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ON CERTAIN PROPERTIES OF CUNTZ–KRIEGER-TYPE ALGEBRAS

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Abstract. This note presents a further study of the class of Cuntz–Krieger-type algebras. A necessary and sufficient condition is identified that ensures that the algebra is purely infinite, the ideal structure is studied, and nuclearity is proved by presenting the algebra as a crossed product of an AF-algebra by an abelian group. The results are applied to examples of Cuntz–Krieger-type algebras, such as higher-rank semigraph C^*-algebras and higher-rank Exel–Laca algebras.

1. Introduction

During the last two decades, Cuntz and Cuntz–Krieger algebras, in the form of graph algebras, have been studied intensively. Recent samples include [10] and [9].

Based on the work of Cuntz and Krieger in [8], in [2] the first author considered a class of so-called Cuntz–Krieger-type algebras relying on a flexible generators-and-relations approach. This class, which is recalled in Section 2, includes (aperiodic) Cuntz–Krieger algebras [8], higher-rank Exel–Laca algebras (see [3]), (aperiodic) higher-rank graph C^*-algebras (see [11], [12]), (aperiodic) ultragraph algebras (see [17]), and (canceling) higher-rank semigraph C^*-algebras (see [4]).

The aim of this article is to analyze these algebras further. Pure infiniteness was introduced by Cuntz in [6, Theorem 1.13] as a fundamental property of his Cuntz

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algebras. In Section 3 we show that a Cuntz–Krieger-type algebra is purely infinite if and only if the projections of its core are infinite (see Theorem 3.2). Applications to higher-rank semigraph C^*-algebras and higher-rank Exel–Laca algebras, stated in Corollaries 3.3 and 3.4, respectively, give quite tractable conditions for checking when those algebras are purely infinite. In Section 4 we study the ideal structure of Cuntz–Krieger-type algebras. The ideal structure for Cuntz–Krieger algebras was first studied by Cuntz in [7]. There is an injection of certain ideals of the core to the ideals of the Cuntz–Krieger-type algebra (see Theorem 4.6). If these certain ideals are all canceling (Definitions 4.8 and 4.11), then this injection is even a lattice isomorphism (see Theorem 4.9, Corollary 4.10, Theorem 4.12, and Corollary 4.13). We give reformulations of such an isomorphism especially for higher-rank semigraph algebras in Corollaries 4.14 and 4.15. In Section 5 we present the stabilized Cuntz–Krieger-type algebras as crossed products of AF-algebras by abelian groups (see Theorem 5.1). This uses Takai's duality and gauge actions (see [16, Theorem 3.4]). Hence Cuntz–Krieger-type algebras are nuclear.

2. Cuntz–Krieger-type algebras

We briefly recall the basic definitions and facts of the class of Cuntz–Krieger-type algebras introduced in [2] and slightly extended in [5]. Assume that we are given an alphabet $\mathcal{A}$, the free nonunital $*$-algebra $\mathbb{F}$ generated by $\mathcal{A}$, a two-sided self-adjoint ideal $\mathbb{I}$ of $\mathbb{F}$, and a closed subgroup H of $\mathbb{T}^{\mathcal{A}}$ ($\mathbb{T}$ denotes the circle). We are interested in the quotient $*$-algebra $\mathbb{F}/\mathbb{I}$ and its universal C^*-algebra $C^*(\mathbb{F}/\mathbb{I})$. Denote the set of words of $\mathbb{F}/\mathbb{I}$ by $W = \{a_1 \cdots a_n \in \mathbb{F}/\mathbb{I} | a_i \in \mathcal{A} \cup \mathcal{A}^*\}$. (We will always write x rather than $x + \mathbb{I}$ in the quotient $\mathbb{F}/\mathbb{I}$ for elements $x \in \mathbb{F}$ if there is no danger of confusion.) An element x of a $*$-algebra is called a *partial isometry* if $xx^*x = x$, and it is called a *projection* if $x^2 = x^* = x$.

We are going to introduce the following properties (A), (B), and (C′) for the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$.

(A) There exists a gauge action $t : H \longrightarrow \operatorname{Aut}(\mathbb{F}/\mathbb{I})$ determined by $t_\lambda(a) = \lambda_a a$ for all $a \in \mathcal{A}$ and $\lambda = (\lambda_b)_{b\in\mathcal{A}} \in H$.

Denote by $(\hat{H}, +, 0)$ the character group of $(H, \cdot, 1)$; note that we write the group operation of $\hat{H}$ additively. The gauge action t induces a so-called balance function $\operatorname{bal} : W\backslash\{0\} \longrightarrow \hat{H}$ from the nonzero words of $\mathbb{F}/\mathbb{I}$ to the character group $\hat{H}$ determined by $\operatorname{bal}(a)((\lambda_b)_{b\in\mathcal{A}}) = \lambda_a \in \mathbb{T}$, $\operatorname{bal}(xy) = \operatorname{bal}(x) + \operatorname{bal}(y)$, and $\operatorname{bal}(x^*) = -\operatorname{bal}(x)$, where $a \in \mathcal{A}$, $(\lambda_b)_{b\in\mathcal{A}} \in H \subseteq \mathbb{T}^{\mathcal{A}}$, and $x, y \in W$ (see [2, Lemma 3.1]).

Define $\mathbb{A}$ to be the linear span in $\mathbb{F}/\mathbb{I}$ of all words $x \in W\backslash\{0\}$ satisfying $\operatorname{bal}(x) = 0$. Actually, $\mathbb{A}$ is a $*$-algebra. Words x with balance $\operatorname{bal}(x) = 0$ are called *zero-balanced*. Write W_n for the set of words with balance $n \in \hat{H}$. Since every element of $\mathbb{F}/\mathbb{I}$ is expressible as a linear combination of words, we may write $\mathbb{F}/\mathbb{I} = \sum_{n\in\hat{H}} \operatorname{lin}(W_n)$. Note, however, that this sum might not be a direct sum.

(B) $\mathbb{A}$ is locally matricial; that is, for all $x_1, \ldots, x_n \in \mathbb{A}$, there exists a finite-dimensional C^*-subalgebra A of $\mathbb{A}$ such that $x_1, \ldots, x_n \in A$.

