CHLODOWSKY–SZASZ–APPELL-TYPE OPERATORS FOR FUNCTIONS OF TWO VARIABLES

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Abstract. This article deals with the approximation properties of the bivariate operators which are the combination of Bernstein–Chlodowsky operators and the Szász operators involving Appell polynomials. We investigate the degree of approximation of the operators with the help of the complete modulus of continuity and the partial moduli of continuity. In the last section of the paper, we introduce the *generalized Boolean sum* (GBS) of these bivariate Chlodowsky–Szasz–Appell-type operators and examine the order of approximation in the Bögel space of continuous functions by means of the mixed modulus of smoothness.

1. Introduction

Appell in [2] introduced a sequence of polynomials $P_n(x)$ of degree n which satisfies the differential equation

$$DP_n(x) = nP_{n-1}(x), \qquad D \equiv \frac{d}{dx},$$

known as Appell polynomials. These polynomials have been studied widely because of their remarkable applications not only in mathematics (see [5]) but also in physics and in chemistry. In [18], Sheffer extended the class of Appell polynomials and called these *zero-type polynomials*. Using Appell polynomials, Jakimovski and Leviatan [14] introduced a generalization of the Favard–Szász

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operators as

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\Big(\frac{k}{n}\Big), \tag{1.1}$$

where $g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k$ is the generating function for the Appell polynomials $p_k(x) \geq 0$ with $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, $R > 1$, and $g(1) \neq 0$.

Subsequently, the Stancu-type generalization of the operators (1.1) was introduced by Atakut and Büyükyazici [4], wherein the authors established some approximation properties. A generalization of the operators given by (1.1) is defined as

$$P_n^*(f;x) = \frac{e^{-b_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(b_n x) f\Big(\frac{k}{c_n}\Big), \tag{1.2}$$

where (b_n), (c_n) denote the unbounded and increasing sequences of positive real numbers such that $b_n \geq 1$, $c_n \geq 1$, and $\lim_{n\to\infty} \frac{1}{c_n} = 0$, $\frac{b_n}{c_n} = 1 + O(\frac{1}{c_n})$ as $n \to \infty$. In the special case $g(z) = 1$, these operators reduce to the modified Szász operators studied by Walczak [20]. Also, for $b_n = n = c_n$, these operators coincide with the operators (1.1). (For more details the reader should consult the recent work on approximation theory in [3], [12].)

On the interval $[0, a_n]$ with $a_n \to \infty$, as $n \to \infty$, the Bernstein–Chlodowsky polynomials are defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \Big(\frac{x}{a_n}\Big)^k \Big(1 - \frac{x}{a_n}\Big)^{n-k} f\Big(k\frac{a_n}{n}\Big), \tag{1.3}$$

where $x \in [0, a_n]$ and $\lim_{n\to\infty} \frac{a_n}{n} = 0$. In approximation theory, Chlodowsky-type generalizations of certain operators have been studied intensively (see [16], [15]).

By combining the Bernstein–Chlodowsky operators (1.3) and the operators (1.2), we introduce the bivariate operators as follows:

$$T_{n,m}(f;x,y) = \sum_{k=0}^{n} \sum_{j=0}^{\infty} \binom{n}{k} \Big(\frac{x}{a_n}\Big)^k \Big(1 - \frac{x}{a_n}\Big)^{n-k} \frac{e^{-b_m y}}{g(1)} p_j(b_m y) f\Big(k\frac{a_n}{n}, \frac{j}{c_m}\Big) \tag{1.4}$$

for all $n, m \in \mathbb{N}$, $f \in C(A)_{a_n}$ with $A_{a_n} = \{(x,y) : 0 \leq x \leq a_n, 0 \leq y < \infty\}$, and $C(A_{a_n}) := \{f : A_{a_n} \to R \text{ is continuous}\}$. Note that the operator (1.4) is the tensorial product of ${}_xB_n$ and ${}_yP_m^*$; that is, $T_{n,m} = {}_xB_n \circ {}_yP_m^*$, where

$${}_xB_n(f;x,y) = \sum_{k=0}^{n} \binom{n}{k} \Big(\frac{x}{a_n}\Big)^k \Big(1 - \frac{x}{a_n}\Big)^{n-k} f\Big(k\frac{a_n}{n}, y\Big),$$

and

$${}_yP_m^*(f;x,y) = \frac{e^{-b_m y}}{g(1)} \sum_{k=0}^{\infty} p_k(b_m y) f\Big(x, \frac{k}{c_m}\Big).$$

Our purpose here is to establish the degree of approximation for the bivariate operators (1.4) by means of the moduli of continuity and the Lipschitz class. The rate of convergence of these operators for a weighted space is studied with the aid of the modulus of continuity introduced in [13]. Subsequently, the GBS case of these operators (1.4) is introduced, and the approximation degree for the GBS operators is obtained by means of the mixed modulus of smoothness.

2. Preliminaries

To examine the approximation properties of the operators (1.4), we give some basic results using the test functions $e_{i,j} = t^i s^j$ $(i, j = 0, 1, 2)$ as follows.

