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ON A CONJECTURE OF THE NORM SCHWARZ INEQUALITY

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Abstract. Let A be a positive invertible matrix, and let B be a normal matrix. Following the investigation of Ando, we show that $\|A\sharp(B^*A^{-1}B)\| \geq \|B\|$, where $\sharp$ denotes the geometric mean, fails in general.

1. Introduction

In the paper [2], Ando considered the following problem. For three matrices A, B, C with $A \geq 0, C \geq 0$, does $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$ imply that $\|A\sharp C\| \geq \|B\|$? Here $A\sharp C$ is the geometric mean of A and C. The inequality $\|A\sharp C\| \geq \|B\|$ was called the *norm Schwarz inequality*. In the case that A is invertible, it is known that $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$ if and only if $C \geq B^*A^{-1}B$, and so the above problem is equivalent to the following. Is $\|A\sharp(B^*A^{-1}B)\| \geq \|B\|$ always true for $A > 0$? Ando showed in [2] that if B satisfies this inequality for any A, then B must be normaloid (i.e., $\|B\| = r(B)$ the spectral radius of B). Then it is natural to wonder whether this norm inequality holds whenever B is normal.

Conjecture. *For any positive invertible matrix A and any normal matrix B in $M_n(\mathbb{C})$, we have*

$$\big\|A\sharp(B^*A^{-1}B)\big\| \geq \|B\|.$$

Ando [2] presented the following four theorems.

(1) If B is normaloid, then the inequality $\|A^{\frac{1}{2}}(B^*A^{-1}B)^{\frac{1}{2}}\| \geq \|B\|$ holds [2, Theorem 2.3].
(2) If B is self-adjoint, then the conjecture is true [2, Theorem 3.4].

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(3) If B is a scalar multiple of a unitary matrix, then the conjecture is true [2, Theorem 3.5].
(4) When $n = 2$, the conjecture is true [2, Theorem 4.1].

The aim of the present article is to construct a counterexample to this conjecture in $M_6(\mathbb{C})$. For this purpose, we introduce some statements which are equivalent to the above conjecture. As a bonus, we can show that if the above conjecture were true, then the inequality

$$A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1} \geq 3I$$

must hold for any positive invertible matrices A, B, and C. Then we can construct a counterexample for this inequality. The idea of constructing a counterexample for this inequality is basically due to Lin, who attributed it to Drury [4]. In the final section, we give another proof of Ando's theorem for 2×2 matrices.

After finishing this work, the author learned from Minghua Lin that he had succeeded in constructing a counterexample to the above conjecture before us. His example consists of 3×3 matrices, and so it is better than ours. The idea of construction, however, is different.

2. Some equivalent conjectures

Throughout this paper, we denote by $M_n(\mathbb{C})$ the space of $n \times n$ matrices. The geometric mean of two positive matrices $A, B \in M_n(\mathbb{C})$ is denoted by $A\sharp B$. If they are invertible, then we can write $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$. For a matrix A we denote its trace and determinant by $\mathrm{Tr}(A)$ and $\det(A)$, respectively. We also denote the operator norm of a matrix A by $\|A\|$.

First, we introduce three conjectures.

Conjecture 1 (see Ando [2])**.** *For any positive invertible matrix A and any normal invertible matrix B in $M_n(\mathbb{C})$, we have*

$$\left\|A\sharp(B^*A^{-1}B)\right\| \geq \|B\|.$$

Conjecture 2. *For any positive invertible matrix S, any unitary matrix U, and any positive invertible matrix D in $M_n(\mathbb{C})$ with $UD = DU$, we have*

$$\left\|D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}\right\| \geq \|D\|.$$

For a unitary matrix U with the spectral decomposition $U = \sum_i z_i P_i$ ($z_i \neq z_j$, $\{P_i\}_i$ are spectral projections), we set

$$E_U(X) = \sum_i P_i X P_i.$$

With respect to the Hilbert–Schmidt inner product $\langle X \mid Y\rangle = \mathrm{Tr}(X^*Y)$ on $M_n(\mathbb{C})$, the map $E_U(\cdot)$ is the orthogonal projection to the commutant of U, that is, to the class $\{X : XU = UX\}$. Also, $E_U(\cdot)$ is a unital, trace-preserving, positive (hence contractive) linear map on $M_n(\mathbb{C})$ such that $E_U(DX) = D \cdot E_U(X)$, $E_U(XD) = E_U(X) \cdot D$ for any $D \geq 0$ with $DU = UD$.