(C′) For every nonzero-balanced word $x \in W\backslash W_0$ and every nonzero projection $e \in \mathbb{A}$ there exists a nonzero projection $p \leq e$ in $\mathbb{A}$ such that $pxp = 0$.

Definition 2.1. A system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ is called a *Cuntz–Krieger-type system*, or $\mathbb{F}/\mathbb{I}$ is called a *Cuntz–Krieger-type $*$-algebra*, if (A), (B), and (C′) are satisfied and there exists a C^*-representation $\pi : \mathbb{F}/\mathbb{I} \longrightarrow A$ which is injective on $\mathbb{A}$.

Throughout this paper, assume that $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ is a Cuntz–Krieger-type system if nothing else is specified. There exists a universal enveloping C^*-algebra $C^*(\mathbb{F}/\mathbb{I})$ for $\mathbb{F}/\mathbb{I}$, and clearly the universal representation $\zeta : \mathbb{F}/\mathbb{I} \longrightarrow C^*(\mathbb{F}/\mathbb{I})$ is injective on $\mathbb{A}$. The enveloping C^*-algebra $C^*(\mathbb{F}/\mathbb{I})$ is called the *Cuntz–Krieger-type algebra associated to* $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$. A $*$-homomorphism $\mathbb{F}/\mathbb{I} \to A$ into a C^*-algebra A is called a *C^*-representation of* $\mathbb{F}/\mathbb{I}$, and it is called $\mathbb{A}$-*faithful* if it is faithful on $\mathbb{A}$. We note that for a system $(\mathcal{A}, H, \mathbb{F}, H)$ satisfying (A), (B), and (C′), an $\mathbb{A}$-faithful representation of $\mathbb{F}/\mathbb{I}$ into a C^*-algebra exists automatically if the word set W consists of partial isometries (see [5, Theorem 3.1]).

We have the following Cuntz–Krieger uniqueness theorem.

Theorem 2.2. *If $\pi : \mathbb{F}/\mathbb{I} \longrightarrow A$ is an $\mathbb{A}$-faithful representation into a C^*-algebra A with a dense image in A, then A is canonically isomorphic to $C^*(\mathbb{F}/\mathbb{I})$ via $\pi(x) \mapsto \zeta(x)$, and so π is essentially the universal map ζ (see [2, Theorem 3.3] and Theorem 2.1 and Corollary 1 of Section 3 of [5]).*

The next lemma states that we usually may assume without loss of generality that ζ is injective. We then usually avoid notating ζ and regard $\mathbb{F}/\mathbb{I}$ as a subset of $C^*(\mathbb{F}/\mathbb{I})$.

Lemma 2.3. *We may assume without loss of generality that the universal representation $\zeta : \mathbb{F}/\mathbb{I} \longrightarrow C^*(\mathbb{F}/\mathbb{I})$ is injective by dividing out the kernel of ζ. The new quotient $\mathbb{F}/\mathbb{I}$ is a Cuntz–Krieger $*$-algebra again ($\mathcal{A}, \mathbb{F}$, and H remain unchanged). $\mathbb{A}$ remains unchanged under this modification.*

In a previous preprint of this paper, we proved the last lemma and the next lemma. However, we have reproved and published them already now in [5, Propositions 2 and 4]. The setting in [5] generalizes the setting of this note by allowing the image of the balance function, here the commutative group $\hat{H}$, to be a noncommutative group. We say that a $*$-algebra X satisfies the C^*-property if, for every $x \in X$, $xx^* = 0$ implies that $x = 0$.

Lemma 2.4. *Representation ζ is injective if and only if $\mathbb{F}/\mathbb{I}$ satisfies the C^*-property. The kernel of ζ is the ideal generated by $\{x \in \mathbb{F}/\mathbb{I} | xx^* = 0\}$.*

Lemma 2.5. *There exists a conditional expectation $F : C^*(\mathbb{F}/\mathbb{I}) \longrightarrow C^*(\mathbb{A}) \subseteq C^*(\mathbb{F}/\mathbb{I})$ determined by $F(\zeta(w)) = 1_{\{\mathrm{bal}(w)=0\}}\zeta(w)$ for words $w \in W$ (see [5, Proposition 2]).*

3. Pure infiniteness

In this section we analyze the pure infiniteness of a Cuntz–Krieger-type algebra $C^*(\mathbb{F}/\mathbb{I})$. We consider a C^*-algebra A to be purely infinite if every nonzero hereditary sub-C^*-algebra of A contains an infinite projection. (This condition is, for instance, stated in [15, Proposition 4.1.1.(v)] and is also used in [14].)

Recall that a projection p in a C^*-algebra A is considered infinite if it is the source projection s^*s of a partial isometry s in A, with range projection ss^* being smaller than p. Recall the following simple lemma.

Lemma 3.1. *If a projection is infinite, then any other projection which is bigger in the Murray–von Neumann order is also infinite.*

Theorem 3.2. *A Cuntz–Krieger-type algebra $C^*(\mathbb{F}/\mathbb{I})$ is purely infinite if and only if every nonzero projection of $\mathbb{A}$ is infinite in $C^*(\mathbb{F}/\mathbb{I})$.*

Proof. We assume that ζ is injective (see Lemma 2.3). Define $A = C^*(\mathbb{F}/\mathbb{I})$. Assume that A is purely infinite. Then for any nonzero projection $e \in \mathbb{A}$, the hereditary C^*-algebra eAe contains some infinite projection p. Since $p \leq e$, it holds that e is infinite in A by Lemma 3.1.

To prove the other direction, assume that every nonzero projection in $\mathbb{A}$ is infinite in A. It is proved in Lemma 1 of [5] that there exists a larger Cuntz–Krieger-type system $S = (\mathcal{A}\times\mathcal{P}, \mathbb{G}, \mathbb{J}, H\times\{1\})$ than $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ such that $\mathbb{G}/\mathbb{J} \cong \mathbb{F}/\mathbb{I}\otimes\mathbb{F}'/\mathbb{I}'$, where $\mathbb{F}'/\mathbb{I}'$ is a commutative unital locally matricial algebra, and the system S satisfies property (C) of [2]. This property is a sharpening of (C′) and states that, for every nonzero-balanced word $x \in W\backslash W_0$ and all nonzero projections $e, e_1, e_2 \in \mathbb{A}$, there exist nonzero projections $p \leq e, p_1 \leq e_1, p_2 \leq e_2$ in $\mathbb{A}$ such that $pxp = 0$ and $p_1xp_2 = 0$. If we can show that $C^*(\mathbb{G}/\mathbb{J}) \cong C^*(\mathbb{F}/\mathbb{I}) \otimes C^*(\mathbb{F}'/\mathbb{I}')$ is purely infinite, then it is not difficult to check that $C^*(\mathbb{F}/\mathbb{I})$ is also purely infinite. (The following fact holds in general: If $A \otimes D$ is purely infinite for two C^*-algebras A and D where D is unital and commutative, then A is purely infinite.)