Lemma 2.1. *We have*

(i) $T_{n,m}(e_{0,0}; x, y) = 1$,

(ii) $T_{n,m}(e_{1,0}; x, y) = x$,

(iii) $T_{n,m}(e_{0,1}; x, y) = \frac{b_m}{c_m} y + \frac{1}{c_m} \frac{g'(1)}{g(1)}$,

(iv) $T_{n,m}(e_{2,0}; x, y) = x^2 + \frac{x}{n}(a_n - x)$,

(v) $T_{n,m}(e_{0,2}; x, y) = \frac{b_m^2}{c_m^2} y^2 + \frac{b_m}{c_m^2}(2\frac{g'(1)}{g(1)} + 1)y + \frac{1}{c_m^2}(\frac{g''(1)}{g(1)} + \frac{g'(1)}{g(1)})$,

(vi) $T_{n,m}(e_{3,0}; x, y) = x^3 + \frac{x(a_n-x)}{n^2}(x(3n-2) + a_n)$,

(vii) $T_{n,m}(e_{0,3}; x, y) = \frac{b_m^3}{c_m^3} y^3 + \frac{b_m^2}{c_m^3}(3\frac{g'(1)}{g(1)} + 4)y^2 + \frac{b_m}{c_m^3}(3\frac{g''(1)}{g(1)} + 8\frac{g'(1)}{g(1)} + 1)y + \frac{1}{c_m^3}(\frac{g'''(1)}{g(1)} + 4\frac{g''(1)}{g(1)} + \frac{g'(1)}{g(1)})$,

(viii) $T_{n,m}(e_{4,0}; x, y) = x^4 + \frac{x^3(a_n-x)}{n^3}(6n^2 - 5n + 2) + \frac{x(a_n-x)^2}{n^3}(2x(3n-2) + a_n) + \frac{x^2 a_n(a_n-x)(n-1)}{n^3}$,

(ix) $T_{n,m}(e_{0,4}; x, y) = \frac{b_m^4}{c_m^4} y^4 + \frac{b_m^3}{c_m^4} y^3(4\frac{g'(1)}{g(1)} + 10) + \frac{b_m^2}{c_m^4} y^2(6\frac{g''(1)}{g(1)} + 30\frac{g'(1)}{g(1)} + 14) + \frac{b_m}{c_m^4} y(4\frac{g'''(1)}{g(1)} + 30\frac{g''(1)}{g(1)} + 28\frac{g'(1)}{g(1)} + 1) + \frac{1}{c_m^4}(\frac{g^{(4)}(1)}{g(1)} + 10\frac{g'''(1)}{g(1)} + 14\frac{g''(1)}{g(1)} + \frac{g'(1)}{g(1)})$.

Proof. By using simple calculations, we can easily prove the above results. Hence the details are omitted. □

As a consequence of Lemma 2.1, we obtain the following lemma.

Lemma 2.2. *For the operator (1.4), we have the following results:*

(i) $T_{n,m}((e_{1,0} - x)^2; x, y) = \frac{x}{n}(a_n - x)$,

(ii) $T_{n,m}((e_{0,1} - y)^2; x, y) = (\frac{b_m}{c_m} - 1)^2 y^2 + (2\frac{b_m}{c_m^2}\frac{g'(1)}{g(1)} - \frac{2}{c_m}\frac{g'(1)}{g(1)} + \frac{b_m}{c_m^2})y + \frac{1}{c_m^2}(\frac{g'(1)}{g(1)} + \frac{g''(1)}{g(1)})$,

(iii) $T_{n,m}((e_{1,0} - x)^4; x, y) = (\frac{3}{n^2} - \frac{6}{n^3})x^4 - \frac{6a_n(n-2)}{n^3}x^3 + a_n^2(\frac{3}{n^2} - \frac{7}{n^3})x^2 + \frac{a_n^3}{n^3}x$,

(iv) $T_{n,m}((e_{0,1} - y)^4; x, y) = (\frac{b_m}{c_m} - 1)^4 y^4 + \{\frac{b_m^3}{c_m^4}(4\frac{g'(1)}{g(1)} + 10) + 6\frac{b_m}{c_m^2}(2\frac{g'(1)}{g(1)} + 1) - 4\frac{b_m^2}{c_m^3}(3\frac{g'(1)}{g(1)} + 4) - \frac{4}{c_m}\frac{g'(1)}{g(1)}\}y^3 + \{\frac{b_m^2}{c_m^4}(6\frac{g''(1)}{g(1)} + 30\frac{g'(1)}{g(1)} + 14) - 4\frac{b_m}{c_m^3}(3\frac{g''(1)}{g(1)} + 8\frac{g'(1)}{g(1)} + 1) + \frac{6}{c_m^2}(\frac{g''(1)}{g(1)} + \frac{g'(1)}{g(1)})\}y^2 + \{\frac{b_m}{c_m^4}(4\frac{g'''(1)}{g(1)} + 30\frac{g''(1)}{g(1)} + 28\frac{g'(1)}{g(1)} + 1) - \frac{4}{c_m^3}(\frac{g'''(1)}{g(1)} + 4\frac{g''(1)}{g(1)} + \frac{g'(1)}{g(1)})\}y + \frac{1}{c_m^4}(\frac{g^{(4)}(1)}{g(1)} + 10\frac{g'''(1)}{g(1)} + 14\frac{g''(1)}{g(1)} + \frac{g'(1)}{g(1)})$.

Lemma 2.3. *Taking into account the conditions on (a_n), (b_n), (c_n), and using Lemma 2.1 and Lemma 2.2, we may write*

(i) $T_{n,m}((e_{1,0} - x)^2; x, y) = O(\frac{a_n}{n})(x^2 + x)$, *as* $n \to \infty$,

(ii) $T_{n,m}((e_{0,1} - y)^2; x, y) \leq \frac{\eta(g)}{c_m}(y^2 + y + 1)$,

(iii) $T_{n,m}((e_{1,0} - x)^4; x, y) = O(\frac{a_n}{n})(x^4 + x^3 + x^2 + x)$, *as* $n \to \infty$,

(iv) $T_{n,m}((e_{0,1}-y)^4;x,y) \leq \frac{\mu(g)}{c_m}(y^4+y^3+y^2+y+1)$,

where $\eta(g)$ *and* $\mu(g)$ *are certain constant depending on* g.