Here we note that if $U^k = I$ for some positive integer k, then the map E_U can also be defined by

$$E_U(X) = \frac{1}{k}\sum_{i=0}^{k-1} U^{*i} X U^i.$$

Conjecture 3. *For any positive invertible matrix S and any unitary matrix U in $M_n(\mathbb{C})$, we have*

$$E_U\big(S\sharp(U^*S^{-1}U)\big) \geq I.$$

The main result in this section is the following.

Theorem 2.1. *All three conjectures above are mutually equivalent.*

Proof. (Conjecture 1 $\Rightarrow$ Conjecture 2) We set $B = UD = DU$ and $A = D^{\frac{1}{2}}SD^{\frac{1}{2}}$. Then we see that

$$\begin{aligned} A\sharp(B^*A^{-1}B) &= (D^{\frac{1}{2}}SD^{\frac{1}{2}})\sharp(D^{\frac{1}{2}}U^*S^{-1}UD^{\frac{1}{2}}) \\ &= D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}. \end{aligned}$$

Since B is normal, applying Conjecture 1, we have

$$\big\|D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}\big\| = \big\|A\sharp(B^*A^{-1}B)\big\| \geq \|B\| = \|D\|.$$

(Conjecture 2 $\Rightarrow$ Conjecture 1) Take a polar decomposition $B = UD = DU$ with unitary U and positive D, and set $S = D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. Then, as shown above, we have $A\sharp(B^*A^{-1}B) = D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}$, and hence Conjecture 2 implies Conjecture 1.

(Conjecture 2 $\Rightarrow$ Conjecture 3) It is enough to show that $e \cdot S\sharp(U^*S^{-1}U) \cdot e \geq e$ for any rank 1 projection e with $Ue = eU$. Indeed, if U has the spectral decomposition $U = \sum_i z_i P_i$ ($z_i \neq z_j$), then we can write $E_U(X) = \sum_i P_i X P_i$. In order to show Conjecture 3, we have to show that $P_i \cdot S\sharp(U^*S^{-1}U) \cdot P_i \geq P_i$ for each i. To do so, it is enough to show that $e \cdot S\sharp(U^*S^{-1}U) \cdot e \geq e$ for any rank 1 projection $e \leq P_i$. Here we remark that a rank 1 projection e satisfies $Ue = eU$ if and only if $e \leq P_i$ for some i.

We set $D = e + \frac{1}{2}(I - e)$. Then, by Conjecture 2, we have

$$\big\|D^{\frac{n}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{n}{2}}\big\| \geq \|D^n\|$$

for any positive integer n. By tending $n \to \infty$, we have

$$\big\|e \cdot S\sharp(U^*S^{-1}U) \cdot e\big\| \geq \|e\| = 1.$$

Then, since e is a rank 1 projection, we conclude that

$$e \cdot S\sharp(U^*S^{-1}U) \cdot e = \big\|e \cdot S\sharp(U^*S^{-1}U) \cdot e\big\| e \geq e.$$

(Conjecture 3 $\Rightarrow$ Conjecture 2) We may assume that $\|D\| = 1$. Take a spectral projection P of D with $DP = P$. Notice that P commutes with U. Then, by

