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SHERMAN TYPE THEOREM ON C^*-ALGEBRAS

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Abstract. In this paper, a new definition of majorization for C^*-algebras is introduced. Sherman's inequality is extended to self-adjoint operators and positive linear maps by applying the method of premajorization used for comparing two tuples of objects. A general result in a matrix setting is established. Special cases of the main theorem are studied. In particular, a HLPK-type inequality is derived.

1. Introduction

We begin this expository section with some elements of majorization theory. An m-tuple $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$ is said to be *majorized* by m-tuple $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ written as $\mathbf{y} \prec \mathbf{x}$ if

$$\sum_{i=1}^{k} y_{[i]} \le \sum_{i=1}^{k} x_{[i]} \quad \text{for } k = 1, \dots, m, \qquad \text{and} \qquad \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} x_i,$$

where $x_{[1]} \ge \dots \ge x_{[m]}$ and $y_{[1]} \ge \dots \ge y_{[m]}$ are the entries of $\mathbf{x}$ and $\mathbf{y}$, respectively, stated in nonincreasing order (see [8, p. 8]).

It is not hard to verify that the majorization relation $\prec$ is a preorder on the space $\mathbb{R}^m$. An $m \times n$ real matrix $\mathbf{S} = (s_{ij})$ is called *column-stochastic* if $s_{ij} \ge 0$ for $i = 1, \dots, m$, $j = 1, \dots, n$, and all column sums of $\mathbf{S}$ are equal to 1; that is, $\sum_{i=1}^{m} s_{ij} = 1$ for $j = 1, \dots, n$. An $m \times n$ real matrix $\mathbf{S} = (s_{ij})$ is called *row-stochastic* if $s_{ij} \ge 0$ for $i = 1, \dots, m$, $j = 1, \dots, n$, and all row sums of $\mathbf{S}$ are

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equal to 1; that is, $\sum_{j=1}^{n} s_{ij} = 1$ for $i = 1, \ldots, m$. An $m \times m$ real matrix $\mathbf{S} = (s_{ij})$ is called *doubly stochastic* if $s_{ij} \geq 0$ for $i, j = 1, \ldots, m$, and all column and row sums of $\mathbf{S}$ are equal to 1; that is, $\sum_{i=1}^{m} s_{ij} = 1 = \sum_{j=1}^{m} s_{ij}$ for $i, j = 1, \ldots, m$ (see [8, pp. 29–30]).

It is well known (see [8, p. 33]) that, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$,

$$\mathbf{y} \prec \mathbf{x} \qquad \text{if and only if } \mathbf{y} = \mathbf{x}\mathbf{S} \tag{1.1}$$

for some doubly stochastic $m \times m$ matrix $\mathbf{S}$. A function $F : J^m \to \mathbb{R}$ with an interval $J \subset \mathbb{R}$ is said to be *Schur-convex* on J^m if, for $\mathbf{x}, \mathbf{y} \in J^m$,

$$\mathbf{y} \prec \mathbf{x} \qquad \text{implies that } F(\mathbf{y}) \leq F(\mathbf{x}).$$

(See [8, pp. 79–154] for applications of Schur-convex functions.) The next result is called the *majorization theorem* or the *HLPK theorem* (see [8, pp. 92–93]).

Theorem A ([6, Theorem 108] and Karamata [7, p. 148])**.** *Let $f : J \to \mathbb{R}$ be a real convex function defined on an interval $J \subset \mathbb{R}$. Then for $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in J^m$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in J^m$,*

$$\mathbf{y} \prec \mathbf{x} \qquad \textit{implies that } \sum_{i=1}^{m} f(y_i) \leq \sum_{i=1}^{m} f(x_i). \tag{1.2}$$

A generalization of Theorem A is as follows.