That is why we may assume without loss of generality in what follows that the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ satisfies property (C) of [2]. To show that $A = C^*(\mathbb{F}/\mathbb{I})$ is purely infinite, we imitate the proof of [14, Proposition 5.11]. Let h be a nonzero positive element of A. We have to show that $\overline{hAh}$ contains an infinite projection. Let $\varepsilon > 0$, and choose $y \geq 0$ in $\mathbb{F}/\mathbb{I}$ such that $\|y - h^2\| \leq \varepsilon$.

By [2, Lemma 2.6] (applied to $\pi = \zeta$), we are provided with a faithful expectation $F : A \to C^*(\mathbb{A})$ such that, for every representation $y = \sum_{\gamma\in\hat{H}} y_\gamma$ (where $y_\gamma \in \mathrm{lin}(W_\gamma)$), there exists a projection $Q \in \mathbb{A}$ satisfying $QyQ = Qy_1Q \in \mathbb{A}$ and $\|Fy\| = \|QyQ\|$.

We may assume without loss of generality that $\|Fh^2\| = 1$. We have

$$\|Fy\| \geq \|Fh^2\| - \varepsilon = 1 - \varepsilon.$$

Let $QyQ \in \mathcal{M}$ for some finite-dimensional C^*-algebra $\mathcal{M} \subseteq \mathbb{A}$. We choose a system of generating matrix units for $\mathcal{M}$ such that the positive element QyQ has diagonal form in $\mathcal{M} = M_{k_1} \oplus \cdots \oplus M_{k_d}$. By projecting on the largest diagonal entry, we can choose a positive operator $R_1 \in \mathcal{M}$ such that $P = R_1QyQR_1$ is a projection and $\|R_1\| \leq (1-\varepsilon)^{-1/2}$. By hypothesis, $P \in \mathbb{A}$ is an infinite projection.

It follows that $\|R_1Qh^2QR_1 - P\| \leq \|R_1^2\|\|Q\|^2\|y-h^2\| \leq \varepsilon/(1-\varepsilon)$. By functional calculus, one obtains $R_2 \in A_+$ so that $R_2R_1Qh^2QR_1R_2$ is a projection and

$$\|R_2R_1Qh^2QR_1R_2 - P\| \leq 2\varepsilon/(1 - \varepsilon).$$

For small ε one can then find an element R_3 in A such that

$$R_3 R_2 R_1 Q h^2 Q R_1 R_2 R_3^* = P.$$

Let $R = R_3 R_2 R_1 Q$ so that $Rh^2 R^* = P$. Consequently, Rh is a partial isometry, whose initial projection hR^*Rh is a projection in hAh and whose final projection is P. Moreover, if V is a partial isometry in A such that $V^*V = P$ and $VV^* < P$, then $(hR^*)V(Rh)$ is a partial isometry in hAh with initial projection hR^*Rh and final projection strictly less than hR^*Rh. □

We shall now apply the last theorem to canceling higher-rank semigraph algebras [4], which are special Cuntz–Krieger-type $*$-algebras.

Corollary 3.3. *A canceling semigraph C^*-algebra $C^*(\mathbb{F}/\mathbb{I})$ (see [4, Definitions 5.1 and 7.2]) is purely infinite if and only if every standard projection (see [4, Definition 5.14]) is infinite in $C^*(\mathbb{F}/\mathbb{I})$.*

Proof. Canceling semigraph algebras are algebras of amenable Cuntz–Krieger systems (see [5]) (this follows from the discussion in [4, Section 7]), which again are Cuntz–Krieger-type $*$-algebras (since the image of the balance map, $\hat{H}$, is an abelian group), and so we can apply Theorem 3.2. We just need to recall that by [4, Corollary 6.4] every nonzero projection in $\mathbb{A}$ is larger or equal than a standard projection in the Murray–von Neumann order, and so is infinite by Lemma 3.1 if every standard projection is infinite. □

The next corollary concerns higher-rank Exel–Laca algebras (see [3]), which are special Cuntz–Krieger-type algebras.

Corollary 3.4. *Let $C^*(\mathbb{F}/\mathbb{I})$ be a higher-rank Exel–Laca algebra (see [3]). Then $C^*(\mathbb{F}/\mathbb{I})$ is purely infinite if and only if every nonzero projection of the form $P_{a_1} \cdots P_{a_n}$ ($a_i \in \mathcal{A}$, $P_a = aa^*$) is infinite in $C^*(\mathbb{F}/\mathbb{I})$.*

Proof. By [3, Corollary 4.14] and [3, Lemma 4.5] every projection $p \in \mathbb{A}$ allows the following estimate in the Murray–von Neumann order:

$$p \precsim xx^* \precsim x^*x = Q_{a_1} \cdots Q_{a_n} \geq P_{b_1} \cdots P_{b_n} \neq 0$$

for some word x in the letters of the alphabet $\mathcal{A}$ and some letters $a_i, b_i \in \mathcal{A}$. Hence the claim follows from Lemma 3.1 and Theorem 3.2. □

4. Ideal structure

In this section we investigate the ideal structure of a Cuntz–Krieger-type algebra $C^*(\mathbb{F}/\mathbb{I})$. We assume that ζ is injective (Lemma 2.3).