Conjecture 3, we compute

$$\begin{aligned}\left\|D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}\right\| &\geq \left\|E_U\left(D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}\right)\right\| \\ &\geq \left\|P \cdot E_U\left(D^{\frac{1}{2}} \cdot S\sharp(U^*S^{-1}U) \cdot D^{\frac{1}{2}}\right) \cdot P\right\| \\ &= \left\|PD^{\frac{1}{2}} \cdot E_U\left(S\sharp(U^*S^{-1}U)\right) \cdot D^{\frac{1}{2}}P\right\| \\ &= \left\|P \cdot E_U(S\sharp U^*S^{-1}U) \cdot P\right\| \\ &\geq \|P\| = 1 = \|D\|. \end{aligned}$$

□

Corollary 2.2. *If Conjecture 1 is true in $M_{3n}(\mathbb{C})$, then, for any positive invertible matrices $A, B, C \in M_n(\mathbb{C})$, we have*

$$A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1} \geq 3I.$$

Proof. Denote by $M_3(M_n(\mathbb{C}))$ the space of 3×3 matrices with entries $M_n(\mathbb{C})$. It is canonically identified with $M_{3n}(\mathbb{C})$. We set $U = \left[\begin{smallmatrix} 0 & 0 & I_n \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{smallmatrix}\right]$. By Theorem 2.1, Conjecture 3 is also true. We will apply Conjecture 3 to these matrices.

It is easy to see that

$$\begin{aligned} S\sharp(U^*S^{-1}U) &= \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \sharp \begin{bmatrix} B^{-1} & 0 & 0 \\ 0 & C^{-1} & 0 \\ 0 & 0 & A^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A\sharp B^{-1} & 0 & 0 \\ 0 & B\sharp C^{-1} & 0 \\ 0 & 0 & C\sharp A^{-1} \end{bmatrix}. \end{aligned}$$

Since $U^3 = I$, we have

$$\begin{aligned} &E_U(S\sharp U^*S^{-1}U) \\ &\quad = \frac{1}{3}\left\{S\sharp(U^*S^{-1}U) + U^* \cdot S\sharp(U^*S^{-1}U) \cdot U + U^{*2} \cdot S\sharp(U^*S^{-1}U) \cdot U^2\right\} \\ &\quad = \frac{1}{3}\operatorname{diag}(A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}, B\sharp C^{-1} + C\sharp A^{-1} + A\sharp B^{-1}, \\ &\qquad C\sharp A^{-1} + A\sharp B^{-1} + B\sharp C^{-1}). \end{aligned}$$

Then, using the assumption that Conjecture 3 is true, we get

$$\frac{A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}}{3} \geq I.$$

□

Therefore, if we can find positive invertible matrices $A, B, C \in M_n(\mathbb{C})$ which do not satisfy

$$A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1} \geq 3I, \tag{†}$$

then we can conclude that Conjecture 1 is not true in $M_{3n}(\mathbb{C})$, and we can construct an explicit counterexample.

Although we will construct a counterexample to the conjecture in the next section, let us show that there is evidence which supports the validity of the conjecture. The following facts state that, if we consider the trace in both sides of the inequalities, then Conjecture 3 and the inequality (†) are true.

Proposition 2.3.

(1) *For any positive invertible matrix S and any unitary matrix U in $M_n(\mathbb{C})$, we have*
$$\frac{1}{n}\operatorname{Tr}\bigl(E_U\bigl(S\sharp(U^*S^{-1}U)\bigr)\bigr) \geq 1.$$
(2) *For any positive invertible matrices $A, B, C \in M_n(\mathbb{C})$, we have*
$$\frac{1}{n}\operatorname{Tr}(A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}) \geq 3.$$

Proof. For a positive invertible matrix $X \in M_n(\mathbb{C})$ with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, we observe by concavity of the function $\log t$
$$\frac{1}{n}\log\det(X) = \frac{1}{n}(\log\lambda_1 + \cdots + \log\lambda_n) \leq \log\frac{1}{n}(\lambda_1 + \cdots + \lambda_n) = \log\frac{1}{n}\operatorname{Tr}(X),$$
and hence
$$\bigl(\det(X)\bigr)^{\frac{1}{n}} \leq \frac{1}{n}\operatorname{Tr}(X).$$