Theorem B ([13, pp. 826–827], [1, p. 93])**.** *Let f be a real convex function defined on an interval $J \subset \mathbb{R}$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}_+^m$, let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$, let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in J^m$, and let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in J^n$. If*

$$\mathbf{y} = \mathbf{x}\mathbf{S} \qquad \textit{and} \qquad \mathbf{a} = \mathbf{b}\mathbf{S}^T \tag{1.3}$$

for some $m \times n$ column-stochastic matrix $\mathbf{S} = (s_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) \leq \sum_{i=1}^{m} a_i f(x_i). \tag{1.4}$$

If f is concave, then the inequality (1.4) is reversed.

Statement (1.4) is referred to as *Sherman's inequality*. (Some applications of Theorem B can be found in [1], [10], and [11].)

As usual, we denote by $\mathbb{B}(H)$ the linear space of all bounded linear operators on a Hilbert space H. For selfadjoint operators $A, B \in \mathbb{B}(H)$, we write $B \leq A$ if $A - B$ is *positive*; that is, $\langle Bh, h\rangle \leq \langle Ah, h\rangle$ for all $h \in H$. In particular, we write $0 \leq A$ if A is *positive*; that is, $0 \leq \langle Ah, h\rangle$ for all $h \in H$.

A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be *positive* in symbol $\Phi \geq 0$ if for self-adjoint operators $A \in \mathcal{A}$,

$$0 \leq A \qquad \text{implies that } 0 \leq \Phi(A).$$

A continuous function $f : J \to \mathbb{R}$ defined on an interval $J \subset \mathbb{R}$ is said to be *operator-convex* if $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$ for any $\lambda \in [0, 1]$ and any self-adjoint operators A, B with spectra in J.

Theorem C ([5, Theorem 2.1, pp. 63–64]). *Let $f : J \to \mathbb{R}$ be an operator-convex function on interval $J \subset \mathbb{R}$. Then the inequality*

$$f\Big(\sum_{i=1}^{m} \Phi_i(A_i)\Big) \leq \sum_{i=1}^{m} \Phi_i f(A_i) \tag{1.5}$$

holds for self-adjoint operators $A_i \in \mathbb{B}(H)$ with spectra in J, $i = 1, \ldots, m$, and positive linear maps $\Phi_i : \mathbb{B}(H) \to \mathbb{B}(K)$, $i = 1, \ldots, m$, such that $\sum_{i=1}^{m} \Phi_i(I_H) = I_K$, where I_H and I_K are the identity maps on Hilbert spaces H and K, respectively.

In this article, we demonstrate a new definition of majorization for C^*-algebras, and we present some results related to Theorems A, B, and C with this new definition. The paper is organized as follows. In Section 2, we collect definitions of right and left premajorizations aimed for comparing two tuples of operators or maps (see [11, pp. 197–198]). In Section 3, we prove operator inequalities similar to (1.2), (1.4), and (1.5) by using the method of premajorization. In Theorem 3.1, we show a general result of the Sherman type. Next, in Section 4, we derive an HLPK result. Section 5 is devoted to recovering a result by Moslehian et al. [9] and the Choi–Davis inequality (see [2], [4]).

2. Right and left premajorizations for C^*-algebras

For a C^*-algebra $\mathcal{A}$ the symbol $\mathcal{A}_{\mathrm{sa}}(J)$ denotes the real space of self-adjoint operators in $\mathcal{A}$ with spectra in a given interval $J \subset \mathbb{R}$.

It is useful here to present some relevant definitions, as follow.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras with unities $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$, respectively. Let $\mathbf{S} = (S_{ij})$ be an $m \times n$ matrix with linear maps $S_{ij} : \mathcal{A} \to \mathcal{B}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. We say that the $m \times n$ matrix $\mathbf{S} = (S_{ij})$ is *column-stochastic* if $S_{ij} \geq 0$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $\sum_{i=1}^{m} S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{B}}$ for $j = 1, \ldots, n$. We say that the matrix $\mathbf{S} = (S_{ij})$ is *row-stochastic* if $S_{ij} \geq 0$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $\sum_{j=1}^{n} S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{B}}$ for $i = 1, \ldots, m$.