Write Σ for the set of two-sided self-adjoint ideals in $\mathbb{F}/\mathbb{I}$. Denote by $\mathcal{I}$ the set of closed two-sided ideals in $C^*(\mathbb{F}/\mathbb{I})$. Suppose that $\mathbb{B}$ is a $*$-subalgebra of $\mathbb{A}$. Write $\Sigma^{\mathbb{B}}$ for the set of self-adjoint two-sided ideals in $\mathbb{B}$. Define

$$\Sigma_{\mathbb{B}} = \{J \cap \mathbb{B} \in \Sigma^{\mathbb{B}} \mid J \in \Sigma\}.$$

For a subset X of $\mathbb{F}/\mathbb{I}$, define $\Sigma(X) \in \Sigma$ to be the two-sided self-adjoint ideal in $\mathbb{F}/\mathbb{I}$ generated by X, and define $\mathcal{I}(X) \in \mathcal{I}$ to be the closed two-sided ideal in $C^*(\mathbb{F}/\mathbb{I})$ generated by X. Denote by $q_X : \mathbb{F}/\mathbb{I} \longrightarrow (\mathbb{F}/\mathbb{I})/\Sigma(X)$ the quotient map.

Lemma 4.1. *For all $J \in \Sigma$, we have $J \cap \mathbb{B} = (\Sigma(J \cap \mathbb{B})) \cap \mathbb{B}$.*

Proof. $J \cap \mathbb{B} \subseteq J \cap \mathbb{B} \cap \mathbb{B} \subseteq (\Sigma(J \cap \mathbb{B})) \cap \mathbb{B} \subseteq \Sigma(J) \cap \mathbb{B} = J \cap \mathbb{B}$. □

Lemma 4.2. *We have $\Sigma_{\mathbb{B}} = \{J \cap \mathbb{B} \in \Sigma^{\mathbb{B}} \mid J \in \Sigma, J = \Sigma(J \cap \mathbb{B})\}$.*

Proof. Given $J \in \Sigma$, consider $I = \Sigma(J \cap \mathbb{B})$. By Lemma 4.2 we have $I = \Sigma(I \cap \mathbb{B})$ and $J \cap \mathbb{B} = I \cap \mathbb{B}$, which proves the claim. □

Lemma 4.3. *We have $\Sigma_{\mathbb{B}} = \{I \in \Sigma^{\mathbb{B}} \mid \Sigma(I) \cap \mathbb{B} = I\}$.*

Proof. Given $I \in \Sigma_{\mathbb{B}}$, we have $I = J \cap \mathbb{B}$ for some ideal $J \in \Sigma$. By Lemma 4.1 we obtain $\Sigma(I) \cap \mathbb{B} = I$. The reverse implication is obvious. □

Lemma 4.4. *We have*

$$\Sigma_{\mathbb{A}} = \{I \in \Sigma^{\mathbb{A}} \mid \forall x, y \in W : \operatorname{bal}(x) + \operatorname{bal}(y) = 0 \Longrightarrow xIy \subseteq I\}. \tag{4.1}$$

Hence $\Sigma_{\mathbb{A}}$ is closed under the lattice operation $I + J$.

Proof. Write $\mathcal{J}$ for the right-handed set of (4.1). Consider $I \in \Sigma_{\mathbb{A}}$,and write it as $I = J \cap \mathbb{A}$ for some $J \in \Sigma$. If $i \in I$ and $x, y \in W$ with $\operatorname{bal}(x) + \operatorname{bal}(y) = 0$, then $xiy \in \mathbb{A} \cap J$. This shows that $\Sigma_{\mathbb{A}} \subseteq \mathcal{J}$.

To prove $\mathcal{J} \subseteq \Sigma_{\mathbb{A}}$, consider $I \in \mathcal{J}$. Since $I \subseteq \mathbb{A}$, $I \subseteq \Sigma(I) \cap \mathbb{A}$. For the reverse inclusion consider $z \in \Sigma(I) \cap \mathbb{A}$. We may write $z = \sum \alpha_k x_k i_k y_k$ for some scalars $\alpha_k \in \mathbb{C}$, some $i_k \in I$, and some (possibly empty) words $x_k, y_k \in W$. We have $F(z) = z$ for the conditional expectation F of Lemma 2.5 as $z \in \mathbb{A}$. Hence $z = \sum \beta_k x_k i_k y_k$ for some scalars $\beta_k \in \mathbb{C}$ such that $\beta_k = 0$ if $\operatorname{bal}(x_k) + \operatorname{bal}(y_k) \neq 0$. This shows that $z \in I$ as $I \in \mathcal{J}$. We have proved that $I = \Sigma(I) \cap \mathbb{A}$, which is in $\Sigma_{\mathbb{A}}$. □

In the next lemma we state a result of Bratteli [1, Theorem 3.3] now for not necessarily separable AF-algebras. We skip the proof, which just consists of a slight adaption of Bratteli's proof.

Lemma 4.5. *Let A be a locally matricial algebra, and let $\overline{A}$ be its C^*-algebraic norm closure. There is a bijection γ between the family of self-adjoint two-sided ideals in A and the family of closed two-sided ideals in $\overline{A}$ through $\gamma(I) = \overline{I}$ and $\gamma^{-1}(I) = I \cap A$.*

Theorem 4.6. *Every $*$-subalgebra $\mathbb{B}$ of $\mathbb{A}$ induces an injective map $\Phi_{\mathbb{B}} : \Sigma_{\mathbb{B}} \longrightarrow \mathcal{I}$ given by $\Phi_{\mathbb{B}}(I) = \mathcal{I}(I)$ for $I \in \Sigma_{\mathbb{B}}$. The inverse map is determined by $\Phi_{\mathbb{B}}^{-1}(D) = D \cap \mathbb{B}$ for $D \in \mathcal{I}$. For all $I, J \in \Sigma_{\mathbb{B}}$, we have*

$$\Phi_{\mathbb{B}}(I + J) = \Phi_{\mathbb{B}}(I) + \Phi_{\mathbb{B}}(J) \quad \text{if } I + J \in \Sigma_{\mathbb{B}},$$
$$\Phi_{\mathbb{B}}(I \cap J) = \Phi_{\mathbb{B}}(I) \cap \Phi_{\mathbb{B}}(J) \quad \text{if } \Phi_{\mathbb{B}}(I) \cap \Phi_{\mathbb{B}}(J) \in \Phi_{\mathbb{B}}(\Sigma_{\mathbb{B}}).$$