3. Main results

In this section, we establish the degree of approximation of the operators given by (1.4) in the space of continuous functions on compact set $I_{ab} := [0,a]\times[0,b] \subset A_{a_n}$. For $f \in C(I_{ab})$, the complete modulus of continuity for the bivariate case is defined as follows:

$$\omega(f;\delta) = \sup\big\{\big|f(t,s)-f(x,y)\big| : (t,s),(x,y) \in I_{ab} \text{ and } \sqrt{(t-x)^2+(s-y)^2} \leq \delta\big\}.$$

The partial moduli of continuity with respect to x and y is given by

$$\omega_1(f;\delta) = \sup\big\{\big|f(x_1,y)-f(x_2,y)\big| : y \in [0,b] \text{ and } |x_1-x_2| \leq \delta\big\},$$

and

$$\omega_2(f;\delta) = \sup\big\{\big|f(x,y_1)-f(x,y_2)\big| : x \in [0,a] \text{ and } |y_1-y_2| \leq \delta\big\}.$$

Clearly, these moduli satisfy the properties of the usual modulus of continuity.

Theorem 3.1. *For all* $(x,y) \in I_{ab}$ *and* $f \in C(I_{ab})$, *we have the following inequality:*

$$\big|T_{n,m}(f;x,y)-f(x,y)\big| \leq 2\omega(f;\delta_{n,m}),$$

where $\delta_{n,m} = (O(\frac{a_n}{n})(x^2+x)+\frac{\eta(g)}{c_m}(y+1)^2)^{1/2}$.

Proof. From the definition of the complete modulus of continuity, we have

$$\begin{aligned}\big|T_{n,m}(f;x,y)-f(x,y)\big| &\leq T_{n,m}\big(\big|f(t,s)-f(x,y)\big|;x,y\big)\\ &\leq T_{n,m}\big(\omega\big(f;\sqrt{(t-x)^2+(s-y)^2}\big);x,y\big)\\ &\leq \omega(f;\delta_{n,m})\Big\{1+\frac{1}{\delta_{n,m}}T_{n,m}\big(\sqrt{(t-x)^2+(s-y)^2};x,y\big)\Big\}.\end{aligned}$$

Using the Cauchy–Schwarz inequaltiy and Lemma 2.3, we have

$$\begin{aligned}&\big|T_{n,m}(f;x,y)-f(x,y)\big|\\ &\quad\leq \omega(f;\delta_{n,m})\Big[1+\frac{1}{\delta_{n,m}}\big\{T_{n,m}\big((e_{1,0}-x)^2+(e_{0,1}-y)^2;x,y\big)\big\}^{1/2}\Big]\\ &\quad\leq \omega(f;\delta_{n,m})\Big[1+\frac{1}{\delta_{n,m}}\big\{T_{n,m}\big((e_{1,0}-x)^2;x,y\big)+T_{n,m}\big((e_{0,1}-y)^2;x,y\big)\big\}^{1/2}\Big]\\ &\quad\leq \omega(f;\delta_{n,m})\Big[1+\frac{1}{\delta_{n,m}}\Big\{O\Big(\frac{a_n}{n}\Big)(x^2+x)+\frac{\eta(g)}{c_m}(y+1)^2\Big\}^{1/2}\Big],\end{aligned}$$

from which the desired result is immediate. □

Theorem 3.2. *For $f \in C(I_{ab})$ and all $(x,y) \in I_{ab}$, the following result holds:*

$$\big|T_{n,m}(f;x,y) - f(x,y)\big| \le 2\big(\omega_1(f;\delta_n) + \omega_2(f;\delta_m)\big),$$

where

$$\delta_n^2 = \frac{x}{n}(a_n - x),$$
$$\delta_m^2 = \Big(\frac{b_m}{c_m} - 1\Big)^2 y^2 + \Big(2\frac{b_m}{c_m^2}\frac{g'(1)}{g(1)} - \frac{2}{c_m}\frac{g'(1)}{g(1)} + \frac{b_m}{c_m^2}\Big)y + \frac{1}{c_m^2}\Big(\frac{g'(1)}{g(1)} + \frac{g''(1)}{g(1)}\Big).$$

Proof. Using the definition of the partial moduli of continuity, Lemma 2.2, and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\big|T_{n,m}(f;x,y) - f(x,y)\big| \\
&\quad \le T_{n,m}\big(\big|f(t,s) - f(x,y)\big|;x,y\big) \\
&\quad \le T_{n,m}\big(\big|f(t,s) - f(x,s)\big|;x,y\big) + T_{n,m}\big(\big|f(x,s) - f(x,y)\big|;x,y\big) \\
&\quad \le T_{n,m}\big(\omega_1\big(f;|t-x|\big);x,y\big) + T_{n,m}\big(\omega_2\big(f;|s-y|\big);x,y\big) \\
&\quad \le \omega_1(f;\delta_n)\Big[1 + \frac{1}{\delta_n}T_{n,m}\big(|t-x|;x,y\big)\Big] + \omega_2(f;\delta_m)\Big[1 + \frac{1}{\delta_m}T_{n,m}\big(|s-y|;x,y\big)\Big] \\
&\quad \le \omega_1(f;\delta_n)\Big[1 + \frac{1}{\delta_n}\big(T_{n,m}\big((e_{1,0}-x)^2;x,y\big)\big)^{1/2}\Big] \\
&\qquad + \omega_2(f;\delta_m)\Big[1 + \frac{1}{\delta_m}\big(T_{n,m}\big((e_{0,1}-y)^2;x,y\big)\big)^{1/2}\Big].
\end{aligned}$$

Choosing $\delta_n^2 = \frac{x}{n}(a_n - x)$ and $\delta_m^2 = (\frac{b_m}{c_m} - 1)^2 y^2 + (2\frac{b_m}{c_m^2}\frac{g'(1)}{g(1)} - \frac{2}{c_m}\frac{g'(1)}{g(1)} + \frac{b_m}{c_m^2})y + \frac{1}{c_m^2}(\frac{g'(1)}{g(1)} + \frac{g''(1)}{g(1)})$, we obtain the required result. □

Now, we establish the degree of approximation for the bivariate operators (1.4) with the aid of the Lipschitz class. For $0 < \gamma_1 \le 1$ and $0 < \gamma_2 \le 1$ and $f \in C(I_{ab})$, we define the Lipschitz class $Lip_M(\gamma_1,\gamma_2)$ for the bivariate case as follows:

$$\big|f(t,s) - f(x,y)\big| \le M|t-x|^{\gamma_1}|s-y|^{\gamma_2}.$$

Theorem 3.3. *Let $f \in Lip_M(\gamma_1,\gamma_2)$. Then we have*

$$\big|T_{n,m}(f;x,y) - f(x,y)\big| \le M\delta_n^{\gamma_1}\delta_m^{\gamma_2},$$

where δ_n and δ_m are the same as in Theorem 3.2.