In the case $\mathcal{A} = \mathcal{B}$ and $m = n$, we say that the $m \times m$ matrix $\mathbf{S} = (S_{ij})$ is *strongly row-stochastic* if $S_{ij} \geq 0$ for $i, j = 1, \ldots, m$, and $\sum_{j=1}^{m} S_{ij} = \mathrm{id}_{\mathcal{A}}$ for $i = 1, \ldots, m$, where $\mathrm{id}_{\mathcal{A}}$ is the identity map on $\mathcal{A}$. In the case $\mathcal{A} = \mathcal{B}$ and $m = n$, we say that the $m \times m$ matrix $\mathbf{S} = (S_{ij})$ is called *strongly doubly stochastic* if $\mathbf{S}$ is column-stochastic and strongly row-stochastic; that is, $S_{ij} \geq 0$ for $i, j = 1, \ldots, m$, and $\sum_{i=1}^{m} S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{A}}$ for $j = 1, \ldots, m$, and $\sum_{j=1}^{m} S_{ij} = \mathrm{id}_{\mathcal{A}}$ for $i = 1, \ldots, m$.

An n-tuple $\mathbf{B} = (B_1, B_2, \ldots, B_n)$ with operators $B_j \in \mathcal{B}_{\mathrm{sa}}(J)$, $j = 1, \ldots, n$, is said to be *right-premajorized* by an m-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_m)$ with operators $A_i \in \mathcal{A}_{\mathrm{sa}}(J)$, $i = 1, \ldots, m$, written as $\mathbf{B} \prec_{\mathrm{rp}} \mathbf{A}$, if there exists an $m \times n$ column-stochastic matrix $\mathbf{S} = (S_{ij})$ such that

$$(B_1, B_2, \ldots, B_n) = (A_1, A_2, \ldots, A_m)\mathbf{S}, \tag{2.1}$$

where the notation (2.1) means that

$$B_j = S_{1j}(A_1) + S_{2j}(A_2) + \cdots + S_{mj}(A_m) \quad \text{for } j = 1, \ldots, n \tag{2.2}$$

with the right action of S_{ij} on A_i (see [11], [3], [12]). Instead of (2.1), we also write

$$(B_1, B_2, \dots, B_n) \prec_{\mathrm{rp}} (A_1, A_2, \dots, A_m) \quad \text{by } \mathbf{S}.$$

Example 2.1. Let $\mathcal{A} = \mathbb{M}_k(\mathbb{R})$ be the unital C^*-algebra of $k \times k$ real matrices, and let $\mathcal{B} = \mathbb{R}$. Take $m = n = 2$. Let $c_{ij} \in \mathbb{R}^k$, $i, j = 1, 2$, be unit vectors; that is, $\|c_{ij}\|^2 = \langle c_{ij}, c_{ij} \rangle = 1$. We define

$$\mathbf{S} = \frac{1}{2} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where $S_{ij} : \mathbb{M}_k(\mathbb{R}) \to \mathbb{R}$ is given by $S_{ij}(A) = \langle A c_{ij}, c_{ij} \rangle$ for $A \in \mathbb{M}_k(\mathbb{R})$, $i, j = 1, 2$. Clearly, S_{ij} are unital positive maps, and therefore $\mathbf{S}$ is column-stochastic.