(1)
$$\begin{aligned}\frac{1}{n}\operatorname{Tr}\bigl(E_U\bigl(S\sharp(U^*S^{-1}U)\bigr)\bigr) &= \frac{1}{n}\operatorname{Tr}\bigl(S\sharp(U^*S^{-1}U)\bigr)\\ &\geq \bigl(\det\bigl(S\sharp(U^*S^{-1}U)\bigr)\bigr)^{\frac{1}{n}} = 1.\end{aligned}$$

(2)
$$\begin{aligned}&\frac{1}{n}\operatorname{Tr}(A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1})\\ &\quad= \frac{1}{n}\operatorname{Tr}(A\sharp B^{-1}) + \frac{1}{n}\operatorname{Tr}(B\sharp C^{-1}) + \frac{1}{n}\operatorname{Tr}(C\sharp A^{-1})\\ &\quad\geq \bigl(\det(A\sharp B^{-1})\bigr)^{\frac{1}{n}} + \bigl(\det(B\sharp C^{-1})\bigr)^{\frac{1}{n}} + \bigl(\det(C\sharp A^{-1})\bigr)^{\frac{1}{n}}\\ &\quad= \det(A)^{\frac{1}{2n}}\det(B)^{-\frac{1}{2n}} + \det(B)^{\frac{1}{2n}}\det(C)^{-\frac{1}{2n}} + \det(C)^{\frac{1}{2n}}\det(A)^{-\frac{1}{2n}}\\ &\quad\geq 3\bigl\{\det(A)^{\frac{1}{2n}}\det(B)^{-\frac{1}{2n}} \times \det(B)^{\frac{1}{2n}}\det(C)^{-\frac{1}{2n}} \times \det(C)^{\frac{1}{2n}}\det(A)^{-\frac{1}{2n}}\bigr\}^{\frac{1}{3}}\\ &\quad= 3.\end{aligned}$$

Here we used the usual arithmetic-geometric inequality $\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}}$. □

By the joint concavity of the geometric mean (see [1, Theorem 2]), we see that
$$\Bigl(\frac{A+B+C}{3}\Bigr)\sharp\Bigl(\frac{B^{-1}+C^{-1}+A^{-1}}{3}\Bigr) \geq \frac{1}{3}(A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1}).$$
Thus, if the inequality (†) were true, we must have
$$\Bigl(\frac{A+B+C}{3}\Bigr)\sharp\Bigl(\frac{B^{-1}+C^{-1}+A^{-1}}{3}\Bigr) \geq I. \tag{‡}$$

Proposition 2.4. *For any positive invertible matrices $A, B, C \in M_n(\mathbb{C})$, the inequality (‡) is true.*

Proof. This is also a direct consequence from the joint concavity of the geometric mean. Indeed,

$$\begin{aligned}
&\Big(\frac{A+B+C}{3}\Big)\sharp\Big(\frac{B^{-1}+C^{-1}+A^{-1}}{3}\Big)\\
&= \Big(\frac{A+B+C}{3}\Big)\sharp\Big(\frac{A^{-1}+B^{-1}+C^{-1}}{3}\Big)\\
&\geq \frac{1}{3}(A\sharp A^{-1}+B\sharp B^{-1}+C\sharp C^{-1}) = 3I.
\end{aligned}$$ □

Finally, we would like to point out the following fact. For any positive invertible matrices $A, B \in M_n(\mathbb{C})$, we can easily see that

$$A\sharp B^{-1} + B\sharp A^{-1} = (A\sharp B^{-1}) + (A\sharp B^{-1})^{-1} \geq 2.$$

3. A counterexample to the conjecture

In this section we will construct a counterexample to Conjecture 1. This example is due to Minghua Lin and Stephen Drury [4].