Proof. Since $f \in Lip_M(\gamma_1,\gamma_2)$, we may write

$$\begin{aligned}
\big|T_{n,m}(f;x,y) - f(x,y)\big| &\le T_{n,m}\big(\big|f(t,s) - f(x,y)\big|;x,y\big) \\
&\le T_{n,m}\big(M|t-x|^{\gamma_1}|s-y|^{\gamma_2};x,y\big) \\
&\le M\,{}_xB_n\big(|t-x|^{\gamma_1};x,y\big)\,{}_yP_m^*\big(|s-y|^{\gamma_2};x,y\big).
\end{aligned}$$

Applying Hölder's inequality with $(p_1, q_1) = (\frac{2}{\gamma_1}, \frac{2}{2-\gamma_1})$ and $(p_2, q_2) = (\frac{2}{\gamma_2}, \frac{2}{2-\gamma_2})$, we have

$$\begin{aligned}\big|T_{n,m}(f;x,y) - f(x,y)\big| &\le M\, {}_xB_n\big((e_{1,0}-x)^2;x,y\big)^{\gamma_1/2}\, {}_xB_n(e_{0,0};x,y)^{(2-\gamma_1)/2}\\ &\quad\times {}_yP_m^*\big((e_{0,1}-y)^2;x,y\big)^{\gamma_2 y/2}\, {}_yP_m^*(e_{0,0};x,y)^{(2-\gamma_2)/2}\\ &= M\delta_n^{\gamma_1}\delta_m^{\gamma_2}.\end{aligned}$$

This proves the theorem. □

Now we estimate the degree of approximation of the bivariate operators (1.4) in a weighted space. Let B_ρ be the space of all functions f defined on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$, $\mathbb{R}_0^+ = [0,\infty)$ having the property $|f(x,y)| \le M_f\rho(x,y)$, where M_f is a positive constant depending only on f and $\rho(x,y) = 1 + x^2 + y^2$ is a weight function. Let C_ρ be the subspace of B_ρ of all continuous functions with the norm $\|f\|_\rho = \sup_{x,y\in\mathbb{R}_0^+} \frac{|f(x,y)|}{\rho(x,y)}$, and let C_ρ^0 be the subspace of all functions $f \in C_\rho$ such that $\lim_{x\to\infty} \frac{|f(x,y)|}{\rho(x,y)}$ exists finitely. For all $f \in C_\rho^0$, the weighted modulus of continuity is defined by

$$\omega_\rho(f;\delta_1,\delta_2) = \sup_{x,y\in\mathbb{R}_0^+}\ \sup_{|h_1|\le\delta_1,|h_2|\le\delta_2} \frac{|f(x+h_1,y+h_2) - f(x,y)|}{\rho(x,y)\rho(h_1,h_2)}. \tag{3.1}$$

Further details of the weighted modulus of continuity can be found in [13].

Theorem 3.4. *If $f \in C_\rho^0$, then, for sufficiently large n, m, the following inequality holds:*

$$\sup_{x,y\in\mathbb{R}_0^+} \frac{|T_{n,m}(f;x,y) - f(x,y)|}{\rho(x,y)^3} \le C\omega_\rho(f;\delta_n,\delta_m),$$

where $\delta_n = (\frac{a_n}{n})^{1/2}$, $\delta_m = (\frac{\sigma(g)}{c_m})^{1/2}$, $\sigma(g) = \max\{\eta(g), \mu(g)\}$, and C is a constant depending on n, m.

Proof. From ([13], p. 577), we may write

$$\begin{aligned}\big|f(t,s) - f(x,y)\big| &\le 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\Big(1 + \frac{|t-x|}{\delta_n}\Big)\Big(1 + \frac{|s-y|}{\delta_m}\Big)\\ &\quad\times\big(1+(t-x)^2\big)\big(1+(s-y)^2\big).\end{aligned}$$

Thus

$$\begin{aligned}&\big|T_{n,m}(f;x,y) - f(x,y)\big|\\ &\quad\le 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{x}{\alpha_n}\Big)^k\Big(1-\frac{x}{\alpha_n}\Big)^{n-k}\Big(1+\frac{1}{\delta_n}\Big|k\frac{a_n}{n} - x\Big|\Big)\\ &\qquad\times\Big(1+\Big(k\frac{a_n}{n}-x\Big)^2\Big)\sum_{j=0}^{\infty}\frac{e^{-b_m y}}{g(1)}p_j(b_m y)\Big(1+\frac{1}{\delta_m}\Big|\frac{j}{c_m} - y\Big|\Big)\Big(1+\Big(\frac{j}{c_m}-y\Big)^2\Big).\end{aligned}$$

Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\big|T_{n,m}(f;x,y)-f(x,y)\big|\\
&\le 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\Big[1+T_{n,m}\big((e_{1,0}-x)^2;x,y\big)\\
&\quad+\frac{1}{\delta_n}\sqrt{T_{n,m}\big((e_{1,0}-x)^2;x,y\big)}\Big]\\
&\quad\times\frac{1}{\delta_n}\sqrt{T_{n,m}\big((e_{1,0}-x)^2;x,y\big)T_{n,m}\big((e_{1,0}-x)^4;x,y\big)}\\
&\quad\times\Big[1+T_{n,m}\big((e_{0,1}-y)^2;x,y\big)+\frac{1}{\delta_m}\sqrt{T_{n,m}\big((e_{0,1}-y)^2;x,y\big)}\\
&\quad\times\frac{1}{\delta_m}\sqrt{T_{n,m}\big((e_{0,1}-y)^2;x,y\big)T_{n,m}\big((e_{0,1}-y)^4;x,y\big)}\Big].
\end{aligned}$$