Let $A_1, A_2 \in \mathbb{M}_k(\mathbb{R})$ be symmetric matrices. Then

$$(B_1, B_2) \prec_{\mathrm{rp}} (A_1, A_2) \quad \text{by } \mathbf{S};$$

that is,

$$(B_1, B_2) = (A_1, A_2)\mathbf{S},$$

where

$$B_1 = \frac{1}{2} S_{11}(A_1) + \frac{1}{2} S_{21}(A_2) = \frac{1}{2} \langle A_1 c_{11}, c_{11} \rangle + \frac{1}{2} \langle A_2 c_{21}, c_{21} \rangle$$

and

$$B_2 = \frac{1}{2} S_{12}(A_1) + \frac{1}{2} S_{22}(A_2) = \frac{1}{2} \langle A_1 c_{12}, c_{12} \rangle + \frac{1}{2} \langle A_2 c_{22}, c_{22} \rangle.$$

We return to definitions. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be unital C^*-algebras. We denote by $P(\mathcal{A}, \mathcal{C})$ the set of all positive linear maps from $\mathcal{A}$ to $\mathcal{C}$. We write $P(\mathcal{A})$ in place of $P(\mathcal{A}, \mathcal{A})$.

An m-tuple $\mathbf{\Phi} = (\Phi_1, \Phi_2, \dots, \Phi_m)$ with $\Phi_i \in P(\mathcal{A}, \mathcal{C})$, $i = 1, \dots, m$, is said to be *left-premajorized* by an n-tuple $\mathbf{\Psi} = (\Psi_1, \Psi_2, \dots, \Psi_n)$ with $\Psi_j \in P(\mathcal{B}, \mathcal{C})$, $j = 1, \dots, n$, written as $\mathbf{\Phi} \prec_{\mathrm{lp}} \mathbf{\Psi}$ if there exists an $n \times m$ row-stochastic matrix $\mathbf{R} = (R_{ji})$ with positive linear maps $R_{ji} : \mathcal{A} \to \mathcal{B}$ for $j = 1, \dots, n$, $i = 1, \dots, m$ such that

$$(\Phi_1, \Phi_2, \dots, \Phi_m) = (\Psi_1, \Psi_2, \dots, \Psi_n)\mathbf{R} \tag{2.3}$$

(see [11], [3], [12]). Here and in the remainder of this article, the notation (2.3) means that

$$\Phi_i = \Psi_1 R_{1i} + \Psi_2 R_{2i} + \cdots + \Psi_n R_{ni} \quad \text{for } i = 1, \dots, m, \tag{2.4}$$

where $\Psi_j R_{ji} = \Psi_j \circ R_{ji}$ denotes the composition of the maps R_{ji} and Ψ_j and is the left action of R_{ji} on Ψ_j. In place of (2.3), we also write

$$(\Phi_1, \Phi_2, \dots, \Phi_m) \prec_{\mathrm{lp}} (\Psi_1, \Psi_2, \dots, \Psi_n) \quad \text{by } \mathbf{R}.$$

Example 2.2. Let $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathbb{M}_k(\mathbb{R})$ be the unital C^*-algebra of $k \times k$ real matrices with even $k = 2l$. We put $m = n = 2$ and

$$\mathbf{R} = \frac{1}{2} \begin{pmatrix} P & Q \\ Q & P \end{pmatrix},$$

where $P : \mathbb{M}_k(\mathbb{R}) \to \mathbb{M}_k(\mathbb{R})$ is the orthoprojector from $\mathbb{M}_k(\mathbb{R})$ onto $\mathbb{M}_l(\mathbb{R}) \oplus \mathbb{M}_l(\mathbb{R})$ and $Q : \mathbb{M}_k(\mathbb{R}) \to \mathbb{M}_k(\mathbb{R})$ is the orthoprojector from $\mathbb{M}_k(\mathbb{R})$ onto the space $\mathbb{D}_k(\mathbb{R})$

of $k \times k$ real diagonal matrices. Clearly, P and Q are unital positive linear maps, and $\mathbf{R}$ is column-stochastic. Moreover, $P^2 = P$, $Q^2 = Q$, and $PQ = Q = QP$. Therefore,

$$\Big(\frac{P+Q}{2}, Q\Big) = (P, Q)\mathbf{R},$$

which means that

$$\Big(\frac{P+Q}{2}, Q\Big) \prec_{\mathrm{lp}} (P, Q) \quad \text{by } \mathbf{R}.$$

3. Sherman-type inequalities

We begin our discussion of Sherman-type inequalities with the following result.