Proof. Step 1. At first we are going to check injectivity of $\Phi_{\mathbb{A}}$. Let $I \in \Sigma_{\mathbb{A}}$, and put $D = \mathcal{I}(I)$. Then $\overline{I} \subseteq \overline{D \cap \mathbb{A}}$ (norm-closures in $C^*(\mathbb{F}/\mathbb{I})$). To prove the reverse inclusion $\overline{D \cap \mathbb{A}} \subseteq \overline{I}$, suppose that $x \in D \cap \mathbb{A}$. Let $\varepsilon > 0$. Since $D = \overline{\Sigma(I)}$, there is some $y \in \Sigma(I)$ such that $\|x - y\| \leq \varepsilon$. Let F be the conditional expectation of Lemma 2.5. Since $Fx = x$, we have

$$\|x - Fy\| = \|Fx - Fy\| \leq \|x - y\| \leq \varepsilon.$$

Choose for y a representation $y = \sum \alpha_i a_i x_i b_i$ for some scalars $\alpha_i \in \mathbb{C}$, some (possibly empty) words $a_i, b_i \in W$, and some elements $x_i \in J$. Since $\mathrm{bal}(x_i) = 0$, either $F(a_i x_i b_i) = a_i x_i b_i$ or $F(a_i x_i b_i) = 0$. Hence $Fy = \sum \beta_i a_i x_i b_i \in \mathbb{A}$ for some scalars $\beta_i \in \mathbb{C}$, and, consequently, $Fy \in \Sigma(I) \cap \mathbb{A} = I$ by Lemma 4.3. Since $\varepsilon > 0$ was arbitrary, $x \in \overline{I}$. We have proved that $\overline{I} = \overline{D \cap \mathbb{A}}$, and so $I = D \cap \mathbb{A}$ by Lemma 4.5. Hence $\Phi_{\mathbb{A}}^{-1}\Phi_{\mathbb{A}}(I) = I$ if we set $\Phi_{\mathbb{A}}^{-1}(D) = D \cap \mathbb{A}$. Hence $\Phi_{\mathbb{A}}$ is injective.

Step 2. In this step we will show that $\Phi_{\mathbb{B}}$ is injective. Define $\mu : \Sigma_{\mathbb{B}} \to \Sigma_{\mathbb{A}}$ by $\mu(I) = \Sigma(I) \cap \mathbb{A}$. The map μ is injective as $\mu^{-1}(J) = J \cap \mathbb{B}$ is an inverse for μ by Lemma 4.3. The identity

$$\Phi_{\mathbb{A}}\big(\mu(I)\big) = \Phi_{\mathbb{A}}\big(\Sigma(I) \cap \mathbb{A}\big) = \overline{\Sigma\big(\Sigma(I) \cap \mathbb{A}\big)} = \overline{\Sigma(I)} = \Phi_{\mathbb{B}}(I)$$

shows that $\Phi_{\mathbb{B}} = \Phi_{\mathbb{A}}\mu$, and so $\Phi_{\mathbb{B}}$ is injective by the proved injectivity of $\Phi_{\mathbb{A}}$. To prove the formula for $\Phi_{\mathbb{B}}^{-1}$, we note that

$$\Phi_{\mathbb{B}}^{-1}(D) = \mu^{-1}\Phi_{\mathbb{A}}^{-1}(D) = (D \cap \mathbb{A}) \cap \mathbb{B} = D \cap \mathbb{B}.$$

Step 3. To prove the lattice rules for $\Phi_{\mathbb{B}}$, we consider $I_1, I_2 \in \Sigma_{\mathbb{B}}$ and set $D_1 = \Phi_{\mathbb{B}}(I_1), D_2 = \Phi_{\mathbb{B}}(I_2)$. If $D_1 \cap D_2 \in \Phi_{\mathbb{B}}(\Sigma_{\mathbb{B}})$, then

$$\Phi_{\mathbb{B}}^{-1}(D_1 \cap D_2) = \Phi_{\mathbb{B}}^{-1}(D_1) \cap \Phi_{\mathbb{B}}^{-1}(D_2) = I_1 \cap I_2,$$

which shows $D_1 \cap D_2 = \Phi_{\mathbb{B}}(I_1 \cap I_2)$. If $I_1 + I_2 \in \Sigma_{\mathbb{B}}$, then

$$\Phi_{\mathbb{B}}(I_1 + I_2) = \overline{\Sigma(I_1 + I_2)} = \overline{\Sigma(D_1 + D_2)} = D_1 + D_2. \qquad \square$$

We need a lemma which is often used in the theory of Cuntz–Krieger-type algebras.

Lemma 4.7. *Let J be a subset of $\mathbb{A}$. Then the gauge actions exist on $(\mathbb{F}/\mathbb{I})/\Sigma(J)$, and so (A) is satisfied for the same H. One has $\mathrm{bal}(q_J(x)) = \mathrm{bal}(x)$ for all words $x \in W$ with $q_J(x) \neq 0$. If π is a representation of $\mathbb{F}/\mathbb{I}$, X is a linear subspace of $\mathbb{A}$, and $J := \ker(\pi|_X)$, then the representation $\tilde{\pi}$ induced by π by dividing out J is injective on $q_J(X)$ ($\pi = \tilde{\pi} q_J$).*

Proof. It is well known that $\mathbb{A}$ is the fixed point algebra of the gauge action t. Hence $t_\lambda(j) = j$ for $j \in J$ and $\lambda \in H$ since $J \subseteq \mathbb{A} = \mathrm{lin}(W_0)$. Since $x \in \Sigma(J)$ allows a representation $x = \sum_i \alpha_i a_i j_i b_i$ for scalars $\alpha_i \in \mathbb{C}$, (possibly empty) words $a_i, b_i \in W$, and elements $j_i \in J$, this shows that $t_\lambda(\Sigma(J)) \subseteq \Sigma(J)$ ($\lambda \in H$). Hence the gauge actions exist on $(\mathbb{F}/\mathbb{I})/\Sigma(J)$. For the last claim, if $\tilde{\pi}(q_J(x)) = 0$ for $x \in X$, then $\pi(x) = 0$, then $x \in \ker(\pi|_X)$, then $x \in J$, and then $q_J(x) = 0$, showing that $\tilde{\pi}$ is injective on $q_J(X)$. $\square$

Definition 4.8. An ideal $I \in \Sigma_{\mathbb{A}}$ is called *canceling* if $\mathbb{F}/\mathbb{I}$ divided by I satisfies property (C′).