Using Lemma 2.3, we have

$$\begin{aligned}
&\big|T_{n,m}(f;x,y)-f(x,y)\big|\\
&\le 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\Big[1+O\Big(\frac{a_n}{n}\Big)(x^2+x)+\frac{1}{\delta_n}\sqrt{O\Big(\frac{a_n}{n}\Big)(x^2+x)}\\
&\quad+\frac{1}{\delta_n}\sqrt{O\Big(\frac{a_n}{n}\Big)(x^2+x)O\Big(\frac{a_n}{n}\Big)(x^4+x^3+x^2+x)}\Big]\\
&\quad\times\Big[1+\frac{\eta(g)}{c_m}(y+1)^2+\frac{1}{\delta_m}\sqrt{\frac{\eta(g)}{c_m}(y+1)^2}\\
&\quad+\frac{1}{\delta_m}\sqrt{\frac{\eta(g)}{c_m}(y+1)^2\frac{\mu(g)}{c_m}(y+1)^4}\Big].
\end{aligned}$$

Taking $\delta_n=(\frac{a_n}{n})^{1/2}$, $\delta_m=(\frac{\sigma(g)}{c_m})^{1/2}$ with $\sigma(g)=\max\{\eta(g),\mu(g)\}$, we reach the desired result. □

4. Construction of GBS operators of the Chlodowsky–Szàsz–Appell type

The continuity and the differentiability of a function in a Bögel space were first examined by Bögel in [8] and [9]. After this, Dobrescu and Matei [11] used the definitions of B-continuity and B-differentiability to obtain the approximating properties of GBS of bivariate Bernstein polynomials. In [6], Badea et al. proved the *Test function theorem* for the functions defined in the Bögel space of continuous functions. In the same space, the quantitative variant of a Korovkin-type theorem was given by Badea and Badea in [7, Theorem 2.2]. Recently, Manjari et al. [19] constructed the GBS operators of a Bernstein–Schurer–Kantorovich type and obtained the degree of approximation for these operators. Agrawal and Ispir [1] established the degree of approximation for the bivariate Chlodowsky–Szasz–Charlier-type operators and the associated GBS operators.

First we give some basic definitions and notation.

A real-valued function f on the rectangle $A = ([a,b] \times [c,d])$ is called B-*continuous* if for every $(x,y) \in A$ there holds

$$\lim_{(u,v)\to(x,y)} \Delta_{(u,v)} f(x,y) = 0,$$

where $\Delta_{(u,v)} f(x,y) = f(x,y) - f(x,v) - f(u,y) + f(u,v)$.

We denote by $C_b(A)$ the space of all B-continuous functions on A. $B(A)$, $C(A)$ denote the space of all bounded functions and the space of all continuous (in the usual sense) functions on A endowed with the sup-norm $\|\cdot\|_\infty$. It is known that $C(A) \subset C_b(A)$ (see [10], page 52).

The mixed modulus of smoothness of $f \in C_b(A)$ is defined as

$$\omega_{\text{mixed}}(f;\delta_1,\delta_2) := \sup\left\{\left|\Delta_{(x+h_1,y+h_2)} f(x,y)\right|\right\},$$

where the supremum is taken over all $(x,y) \in A$, $(h_1,h_2) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that $(x+h_1, y+h_2) \in A$, $0 < |h_1| \le \delta_1$, $0 < |h_2| \le \delta_2$, and where $\Delta_{u,v} f(x,y)$ is as defined above. The mixed modulus of continuity involving upper bounds and the total modulus of continuity were introduced by Marchaud [17, p. 410].

A real-valued function defined on A is called a *uniformly* B-*continuous function* if and only if

$$\lim_{\delta_1,\delta_2\to 0} \omega_{\text{mixed}}(f;\delta_1,\delta_2) = 0.$$

Furthermore, for all nonnegative numbers λ_1, λ_2, there holds

$$\omega_{\text{mixed}}(f;\lambda_1\delta_1,\lambda_2\delta_2) \le \big(1 +]\lambda_1[\big)\big(1 +]\lambda_2[\big)\omega_{\text{mixed}}(f;\delta_1,\delta_2),$$

where $]\lambda[$ denotes the largest integer which is smaller than λ.

A function $f : A \longrightarrow R$ is called *Bögel differentiable* if, for every $(x,y) \in A$,

$$\lim_{(u,v)\to(x,y)} \frac{\Delta_{(u,v)} f(x,y)}{(u-x)(v-y)} = D_B f(x,y) < \infty.$$

Here D_B is called the B-*derivative* of f, and the space of all B-differentiable functions is denoted by $D_b(A)$.

In this section, we introduce the GBS case of the operators defined in (1.4).

For every $f \in C_b(A)$, the GBS operator associated with the operator $T_{n,m}(f;x,y)$ is defined as follows:

$$U_{n,m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{\infty} \binom{n}{k}\left(\frac{x}{a_n}\right)^k \left(1-\frac{x}{a_n}\right)^{n-k} \frac{e^{-b_m y}}{g(1)} p_j(b_m y)$$
$$\times \left[f\left(k\frac{a_n}{n}, y\right) + f\left(x, \frac{j}{c_m}\right) - f\left(k\frac{a_n}{n}, \frac{j}{c_m}\right)\right]. \tag{4.1}$$

Let $I_{cd} := [0,c] \times [0,d] \subset A_{a_n}$.