Theorem 3.1. *Let $f : J \to \mathbb{R}$ be an operator-convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be unital C^*-algebras. Suppose that A_i and B_j are self-adjoint operators in $\mathcal{A}_{\mathrm{sa}}(J)$ and $\mathcal{B}_{\mathrm{sa}}(J)$, respectively, for $i = 1, \dots, m$, $j = 1, \dots, n$. Suppose that $\Phi_i : \mathcal{A} \to \mathcal{C}$ and $\Psi_j : \mathcal{B} \to \mathcal{C}$ are positive linear maps for $i = 1, \dots, m$, $j = 1, \dots, n$. Assume also that*

$$(B_1, B_2, \dots, B_n) \prec_{\mathrm{rp}} (A_1, A_2, \dots, A_m) \quad \textit{by } \mathbf{S}, \tag{3.1}$$

and that

$$(\Phi_1, \Phi_2, \dots, \Phi_m) \prec_{\mathrm{lp}} (\Psi_1, \Psi_2, \dots, \Psi_n) \quad \textit{by } \mathbf{S}^T \tag{3.2}$$

for some $m \times n$ column-stochastic matrix $\mathbf{S} = (S_{ij})$ with positive linear maps $S_{ij} : \mathcal{A} \to \mathcal{B}$. Then the following Sherman-type inequality holds:

$$\sum_{j=1}^{n} \Psi_j\big(f(B_j)\big) \le \sum_{i=1}^{m} \Phi_i\big(f(A_i)\big). \tag{3.3}$$

Proof. From (3.1) and (2.2), we have

$$B_j = \sum_{i=1}^{m} S_{ij}(A_i) \quad \text{for } j = 1, \dots, n. \tag{3.4}$$

Likewise, by virtue of (3.2) and (2.4), we get

$$\Phi_i = \sum_{j=1}^{n} \Psi_j S_{ij} \quad \text{for } i = 1, \dots, m. \tag{3.5}$$

Remember that f is operator-convex, and that $S_{ij} \ge 0$ for $i = 1, \dots, m$, $j = 1, \dots, n$, and $\sum_{i=1}^{m} S_{ij}(I_{\mathcal{A}}) = I_{\mathcal{B}}$ for $j = 1, \dots, n$, where $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are unities of the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Then it follows from (3.4) and Theorem C (by Jensen's operator inequality) that

$$f(B_j) = f\Big(\sum_{i=1}^{m} S_{ij}(A_i)\Big) \le \sum_{i=1}^{m} S_{ij}\big(f(A_i)\big) \quad \text{for } j = 1, \dots, n.$$

Simultaneously, Ψ_j, $j = 1, \dots, n$, are positive linear maps. Therefore, we have

$$\Psi_j\big(f(B_j)\big) \le \Psi_j\Big(\sum_{i=1}^{m} S_{ij}\big(f(A_i)\big)\Big) = \sum_{i=1}^{m} \Psi_j S_{ij}\big(f(A_i)\big) \quad \text{for } j = 1, \dots, n.$$

For this reason, we obtain

$$\sum_{j=1}^{n} \Psi_j\big(f(B_j)\big) \le \sum_{j=1}^{n}\Big(\sum_{i=1}^{m} \Psi_j S_{ij}\big(f(A_i)\big)\Big) = \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} \Psi_j S_{ij}\big(f(A_i)\big)\Big)$$
$$= \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} \Psi_j S_{ij}\Big)\big(f(A_i)\big) = \sum_{i=1}^{m} \Phi_i\big(f(A_i)\big),$$

the last equality being a consequence of (3.5). This completes the proof. □

Remark 3.2. For $n = 1$ in Theorem 3.1, we get Theorem C.

Remark 3.3. Premajorization relations (3.1)–(3.2) in Theorem 3.1 are the operator-map counterpart of the *weighted majorization* (1.3) adequate in the scalar context (see Theorem B).