The proof of the next theorem will reveal that I is canceling if and only if $\mathbb{F}/\mathbb{I}$ divided by I is a Cuntz–Krieger-type $*$-algebra. Write $\Omega_{\mathbb{A}} \subseteq \Sigma_{\mathbb{A}}$ for the family of all canceling ideals.

Theorem 4.9. *We have $\Phi_{\mathbb{A}}(\Omega_{\mathbb{A}}) = \{D \in \mathcal{I} \mid D \cap \mathbb{A} \in \Omega_{\mathbb{A}}\}$.*

Proof. Define $\mathcal{J} = \{D \in \mathcal{I} \mid D \cap \mathbb{A} \in \Omega_{\mathbb{A}}\}$. To prove $\Phi_A(\Omega_{\mathbb{A}}) \subseteq \mathcal{J}$, consider an element $I \in \Omega_{\mathbb{A}}$, and note that $\Phi_{\mathbb{A}}^{-1}(\Phi_{\mathbb{A}}(I)) = I = \Phi_{\mathbb{A}}(I) \cap \mathbb{A} \in \Omega_{\mathbb{A}}$ by Theorem 4.6. Hence $\Phi_{\mathbb{A}}(I) \in \mathcal{J}$.

To prove $\mathcal{J} \subseteq \Phi_A(\Omega_{\mathbb{A}})$, consider an element $D \in \mathcal{J}$. Define $J = \Sigma(D \cap \mathbb{A})$. Write $\pi : \mathbb{F}/\mathbb{I} \longrightarrow C^*(\mathbb{F}/\mathbb{I})/D$ for the canonical quotient map. Write $C^*(J)$ for the norm closure of J in $C^*(\mathbb{F}/\mathbb{I})$. As J is a two-sided self-adjoint ideal in $\mathbb{F}/\mathbb{I}$ by definition, $C^*(J)$ is a two-sided closed ideal in the norm closure $C^*(\mathbb{F}/\mathbb{I})$ of $\mathbb{F}/\mathbb{I}$. Since $C^*(J) \subseteq D$, π induces a homomorphism $\tilde{\pi} : (\mathbb{F}/\mathbb{I})/J \longrightarrow C^*(\mathbb{F}/\mathbb{I})/D$. There is also a canonical homomorphism $\sigma : (\mathbb{F}/\mathbb{I})/J \longrightarrow C^*(\mathbb{F}/\mathbb{I})/C^*(J)$. Hence, by introducing a further quotient map λ, we obtain a commutative diagram

$$\begin{array}{ccc} (\mathbb{F}/\mathbb{I})/J & \xrightarrow{\tilde{\pi}} & C^*(\mathbb{F}/\mathbb{I})/D \\ & {\scriptstyle\sigma}\searrow & \uparrow{\scriptstyle\lambda} \\ & & C^*(\mathbb{F}/\mathbb{I})/C^*(J) \end{array}$$

Since $D \cap \mathbb{A} = \ker(\pi|_{\mathbb{A}})$, by Lemma 4.7 the algebra $(\mathbb{F}/\mathbb{I})/J$ is invariant under the gauge actions and $\tilde{\pi}$ is injective on $q_J(\mathbb{A})$, which is the new core "$\mathbb{A}$" for the algebra $(\mathbb{F}/\mathbb{I})/J$ since $\mathrm{bal}(q_J(x)) = \mathrm{bal}(x)$. Then $(\mathbb{F}/\mathbb{I})/J$ is an algebra which satisfies (A) and (B), and there exists an $\mathbb{A}$-faithful C^*-representation $\tilde{\pi}$. Since J is generated by the canceling ideal $D \cap \mathbb{A} \in \Omega_{\mathbb{A}}$, by Definition 4.8 $(\mathbb{F}/\mathbb{I})/J$ satisfies also (C′), and so is a Cuntz–Krieger $*$-algebra.

Hence, by Theorem 2.2, the images of $\tilde{\pi}$ and σ are canonically isomorphic, and so λ is proved to be an isomorphism. By the definition of λ this implies $C^*(J) = D$. Since $D \in \mathcal{J}$, $D \cap \mathbb{A} \in \Omega_{\mathbb{A}}$, then $D = C^*(J) = \Phi_{\mathbb{A}}(D \cap \mathbb{A}) \in \Phi_{\mathbb{A}}(\Omega_{\mathbb{A}})$ as we wanted to show. □

Corollary 4.10. *If all ideals in $\Sigma_{\mathbb{A}}$ are canceling, then $\Phi_{\mathbb{A}}$ is a lattice isomorphism.*

Proof. Since all ideals in $\Sigma_{\mathbb{A}}$ are canceling, $\Omega_{\mathbb{A}} = \Sigma_{\mathbb{A}}$. By Theorem 4.9, $\Phi_{\mathbb{A}}$ is surjective. By Theorem 4.6 and Lemma 4.4, $\Phi_{\mathbb{A}}$ is an injective lattice homomorphism. □

We aim to generalize Theorem 4.9 by allowing $\mathbb{A}$ to be a smaller algebra $\mathbb{B}$. The sense of the next definition will become clear in Corollary 4.13 or in the proof of Corollary 4.14.

Definition 4.11. An ideal $I \in \Sigma_{\mathbb{B}}$ is called $\mathbb{B}$-*canceling* if $X := (\mathbb{F}/\mathbb{I})/\Sigma(I)$ satisfies property (C′), and every arbitrarily given C^*-representation of X is injective on $q_I(\mathbb{A})$ if and only if it is injective on $q_I(\mathbb{B})$.

Note that canceling is the same as $\mathbb{A}$-cancelling. Write $\Omega_{\mathbb{B}} \subseteq \Sigma_{\mathbb{B}}$ for the family of $\mathbb{B}$-cancelling ideals. The next theorem and corollary generalize the last ones.