Theorem 4.1. *For every $f \in C_b(I_{cd})$ and for all $(x,y) \in I_{cd}$, we have the following inequality for the operator defined in (4.1):*

$$\left|U_{n,m}(f;x,y) - f(x,y)\right| \le 4\omega_{\text{mixed}}(f;\delta_n,\delta_m), \tag{4.2}$$

where $\delta_n = (\frac{a_n}{n}(c^2+c))^{1/2}$, $\delta_m := \delta_m(g) = (\frac{\rho(g)}{c_m})^{1/2}$, and $\rho(g)$ is a constant depending on g.

Proof. Using the definition of $\omega_{\text{mixed}}(f;\delta_n,\delta_m)$ and the elementary inequality, we have

$$\omega_{\text{mixed}}(f;\lambda_1\delta_n,\lambda_2\delta_m)\le(1+\lambda_1)(1+\lambda_2)\omega_{\text{mixed}}(f;\delta_n,\delta_m),\quad \lambda_1,\lambda_2>0.$$

Therefore,

$$\begin{aligned}\big|\Delta_{(x,y)}f(t,s)\big| &\le \omega_{\text{mixed}}\big(f;|t-x|,|s-y|\big)\\ &\le \Big(1+\frac{|t-x|}{\delta_n}\Big)\Big(1+\frac{|s-y|}{\delta_m}\Big)\omega_{\text{mixed}}(f;\delta_n,\delta_m) \end{aligned}\tag{4.3}$$

for every $(x,y),(t,s)\in I_{cd}$ and for any $\delta_n,\delta_m>0$. Further, by the definition of $\Delta_{(x,y)}f(t,s)$, we get

$$f(x,s)+f(t,y)-f(t,s)=f(x,y)-\Delta_{(x,y)}f(t,s).$$

Applying the operator defined in (1.4) to both sides of the above equality, we get

$$U_{n,m}(f;x,y)=f(x,y)T_{n,m}(e_{0,0};x,y)-T_{n,m}\big(\Delta_{(x,y)}f(t,s);x,y\big).$$

Since $T_{n,m}(e_{0,0};x,y)=1$, applying the Cauchy–Schwarz inequality,

$$\begin{aligned}&\big|U_{n,m}(f;x,y)-f(x,y)\big|\\ &\quad\le T_{n,m}\big(\big|\Delta_{(x,y)}f(t,s)\big|;x,y\big)\\ &\quad\le \Big(T_{n,m}(e_{0,0};x,y)+\delta_n^{-1}\sqrt{T_{n,m}\big((e_{1,0}-x)^2;x,y\big)}+\delta_m^{-1}\sqrt{T_{n,m}\big((e_{0,1}-y)^2;x,y\big)}\\ &\qquad+\delta_n^{-1}\delta_m^{-1}\sqrt{T_{n,m}\big((e_{1,0}-x)^2;x,y\big)}\sqrt{T_{n,m}\big((e_{0,1}-y)^2;x,y\big)}\Big)\omega_{\text{mixed}}(f;\delta_n,\delta_m).\end{aligned}\tag{4.4}$$

By Lemma 2.2 and for all $(x,y)\in I_{cd}$, we have

$$\begin{aligned}T_{n,m}\big((e_{1,0}-x)^2;x,y\big)&=\frac{x(a_n-x)}{n}\\ &\le\frac{a_n}{n}(x^2+x)\le\frac{a_n}{n}(c^2+c).\end{aligned}\tag{4.5}$$

Similarly,

$$\begin{aligned}T_{n,m}\big((e_{0,1}-y)^2;x,y\big)&\le\frac{\eta(g)}{c_m}(y^2+y+1)\\ &\le\frac{\eta(g)}{c_m}(d^2+d+1)=\frac{\rho(g)}{c_m},\end{aligned}\tag{4.6}$$

where $\rho(g)$ is a constant depending on g. Choosing $\delta_n=(\frac{a_n}{n}(c^2+c))^{1/2}$ and $\delta_m:=\delta_m(g)=(\frac{\rho(g)}{c_m})^{1/2}$, we get the required result. □

In our next theorem, we obtain the degree of approximation of the $U_{n,m}$ in terms of the Lipschitz-class which is defined as follows:

$$\begin{aligned}Lip_M(\xi_1,\xi_2)=\Big\{&f\in C_b(I_{cd}):\big|\Delta_{(x,y)}f(t,s)\big|\le M|t-x|^{\xi_1}|s-y|^{\xi_2},\\ &\text{for }(t,s),(x,y)\in I_{cd}\Big\},\end{aligned}$$

where $f\in C_b(I_{cd})$ and $0<\xi_1,\xi_2\le 1$.

Theorem 4.2. *For $f \in Lip_M(\xi_1, \xi_2)$ and $(x, y) \in I_{cd}$, we have*

$$\left|U_{n,m}(f; x, y) - f(x, y)\right| \leq M\delta_n^{\xi_1/2}\delta_m^{\xi_2/2},$$

where $\delta_n = \|_x B_n((t-x)^2; \cdot)\|_\infty$, $\delta_m = \|_y P_m^*((s-y)^2; \cdot)\|_\infty$, and M is a certain positive constant.