We demonstrate a specialization of Theorem 3.1 with $\mathcal{B} = \mathcal{C}$. Such a result gives a motivation for the definition of a strongly row-stochastic (operator) matrix (see the beginning of Section 2).

Corollary 3.4. *Let $f : J \to \mathbb{R}$ be an operator-convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Suppose that A_i and B_j are self-adjoint operators in $\mathcal{A}_{\mathrm{sa}}(J)$ and $\mathcal{B}_{\mathrm{sa}}(J)$, respectively, with spectra in J for $i = 1, \dots, m$, $j = 1, \dots, n$. Assume that*

$$(B_1, B_2, \dots, B_n) \prec_{\mathrm{rp}} (A_1, A_2, \dots, A_m) \quad \textit{by } \mathbf{S} \tag{3.6}$$

for some $m \times n$ column-stochastic matrix $\mathbf{S} = (S_{ij})$ with positive linear maps $S_{ij} : \mathcal{A} \to \mathcal{B}$. Then the following Sherman-type inequality holds:

$$\sum_{j=1}^{n} f(B_j) \le \sum_{i=1}^{m} \Phi_i\big(f(A_i)\big), \tag{3.7}$$

where $\Phi_i = \sum_{j=1}^{n} S_{ij}$ is the ith row sum of the matrix $\mathbf{S}$.

Proof. We define $\mathcal{C} = \mathcal{B}$. We introduce positive linear maps $\Phi_i : \mathcal{A} \to \mathcal{B}$ and $\Psi_j : \mathcal{B} \to \mathcal{B}$ for $i = 1, \dots, m$, $j = 1, \dots, n$ such that

$$\Phi_i = \sum_{j=1}^{n} S_{ij} \qquad \text{and} \qquad \Psi_j = \mathrm{id}_{\mathcal{B}}.$$

Thus Φ_i is the ith row sum of $\mathbf{S}$. It is easily seen that

$$(\Phi_1, \Phi_2, \dots, \Phi_m) \prec_{\mathrm{lp}} (\Psi_1, \Psi_2, \dots, \Psi_n) \quad \text{by } \mathbf{S}^T. \tag{3.8}$$

On account of (3.6) and (3.8), we are allowed to use Theorem 3.1. Therefore, inequality (3.7) is a direct consequence of (3.3). This finishes the proof. □

4. HLPK-TYPE INEQUALITY

In the case $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and $m = n$, where $\mathcal{A}$ is an unital C^*-algebra with unity $I_{\mathcal{A}}$, we now give definitions. We say that an operator m-tuple $\mathbf{B} = (B_1, B_2, \ldots, B_m)$ with $B_j \in \mathcal{A}_{\mathrm{sa}}(J)$, $j = 1, \ldots, m$, is *strongly majorized* by an operator m-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_m)$ with $A_i \in \mathcal{A}_{\mathrm{sa}}(J)$, $i = 1, \ldots, m$, written as $\mathbf{B} \prec_{\mathrm{str}} \mathbf{A}$ if there exists an $m \times m$ strongly doubly stochastic matrix $\mathbf{S} = (S_{ij})$ such that

$$(B_1, B_2, \ldots, B_m) = (A_1, A_2, \ldots, A_m)\mathbf{S} \tag{4.1}$$

in the sense that

$$B_j = S_{1j}(A_1) + S_{2j}(A_2) + \cdots + S_{mj}(A_m) \quad \text{for } j = 1, \ldots, m$$

with the right action of S_{ij} on A_i (see [11], [3], [12]).