Theorem 4.12. *We have $\Phi_{\mathbb{B}}(\Omega_{\mathbb{B}}) = \{D \in \mathcal{I} \mid D \cap \mathbb{B} \in \Omega_{\mathbb{B}}\}$.*

Proof. This is proved exactly like Theorem 4.9. One just replaces $\mathbb{A}$ by $\mathbb{B}$ and $\Omega_{\mathbb{A}}$ by $\Omega_{\mathbb{B}}$ everywhere. □

Corollary 4.13. *If all ideals in $\Sigma_{\mathbb{B}}$ are $\mathbb{B}$-canceling, then $\Phi_{\mathbb{B}}$ is a bijection.*

Proof. Since all ideals in $\Sigma_{\mathbb{B}}$ are $\mathbb{B}$-canceling, $\Omega_{\mathbb{B}} = \Sigma_{\mathbb{B}}$. By Theorem 4.12 $\Phi_{\mathbb{B}}$ is surjective, and by Theorem 4.6 $\Phi_{\mathbb{B}}$ is injective. □

We will now apply Corollary 4.13 to canceling higher-rank semigraph algebras (see [4, Definitions 5.1, 7.2]).

Corollary 4.14. *Let $\mathbb{F}/\mathbb{I}$ be a canceling semigraph algebra (see [4, Definitions 5.1 and 7.2]), and let $\mathbb{B}$ be the $*$-subalgebra of $\mathbb{A}$ generated by the standard projections (see [4, Definition 5.14]). Then every quotient of $\mathbb{F}/\mathbb{I}$ by an ideal in $\Sigma_{\mathbb{B}}$ is a semigraph algebra by [4, Lemma 8.1]. Now if every such quotient is canceling (as a semigraph algebra), then $\Phi_{\mathbb{B}}$ is a bijection.*

Proof. A C^*-representation of a canceling semigraph algebra is injective on $\mathbb{A}$ if and only if it is injective on $\mathbb{B}$ by [4, Corollary 6.4]. If I is an ideal in $\Sigma_{\mathbb{B}}$, then the image of q_I is a semigraph algebra by [4, Lemma 8.1]. The set of standard projections (see [4, Definition 5.14]) in the semigraph algebra $q_I(\mathbb{F}/\mathbb{I})$ are the image of the standard projections in $\mathbb{F}/\mathbb{I}$, and so $q_I(\mathbb{B})$ is the $*$-algebra generated by the standard projections in $q_I(\mathbb{F}/\mathbb{I})$. Note also that $q_I(\mathbb{A})$ is the core, or the "$\mathbb{A}$," of $q_I(\mathbb{F}/\mathbb{I})$. Hence, by [4, Corollary 6.4], a C^*-representation of $q_I(\mathbb{F}/\mathbb{I})$ is injective on $q_I(\mathbb{A})$ if and only if it is injective on $q_I(\mathbb{B})$. If we assume that $q_I(\mathbb{F}/\mathbb{I})$ is canceling (as a semigraph algebra), then it is a Cuntz–Krieger-type $*$-algebra, and so satisfies (C$'$), and by Definition 4.11 I is $\mathbb{B}$-canceling.

If we assume that $q_I(\mathbb{F}/\mathbb{I})$ is canceling for every $I \in \Sigma_{\mathbb{B}}$, then $\Sigma_{\mathbb{B}}$ consists of $\mathbb{B}$-canceling ideals only, and so $\Sigma_{\mathbb{B}} = \Omega_{\mathbb{B}}$. The claim follows thus by Corollary 4.13. □

Corollary 4.15. *If every quotient of a canceling semigraph algebra $\mathbb{F}/\mathbb{I}$ by an ideal in $\Sigma_{\mathbb{A}}$ is canceling (as a semigraph algebra), then $\Phi_{\mathbb{A}}$ is a lattice isomorphism.*

Proof. We repeat the last three sentences of the proof of Corollary 4.14 and replace $\mathbb{B}$ by $\mathbb{A}$ everywhere. □

5. Crossed product representation and nuclearity

By using the Cuntz–Krieger uniqueness theorem, Theorem 2.2, we can extend each gauge action $t_\lambda \in \mathrm{Aut}(\mathbb{F}/\mathbb{I})$ to a gauge action $\theta_\lambda \in \mathrm{Aut}(C^*(\mathbb{F}/\mathbb{I}))$ $(\lambda \in H)$. We may thus apply Takai's duality theorem [16] and obtain the following result.

Theorem 5.1. *By Takai's duality theorem, we have*

$$C^*(\mathbb{F}/\mathbb{I}) \otimes \mathcal{K}\big(L^2(\mathcal{H})\big) \cong C^*(\mathbb{F}/\mathbb{I}) \rtimes_\theta H \rtimes_{\widehat{\theta}} \widehat{H}.$$

Moreover, $C^*(\mathbb{F}/\mathbb{I}) \rtimes_\theta H$ is the norm closure of a locally matricial algebra. Hence $C^*(\mathbb{F}/\mathbb{I})$ is nuclear.

Proof. The nuclearity is concluded from the observation that $C^*(\mathbb{F}/\mathbb{I})$ is then evidently the corner of a crossed product of a (possibly nonseparable) AF-algebra by an abelian group.

We assume that ζ is injective (Lemma 2.3).

Step 1. In the first step we follow the idea in [13, Lemma 3.1]. We denote the crossed product $C^*(\mathbb{F}/\mathbb{I}) \rtimes_\theta H$ by A. Let $\mathcal{M}(A)$ be the multiplier algebra of A. Let $(U_\lambda)_{\lambda\in H} \subseteq \mathcal{M}(A)$ be the unitaries inducing the actions $(\theta_\lambda)_{\lambda\in H}$. Let

$$\chi(F) := \int_H F(\lambda) U_\lambda \, d\lambda \quad \forall F \in \widehat{H},$$

where we integrate in $\mathcal{M}(A)$, and where $d\lambda$ denotes the normalized Haar measure on H. It is easy to see that $\big(\chi(F)\big)_{F\in\widehat{H}}$ forms a family of mutually orthogonal projections in $\mathcal{M}(A)$.