Proof. By the definition of $U_{n,m}(f; \cdot, \cdot)$ and using the linearity of the operator $T_{n,m}(f; \cdot, \cdot)$, we may write

$$\begin{aligned} U_{n,m}(f; \cdot, \cdot) &= T_{n,m}\big(f(x, s) + f(t, y) - f(t, s); x, y\big) \\ &= T_{n,m}\big(f(x, y) - \Delta_{(x,y)}f(t, s); x, y\big) \\ &= f(x, y)T_{n,m}(e_{0,0}; x, y) - T_{n,m}\big(\Delta_{(x,y)}f(t, s); x, y\big). \end{aligned}$$

By our hypothesis, we get

$$\begin{aligned} \left|U_{n,m}(f; x, y) - f(x, y)\right| &\leq T_{n,m}\big(\left|\Delta_{(x,y)}f(t, s)\right|; x, y\big) \\ &\leq MT_{n,m}\big(|t-x|^{\xi_1}|s-y|^{\xi_2}; x, y\big) \\ &= MT_{n,m}\big(|t-x|^{\xi_1}; x, y\big)T_{n,m}\big(|s-y|^{\xi_2}; x, y\big). \end{aligned}$$

Now, applying the Hölder's inequality with $(p_1, q_1) = (2/\xi_1, 2/(2-\xi_1))$ and $(p_2, q_2) = (2/\xi_2, 2/(2-\xi_2))$, we obtain

$$\begin{aligned} \left|U_{n,m}(f; x, y) - f(x, y)\right| &\leq M {}_xB_n\big((t-x)^2; x\big)^{\xi_1/2} {}_xB_n(e_0; x)^{(2-\xi_1)/2} \\ &\quad \times {}_yP_m^*\big((s-y)^2; y\big)^{\xi_2/2} {}_yP_m^*(e_0; y)^{(2-\xi_2)/2}. \end{aligned}$$

Taking $\delta_n = \|_x B_n((t-x)^2; \cdot)\|_\infty$ and $\delta_m = \|_y P_m^*((s-y)^2; \cdot)\|_\infty$, we get the desired result. ☐

Theorem 4.3. *If $f \in D_b(I_{cd})$ and $D_Bf \in B(I_{cd})$, then, for each $(x, y) \in I_{cd}$, we get*

$$\begin{aligned} &\left|U_{n,m}(f; x, y) - f(x, y)\right| \\ &\quad \leq C\big\{3\|D_Bf\|_\infty + 2\omega_{\text{mixed}}(f; \delta_n, \delta_m)\sqrt{x^2+x}\sqrt{y^2+y+1}\big\}\delta_n\delta_m \\ &\qquad + \big\{\omega_{\text{mixed}}(f; \delta_n, \delta_m)(\delta_m\sqrt{x^4+x^3+x^2+x}\sqrt{y^2+y+1} \\ &\qquad + \delta_n\sqrt{y^4+y^3+y^2+y+1}\sqrt{x^2+x})\big\}, \end{aligned}$$

where $\delta_n = \sqrt{\frac{a_n}{n}}$, $\delta_m = \sqrt{\frac{\sigma(g)}{c_m}}$, $\sigma(g) = \max\{\eta(g), \mu(g)\}$, and C is a constant depending on n, m only.

Proof. By our hypothesis,

$$\Delta_{(x,y)}f(t, s) = (t-x)(s-y)D_Bf(\alpha, \beta) \quad \text{with } x < \alpha < t; y < \beta < s.$$

Clearly,

$$D_Bf(\alpha, \beta) = \Delta_{(x,y)}D_Bf(\alpha, \beta) + D_Bf(\alpha, y) + D_Bf(x, \beta) - D_Bf(x, y).$$

Since $D_B f \in B(I_{cd})$, from the above equalities, we have

$$\begin{aligned}
&\big|T_{n,m}\big(\Delta_{(x,y)} f(t,s);x,y\big)\big| \\
&\quad= \big|T_{n,m}\big((t-x)(s-y)D_B f(\alpha,\beta);x,y\big)\big| \\
&\quad\le T_{n,m}\big(|t-x||s-y|\big|\Delta_{(x,y)} D_B f(\alpha,\beta)\big|;x,y\big) \\
&\qquad+ T_{n,m}\big(|t-x||s-y|\big(\big|D_B f(\alpha,y)\big| + \big|D_B f(x,\beta)\big| + \big|D_B f(x,y)\big|\big);x,y\big) \\
&\quad\le T_{n,m}\big(|t-x||s-y|\omega_{\text{mixed}}\big(D_B f;|\alpha-x|,|\beta-y|\big);x,y\big) \\
&\qquad+ 3\|D_B f\|_\infty T_{n,m}\big(|t-x||s-y|;x,y\big).
\end{aligned} \tag{4.7}$$

By the properties of the mixed modulus of smoothness ω_{mixed}, we can write

$$\begin{aligned}
&\omega_{\text{mixed}}\big(D_B f;|\alpha-x|,|\beta-y|\big) \\
&\quad\le \omega_{\text{mixed}}\big(D_B f;|t-x|,|s-y|\big) \\
&\quad\le \big(1+\delta_n^{-1}|t-x|\big)\big(1+\delta_m^{-1}|s-y|\big)\omega_{\text{mixed}}(D_B f;\delta_n,\delta_m).
\end{aligned} \tag{4.8}$$

Combining (4.7), (4.8), and using the Cauchy–Schwarz inequality, we find

$$\begin{aligned}
&\big|U_{n,m}(f;x,y) - f(x,y)\big| \\
&\quad= \big|T_{n,m}\Delta_{(x,y)} f(t,s);x,y\big| \\
&\quad\le 3\|D_B f\|_\infty \sqrt{T_{n,m}\big((t-x)^2(s-y)^2;x,y\big)} + \Big(T_{n,m}\big(|t-x||s-y|;x,y\big) \\
&\qquad+ \delta_n^{-1} T_{n,m}\big((t-x)^2|s-y|;x,y\big) + \delta_m^{-1} T_{n,m}\big(|t-x|(s-y)^2;x,y\big) \\
&\qquad+ \delta_n^{-1}\delta_m^{-1} T_{n,m}\big((t-x)^2(s-y)^2;x,y\big)\Big)\omega_{\text{mixed}}(D_B f;\delta_n,\delta_m) \\
&\quad\le 3\|D_B f\|_\infty \sqrt{T_{n,m}\big((t-x)^2(s-y)^2;x,y\big)} + \Big(\sqrt{T_{n,m}\big((t-x)^2(s-y)^2;x,y\big)} \\
&\qquad+ \delta_n^{-1}\sqrt{T_{n,m}\big((t-x)^4(s-y)^2;x,y\big)} + \delta_m^{-1}\sqrt{T_{n,m}\big((t-x)^2(s-y)^4;x,y\big)} \\
&\qquad+ \delta_n^{-1}\delta_m^{-1} T_{n,m}\big((t-x)^2(s-y)^2;x,y\big)\Big)\omega_{\text{mixed}}(D_B f;\delta_n,\delta_m).
\end{aligned} \tag{4.9}$$