Instead of (4.1), we also write

$$(B_1, B_2, \ldots, B_m) \prec_{\mathrm{str}} (A_1, A_2, \ldots, A_m) \quad \text{by } \mathbf{S}. \tag{4.2}$$

It is interesting that (4.2) implies that

$$B_1 + B_2 + \cdots + B_m = A_1 + A_2 + \cdots + A_m.$$

Indeed, we have

$$\begin{aligned} \sum_{j=1}^m B_j &= \sum_{j=1}^m \sum_{i=1}^m S_{ij}(A_i) = \sum_{i=1}^m \sum_{j=1}^m S_{ij}(A_i) \\ &= \sum_{i=1}^m \Big(\sum_{j=1}^m S_{ij} \Big)(A_i) = \sum_{i=1}^m \mathrm{id}_{\mathcal{A}}(A_i) = \sum_{i=1}^m A_i. \end{aligned}$$

We apply Corollary 3.4 to prove the following majorization theorem for C^*-algebras. An alternative proof of Theorem 4.1 can be done via the operator convexity and the above definition of strong majorization.

Theorem 4.1. *Let $f : J \to \mathbb{R}$ be an operator-convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ be a unital C^*-algebra. Suppose that A_i and B_j are self-adjoint operators in $\mathcal{A}_{\mathrm{sa}}(J)$ with spectra in J for $i, j = 1, \ldots, m$. Assume that*

$$(B_1, B_2, \ldots, B_m) \prec_{\mathrm{str}} (A_1, A_2, \ldots, A_m). \tag{4.3}$$

Then the following HLPK-type inequality holds:

$$\sum_{j=1}^m f(B_j) \le \sum_{i=1}^m f(A_i). \tag{4.4}$$

Proof. We define $n = m$ and $\mathcal{C} = \mathcal{B} = \mathcal{A}$. It follows from (4.3) that

$$(B_1, B_2, \ldots, B_m) \prec_{\mathrm{rp}} (A_1, A_2, \ldots, A_m) \quad \text{by } \mathbf{S}$$

for some $m \times m$ strongly doubly stochastic matrix $\mathbf{S} = (S_{ij})$.

We consider the special maps

$$\Phi_i = \mathrm{id}_{\mathcal{A}} \qquad \text{and} \qquad \Psi_j = \mathrm{id}_{\mathcal{A}} \quad \text{for } i, j = 1, \ldots, m.$$

However, $\mathbf{S}$ is strongly doubly stochastic, and so

$$\mathrm{id}_{\mathcal{A}} = \sum_{j=1}^{m} S_{ij} \quad \text{for } i = 1, \dots, m.$$

In conclusion, we are permitted to utilize Corollary 3.4 with $\Phi_i = \mathrm{id}_{\mathcal{A}}$, $i = 1, \dots, m$. Then, by (3.7), we obtain

$$\sum_{j=1}^{m} f(B_j) \le \sum_{i=1}^{m} \Phi_i\big(f(A_i)\big) = \sum_{i=1}^{m} f(A_i),$$

as required. □

5. Further applications

We continue to study special cases of Theorem 3.1. A specialization of Theorem 3.1 for $m = n = 2$ and $\mathcal{A} = \mathcal{B} = \mathcal{C}$ corresponds in some sense to [9, Theorem 2.1], due to Moslehian et al., as follows.

Corollary 5.1 ([9, Theorem 2.1])**.** *Let $f : J \to \mathbb{R}$ be an operator-convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ be a unital C^*-algebra. Suppose that A_i and B_j are self-adjoint operators in $\mathcal{A}_{\mathrm{sa}}(J)$ with spectra in J for $i, j = 1, 2$. If*

$$(B_1, B_2) \prec_{\mathrm{str}} (A_1, A_2), \tag{5.1}$$

then

$$f\big(\Phi(B_1)\big) + f\big(\Phi(B_2)\big) \le \Phi\big(f(A_1)\big) + \Phi\big(f(A_2)\big), \tag{5.2}$$

where $\Phi : \mathcal{A} \to \mathcal{A}$ is a unital positive linear map.