Recall that $\mathrm{bal}(a)_\lambda a = \lambda_a a = \theta_\lambda(a)$ for $a \in \mathcal{A}$ and $\lambda \in H$, and we write the group operation of $\hat{H}$ additively. Notice that

$$\chi(F)a = a\chi\big(F + \mathrm{bal}(a)\big) \quad \forall a \in \mathcal{A}\ \forall F \in \widehat{H}. \tag{5.1}$$

Notice that $a\chi(F) \in A$ for all $a \in \mathcal{A}$ and $F \in \widehat{H}$. By an application of the Stone–Weierstrass theorem, the linear span of $\widehat{H}$ is dense in $L^1(H)$. Hence A is the norm closure of

$$B := \mathrm{lin}\{\chi(F)x \mid x \in W, F \in \widehat{H}\}.$$

Step 2. It remains to show that B is locally matricial. Consider a finite subset

$$\Gamma = \big\{\chi(F_1)x_1, \chi(F_2)x_2, \ldots, \chi(F_n)x_n\big\}$$

for some fixed nonzero $x_1, \ldots, x_n \in W$ and $F_1, \ldots, F_n \in \widehat{H}$. By enlarging Γ, if necessary, we can assume that Γ is self-adjoint (possible by identity (5.1)).

Let ω be the set of nonzero words in the alphabet Γ. By identity (5.1) each $y \in \omega$ has a representation

$$y = \chi(F_{j_1})x_{j_1}\chi(F_{j_2})x_{j_2}\cdots\chi(F_{j_m})x_{j_m} = \chi(F_{j_1})x_{j_1}x_{j_2}\cdots x_{j_m}$$

for some $1 \le j_1, \ldots, j_m \le n$. Since $y \ne 0$, we necessarily have

$$F_{j_{k+1}} = F_{j_k} + \mathrm{bal}(x_{j_k}) \quad \forall k = 1, \ldots, m-1.$$

Let

$$K = \big\{x_{j_1}x_{j_2}\cdots x_{j_m} \in \mathbb{F}/\mathbb{I} \mid m \ge 1, 1 \le j_1, \ldots, j_{m+1} \le n,\\ F_{j_{k+1}} = F_{j_k} + \mathrm{bal}(x_{j_k})\ \forall k = 1, \ldots, m\big\}.$$

Notice that

$$\omega \subseteq \Gamma \cup \big\{\chi(F_1), \ldots, \chi(F_n)\big\}K\Gamma$$

(products in A). Thus, if we can show that K lies in some finite-dimensional space $\mathcal{M}_n$, then $\mathrm{lin}(\omega) = \mathrm{Alg}^*(\Gamma)$ is a subspace of the finite-dimensional space

$$\mathrm{lin}\big(\Gamma \cup \big\{\chi(F_1), \ldots, \chi(F_n)\big\}\mathcal{M}_n\Gamma\big),$$

and we are done.

We construct $\mathcal{M}_n$ by induction. Let $\gamma \subseteq \{1, \ldots, n\}$ and

$$L_\gamma := \{x_{j_1}x_{j_2}\cdots x_{j_m} \in K \mid \{F_{j_1}, F_{j_2}, \ldots, F_{j_{m+1}}\} \subseteq \{F_i \mid i \in \gamma\}\}.$$

If $|\gamma| = 1$, then all x_{j_k} of $x_{j_1}x_{j_2}\cdots x_{j_m} \in L_\gamma$ are zero-balanced. Let $\mathcal{M}_1 \subseteq \mathbb{A}$ be a finite-dimensional $*$-algebra containing $\{x_i \in \mathbb{A} \mid 1 \leq i \leq n, \mathrm{bal}(x_i) = 0\}$. Then it is clear that $L_\gamma \subseteq \mathcal{M}_1$.

By induction hypothesis on $N = 1, \ldots, n-1$, we assume that there exists a finite-dimensional vector space $\mathcal{M}_N$ such that $L_\gamma \subseteq \mathcal{M}_N$ for all $\gamma \subseteq \{1, \ldots, n\}$ with $|\gamma| = N$.

Let $\delta \subseteq \{1, \ldots, n\}$ with $|\delta| = N + 1$. Let $x = x_{j_1}x_{j_2}\cdots x_{j_m} \in L_\delta$. Let

$$\{1 \leq i \leq m+1 \mid F_{j_i} = F_{j_1}\} =: \{1 = i_1 \leq \cdots \leq i_M \leq m+1\}.$$

For $k = 1, \ldots, M-1$ let

$$y_k = \prod_{t=i_k}^{i_{k+1}-1} x_{j_t}.$$

Since y_k is a partial word of the word $x = x_{j_1}x_{j_2}\cdots x_{j_m}$ which lives in K, we get

$$\mathrm{bal}(y_k) = \sum_{t=i_k}^{i_{k+1}-1} \mathrm{bal}(x_{j_t}) = \sum_{t=i_k}^{i_{k+1}-1} F_{j_{t+1}} - F_{j_t} = F_{j_{i_{k+1}}} - F_{j_{i_k}} = F_{j_1} - F_{j_1} = 0.$$

Hence y_k is zero-balanced and lives in $\mathbb{A}$. We have

$$x = y_1 y_2 \cdots y_{M-1} x_{j_{i_M}} x_{j_{i_M+1}} \cdots x_{j_m}.$$

Notice that, for all $k = 1, \ldots, M$, both the "middle term" of y_k, that is,

$$x_{j_{i_k+1}} x_{j_{i_k+2}} \cdots x_{j_{i_{k+1}-2}},$$

and the "end term" of x, that is, $x_{j_{i_M+1}} \cdots x_{j_m}$, lie in $L_{\delta\setminus\{j_1\}} \subseteq \mathcal{M}_N$ (the inclusion is by the induction hypothesis). Thus $y_1, \ldots, y_{M-1}$ lie in the finite-dimensional vector space

$$Y = \Big(\sum_{s=1}^{n} \mathbb{C}x_s + \sum_{s,t=1}^{n} \mathbb{C}x_s x_t + \sum_{s,t=1}^{n} x_s \mathcal{M}_N x_t\Big) \cap \mathbb{A}.$$

Hence $Z = \mathrm{Alg}^*(Y)$ is a finite-dimensional vector space since $Y \subseteq \mathbb{A}$. Thus $y_1 \cdots y_{M-1} \in Z$, and x lies in the finite-dimensional vector space

$$\mathcal{M}_{N+1} = Z + \sum_{s=1}^{n} Z x_s + \sum_{s=1}^{n} Z x_s \mathcal{M}_N.$$

Notice that the choice of $\mathcal{M}_{N+1}$ is independent of δ and $x \in L_\delta$. This completes the induction. If $N+1 = n$, then the proof is complete since then $K = L_{\{1,\ldots,n\}} \subseteq \mathcal{M}_n$. □

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