In the inequality

$$A\sharp B^{-1} + B\sharp C^{-1} + C\sharp A^{-1} \geq 3I,$$

if we set $A = X^2$, $B = Y^{-2}$, and $C = I$, then we obtain

$$X^2\sharp Y^2 + X^{-1} + Y^{-1} \geq 3I.$$

We show that there are two positive-definite matrices X and Y such that they do not satisfy this inequality. This means that there are 6×6 matrices which do not satisfy Conjecture 1.

The following fact is well known for the specialists. We include its proof for completeness.

Lemma 3.1 ([3], Proposition 4.1.12)**.** *For 2×2 matrices $X > 0$ and $Y > 0$, we have*

$$X\sharp Y = \frac{(\det(X)\det(Y))^{\frac{1}{4}}}{\det(\frac{1}{\sqrt{\det(X)}}X + \frac{1}{\sqrt{\det(Y)}}Y)^{\frac{1}{2}}}\Big(\frac{1}{\sqrt{\det(X)}}X + \frac{1}{\sqrt{\det(Y)}}Y\Big).$$

In particular, if $\det(X) = \det(Y)$, we have

$$X\sharp Y = \sqrt{\frac{\det(X)}{\det(X+Y)}}(X+Y).$$

Proof. Applying the Cayley–Hamilton theorem to the matrix $(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}}$, we have

$$X^{-\frac{1}{2}}YX^{-\frac{1}{2}} - \mathrm{Tr}\big((X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}}\big)(X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}} + \Big(\frac{\det(Y)}{\det(X)}\Big)^{\frac{1}{2}} = 0.$$

By multiplying $X^{\frac{1}{2}}$ from both sides, we see that

$$Y - \mathrm{Tr}\big((X^{-\frac{1}{2}}YX^{-\frac{1}{2}})^{\frac{1}{2}}\big)X\sharp Y + \Big(\frac{\det(Y)}{\det(X)}\Big)^{\frac{1}{2}}X = 0.$$

Hence we can write

$$X\sharp Y = c\Big(\frac{1}{\sqrt{\det(X)}}X + \frac{1}{\sqrt{\det(Y)}}Y\Big).$$

By taking the determinants, we have

$$\big(\det(X)\det(Y)\big)^{\frac{1}{2}} = c^2 \det\Big(\frac{1}{\sqrt{\det(X)}}X + \frac{1}{\sqrt{\det(Y)}}Y\Big),$$

and so we are done. □

Set

$$X = \frac{1}{5^2}\begin{bmatrix} 50 & 5 \\ 5 & 1 \end{bmatrix}, \qquad Y = \frac{1}{5^2}\begin{bmatrix} 50 & -5 \\ -5 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here we remark that $\det(X) = \det(Y) = \frac{1}{5^2}$ and

$$X^2 = \frac{1}{5^4}\begin{bmatrix} 2525 & 255 \\ 255 & 26 \end{bmatrix}, \qquad Y^2 = \frac{1}{5^4}\begin{bmatrix} 2525 & -255 \\ -255 & 26 \end{bmatrix}.$$

By Lemma 3.1, we know that

$$X^2\sharp Y^2 = \sqrt{\frac{\det(X^2)}{\det(X^2+Y^2)}}(X^2+Y^2).$$

Since $X^2 + Y^2 = \frac{1}{5^4}\left[\begin{smallmatrix} 5050 & 0 \\ 0 & 52 \end{smallmatrix}\right]$, we compute

$$P(X^2\sharp Y^2)P = \frac{\frac{1}{5^2}}{\frac{1}{5^4}(5050 \times 52)^{\frac{1}{2}}} \times \frac{5050}{5^4}P = \sqrt{\frac{101}{650}}P.$$

Since

$$X^{-1} = \begin{bmatrix} 1 & -5 \\ -5 & 50 \end{bmatrix}, \qquad Y^{-1} = \begin{bmatrix} 1 & 5 \\ 5 & 50 \end{bmatrix},$$

we see that

$$P(X^2\sharp Y^2 + X^{-1} + Y^{-1})P = \sqrt{\frac{101}{650}}P + 2P < 3P.$$

Therefore, we conclude that the matrices X and Y do not satisfy the inequality

$$X^2\sharp Y^2 + X^{-1} + Y^{-1} \geq 3I.$$

4. THE CONJECTURE FOR 2×2 MATRICES

In [2, Theorem 4.1], Ando showed that Conjecture 1 is true for 2×2-matrices. In this section, we give another proof for this result. In Section 2, we saw that Conjecture 1 is equivalent to Conjecture 3. Thus it is enough to show the following.