Proof. We set $\mathcal{C} = \mathcal{B} = \mathcal{A}$ and $n = m = 2$. By virtue of (5.1) there exists a 2×2 strongly doubly stochastic matrix $\mathbf{S} = \left(\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right)$ such that

$$(B_1, B_2) = (A_1, A_2)\mathbf{S}. \tag{5.3}$$

Denote

$$\mathbf{S}_0 = \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}.$$

It follows from (5.3) that

$$\big(\Phi(B_1), \Phi(B_2)\big) = (B_1, B_2)\mathbf{S}_0 = (A_1, A_2)\mathbf{S}\mathbf{S}_0, \tag{5.4}$$

where

$$\mathbf{S}\mathbf{S}_0 = \begin{pmatrix} S_{11}\Phi, & S_{12}\Phi \\ S_{21}\Phi, & S_{22}\Phi \end{pmatrix}$$

is column-stochastic because Φ is unital and $\mathbf{S}$ is strongly doubly (hence column-) stochastic.

In consequence, by (5.4),

$$\big(\Phi(B_1), \Phi(B_2)\big) \prec_{\mathrm{rp}} (A_1, A_2) \quad \text{by } \mathbf{S}\mathbf{S}_0. \tag{5.5}$$

On the other hand, by defining

$$(\Phi_1, \Phi_2) = (\Phi, \Phi) \qquad \text{and} \qquad (\Psi_1, \Psi_2) = (\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}}), \tag{5.6}$$

we obtain

$$(\Phi,\Phi) \prec_{\mathrm{lp}} (\mathrm{id}_{\mathcal{A}},\mathrm{id}_{\mathcal{A}}) \quad \text{by } (\mathbf{SS}_0)^T. \tag{5.7}$$

In fact,

$$(\mathbf{SS}_0)^T = \begin{pmatrix} S_{11}\Phi, & S_{21}\Phi \\ S_{12}\Phi, & S_{22}\Phi \end{pmatrix},$$

and, therefore, for $j=1,2$,

$$\mathrm{id}_{\mathcal{A}} S_{j1}\Phi + \mathrm{id}_{\mathcal{A}} S_{j2}\Phi = S_{j1}\Phi + S_{j2}\Phi = (S_{j1}+S_{j2})\Phi = \mathrm{id}_{\mathcal{A}}\Phi = \Phi,$$

proving (5.7). By making use of (5.5), (5.6), (5.7), and of Theorem 3.1 for $m = n = 2$ and, $\mathcal{A} = \mathcal{B} = \mathcal{C}$, we deduce from (3.3) that

$$f\big(\Phi(B_1)\big) + f\big(\Phi(B_2)\big) \le \Phi f(A_1) + \Phi f(A_2),$$

as claimed. □

Corollary 5.2 (Choi–Davis inequality [2, Theorem 2.1], [4, p. 44])**.** *Let $f : J \to \mathbb{R}$ be an operator-convex function on interval $J \subset \mathbb{R}$. Let $\mathcal{A}$ be a unital C^*-algebra. Suppose that A is a self-adjoint operator in $\mathcal{A}_{\mathrm{sa}}(J)$. Then*

$$f\big(\Phi(A)\big) \le \Phi\big(f(A)\big), \tag{5.8}$$

where $\Phi : \mathcal{A} \to \mathcal{A}$ is a unital positive linear map.

Proof. We define $A_1 = A_2 = A$, $B_1 = B_2 = A$, and $\mathbf{S} = \left(\begin{smallmatrix} \mathrm{id}_{\mathcal{A}} & 0 \\ 0 & \mathrm{id}_{\mathcal{A}} \end{smallmatrix}\right)$. Clearly,

$$(A,A) \prec_{\mathrm{str}} (A,A) \quad \text{by } \mathbf{S},$$

and $\mathbf{S}$ is strongly doubly stochastic. In other words, (5.1) is met. According to Corollary 5.1, from (5.2), we infer that

$$f\big(\Phi(A)\big) + f\big(\Phi(A)\big) \le \Phi\big(f(A)\big) + \Phi\big(f(A)\big),$$

which gives (5.8), as desired. □

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