Since for $(x,y),(t,s) \in I_{cd}$ and $i,j \in \{1,2\}$ we have

$$T_{n,m}\big((t-x)^{2i}(s-y)^{2j};x,y\big) = {}_xB_n\big((t-x)^{2i};x\big)\, {}_yP_m^*\big((s-y)^{2j};y\big), \tag{4.10}$$

and from Lemma 2.3,

$$\begin{aligned}
{}_xB_n\big((t-x)^2;x\big) &= O\Big(\frac{a_n}{n}\Big)(x^2+x), \\
{}_xB_n\big((t-x)^4;x\big) &= O\Big(\frac{a_n}{n}\Big)(x^4+x^3+x^2+x), \\
{}_yP_m^*\big((s-y)^2;y\big) &\le \frac{\eta(g)}{c_m}(y^2+y+1) \\
{}_yP_m^*\big((s-y)^4;y\big) &\le \frac{\mu(g)}{c_m}(y^4+y^3+y^2+y+1),
\end{aligned}$$

combining (4.9) and (4.10), on choosing $\delta_n = \sqrt{\frac{a_n}{n}}$, $\delta_m = \sqrt{\frac{\sigma(g)}{c_m}}$ and $\sigma(g) = \max(\eta(g),\mu(g))$, we reach the required result. $\square$

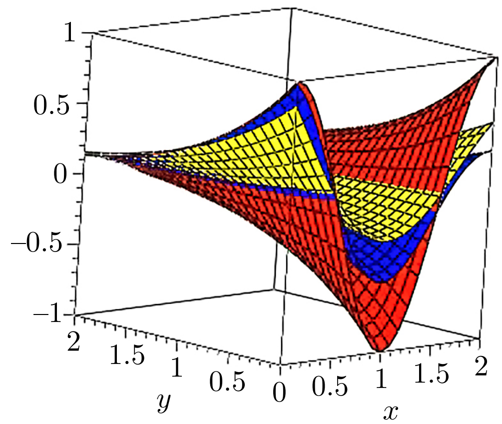

FIGURE 1. The convergence of $T_{n,m}(f;x,y)$ to $f(x,y)$ (red f, blue $T_{40,40}$, yellow $T_{5,5}$).

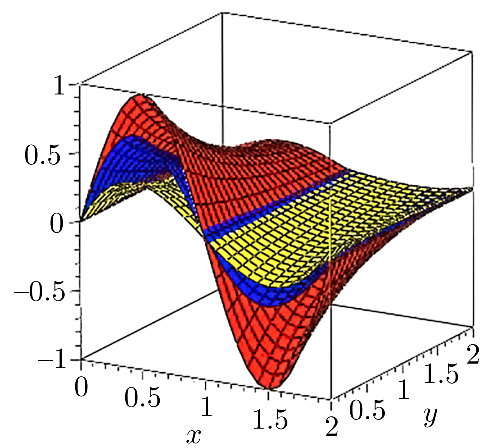

FIGURE 2. The convergence of $T_{n,m}(f;x,y)$ to $f(x,y)$ (red f, blue $T_{40,40}$, yellow $T_{5,5}$).

5. Numerical examples

In this section, we give some numerical results regarding the approximation properties of Chlodowsky–Szász–Appell operators defined in (1.4).

Example 5.1. Let us consider the function $f(x,y) = e^{-y}\cos(\pi x)$, $g(u) = u$ and $a_n = \sqrt{n}$, $b_n = n$, $c_n = n + \frac{1}{\sqrt{n}}$. For $n = m = 5$ and $n = m = 40$ the convergence of $T_{n,m}(f;x,y)$ to $f(x,y)$ is illustrated in Figure 1.

Example 5.2. Let us consider the function $f(x,y) = e^{-y}\sin(\pi x)$, $g(u) = u$ and $a_n = \sqrt{n}$, $b_n = n$, $c_n = n + e^{-n}$. For $n = m = 5$ and $n = m = 40$ the convergence of $T_{n,m}(f;x,y)$ to $f(x,y)$ is illustrated in Figure 2.

We notice from the above examples that for $n = m = 40$ the approximation of the operator $T_{n,m}$ to the function f is better than $n = m = 5$.

Table 1. Error of approximation for $T_{n,m}$

$n=m$	$a_n=\sqrt{n}$, $b_n=n$, $c_n=n+\frac{1}{\sqrt{n}}$	$a_n=\sqrt{n}$, $b_n=n$, $c_n=n+e^{-n}$
20	3.9062769320	3.9678794400
50	2.6253029800	2.6362709420
100	1.9595674840	1.9624455190
500	0.9975826189	0.9977048873
1000	0.7505644223	0.7505954100
1500	0.6370476284	0.6370614836
2000	0.5677575415	0.5677653628
2500	0.5196173630	0.5196223806
3000	0.4835609538	0.4835644444

Example 5.3. If $f\in C(I_{ab})$, then

$$\begin{aligned}\left|T_{n,m}(f;x,y)-f(x,y)\right| &\leq 2\big(\omega_1(f;\delta_n)+\omega_2(f;\delta_m)\big)\\ &\leq 2\big(\|f^{(1,0)}\|_\infty\delta_n+\|f^{(0,1)}\|_\infty\delta_m\big),\end{aligned}$$

where δ_n and δ_m are defined in Theorem 3.2.

In Table 1 we compute the error of approximation of $f(x,y)=xye^{-y}$ by using the above relation for $I_{ab}=[0,4]\times[0,4]$.

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