Theorem 4.1. *For any positive invertible 2×2 matrix S and any unitary 2×2 matrix U, we have*

$$E_U\big(S\sharp(U^*S^{-1}U)\big) \geq I.$$

Proof. Without loss of generality we may assume that U is a diagonal matrix of the form $U = \left[\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right]$ with $|z| = 1$ because $(wU)^*S^{-1}(wU) = U^*S^{-1}U$ for any complex number w with $|w| = 1$. In the case that $z = 1$, U becomes the identity, and so the statement is obvious. Therefore, we have only to consider the case where $z \neq 1$ and $U \neq I$. Here we remark that in this case, the map E_U is defined by

$$E_U\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = \begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix}.$$

We can also assume that $S = \left[\begin{smallmatrix} a & b \\ \bar{b} & c \end{smallmatrix}\right]$ with $\det(S) = ac - |b|^2 = 1$ since

$$(\alpha S)\sharp\big\{U^*(\alpha S)^{-1}U\big\} = S\sharp(U^*S^{-1}U)$$

for any positive number α. Then we see that

$$S^{-1} = \begin{bmatrix} c & -b \\ -\bar{b} & a \end{bmatrix}, \qquad U^*S^{-1}U = \begin{bmatrix} c & -bz \\ -\overline{bz} & a \end{bmatrix},$$
$$S + U^*S^{-1}U = \begin{bmatrix} a+c & b(1-z) \\ \overline{b(1-z)} & a+c \end{bmatrix}.$$

Then we compute

$$\begin{aligned} \det(S + U^*S^{-1}U) &= (a+c)^2 - \big|b(1-z)\big|^2 \\ &= 2\big(ac - |b|^2\big) + a^2 + c^2 + 2|b|^2 \operatorname{Re} z \\ &= a^2 + c^2 + 2\big(1 + |b|^2 \operatorname{Re} z\big). \end{aligned}$$

Then, since $\det(S) = \det(U^*S^{-1}U) = 1$, by Lemma 3.1, we have

$$\begin{aligned} S\sharp U^*S^{-1}U &= \sqrt{\frac{\det(S)}{\det(S + U^*S^{-1}U)}}(S + U^*S^{-1}U) \\ &= \frac{1}{\sqrt{a^2 + c^2 + 2(1 + |b|^2 \operatorname{Re} z)}} \begin{bmatrix} a+c & b(1-z) \\ \overline{b(1-z)} & a+c \end{bmatrix}, \end{aligned}$$

and hence

$$E_U(S\sharp U^*S^{-1}U) = \frac{1}{\sqrt{a^2 + c^2 + 2(1 + |b|^2 \operatorname{Re} z)}} \begin{bmatrix} a+c & 0 \\ 0 & a+c \end{bmatrix}.$$

On the other hand, we see that

$$\begin{aligned}(a+c)^2-\bigl\{a^2+c^2+2\bigl(1+|b|^2\operatorname{Re} z\bigr)\bigr\} &= 2\bigl\{(ac-1)-|b|^2\operatorname{Re} z\bigr\}\\ &= 2\bigl(|b|^2-|b|^2\operatorname{Re} z\bigr)\\ &= 2|b|^2(1-\operatorname{Re} z)\geq 0.\end{aligned}$$

Here we used the fact that $ac-1=|b|^2$. Therefore, we conclude

$$E_U(S\sharp U^*S^{-1}U)\geq I. \qquad \square$$

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