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NEW MODULI FOR BANACH SPACES

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Abstract. Modifying the moduli of supporting convexity and supporting smoothness, we introduce new moduli for Banach spaces which occur, for example, as lengths of catheti of right-angled triangles (defined via so-called *quasiorthogonality*). These triangles have two boundary points of the unit ball of a Banach space as endpoints of their hypotenuse, and their third vertex lies in a supporting hyperplane of one of the two other vertices. Among other things, it is our goal to quantify via such triangles the local deviation of the unit sphere from its supporting hyperplanes. We prove respective Day–Nordlander-type results involving generalizations of the modulus of convexity and the modulus of Banaś.

1. Introduction

The modulus of convexity (going back to [8]; see also [13]) and the modulus of smoothness (defined in [9]; see also [14]) are well-known classical constants from Banach space theory. For these two notions various interesting applications were found, and a large variety of natural refinements, generalizations, and modifications created an impressive set of interesting results and problems (see, e.g., [5], [6], [15], [17], [18], [21] to cite only references close to our discussion here). Inspired by [5], two further constants in this direction were introduced and investigated in [15], namely, the modulus of supporting convexity and the modulus of

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supporting smoothness. These moduli suitably quantify the local deviation of the boundary of the unit ball of a real Banach space from its supporting hyperplanes near to arbitrarily chosen touching points. Using the concept of right-angled triangles in terms of so-called *quasiorthogonality* (which is closely related to the concept of Birkhoff–James orthogonality), we modify and complete the framework of moduli defined in [8], [5], and [15] by introducing and studying new related constants. These occur as lengths of catheti of such triangles, whose hypotenuse connects two boundary points of the unit ball and whose third vertex lies in the related supporting hyperplane. We prove Day–Nordlander-type results referring to these moduli, yielding even generalizations of the constants introduced in [8], [5], and [15]. Respective results on Hilbert spaces are obtained, too. At the end, we discuss some conjectures and questions which refer to further related inequalities between such moduli (for general Banach spaces, but also for Hilbert spaces), possible characterizations of inner product spaces, and Milman's moduli.

This article is organized as follows. After presenting our notation and basic definitions in Section 2, we clarify the geometric position of the mentioned right-angled triangles close to a point of the unit sphere of a Banach space and its corresponding supporting hyperplane in Section 3. This yields a clear geometric presentation of the new moduli, but also of further moduli already discussed in the literature. In Section 4, we particularly study properties of the catheti of these triangles, yielding the announced results of the Day–Nordlander type as well as results on Hilbert spaces. In a similar way, we study properties of the hypotenuses in Section 5, obtaining again Day–Nordlander-type results and further new geometric inequalities. In Section 6, our notions and results are put into a more general framework connected with concepts like monotone operators, dual mappings of unit spheres, and their monotonicity. And in Section 7, some open questions and conjectures on the topics shortly described above are collected.

2. Notation and basic definitions

In the rest of this article, we will need the following notation. Let X be a *real Banach space*, and let X^* be its *conjugate space*. We use H to denote a *Hilbert space*. For a set $A \subset X$, we denote by ∂A and $\operatorname{int} A$ the *boundary* and the *interior* of A, respectively. We use $\langle p, x\rangle$ to denote the *value of a functional* $p \in X^*$ *at a vector* $x \in X$. For $R > 0$ and $c \in X$, we denote by $\mathfrak{B}_R(c)$ the *closed ball with center* c *and radius* R, and we denote by $\mathfrak{B}^*_R(c)$ the respective *ball in the conjugate space*. Thus, $\partial\mathfrak{B}_1(o)$ denotes the *unit sphere* of X. By definition, we put $J_1(x) = \{p \in \partial\mathfrak{B}^*_1(o) : \langle p, x\rangle = \|x\|\}$.

We will use the notation xy for the *segment* with the (distinct) endpoints x and y for the *line* passing through these points, for (oriented) *arcs* from $\partial\mathfrak{B}_R(c)$, as well as for the *vector* from x to y (the respective meaning will always be clear from the context). Further on, abbreviations like abc and $abcd$ are used for *triangles* and *4-gons* as convex hulls of these three or four points.

It is well known that the *Birkhoff–James orthogonality* (see [11, Chapter 2, Section 1] and [3]) is not a symmetric relation. Due to this, we say that y is *quasiorthogonal* to the vector $x \in X \setminus \{o\}$, and we write $y \urcorner x$ if there exists a functional $p \in J_1(x)$ such that $\langle p, y\rangle = 0$. Note that the following conditions are equivalent:

- y is quasiorthogonal to x;
- for any $\lambda \in \mathbb{R}$, the vector $x + \lambda y$ lies in the supporting hyperplane to the ball $\mathfrak{B}_{\|x\|}(o)$ at x;
- for any $\lambda \in \mathbb{R}$, the inequality $\|x + \lambda y\| \geq \|x\|$ holds (i.e., x is orthogonal to y in the sense of Birkhoff–James).

Thus, y is quasiorthogonal to x if and only if x is the Birkhoff–James orthogonal to y.

Let

$$\delta_X(\varepsilon) := \inf\Big\{1 - \frac{\|x+y\|}{2} : x, y \in \mathfrak{B}_1(o), \|x - y\| \geq \varepsilon\Big\},$$

and let

$$\rho_X(\tau) := \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\Big\}.$$

The functions $\delta_X(\cdot) : [0,2] \to [0,1]$ and $\rho_X(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ are referred to as the *modulus of convexity of* X and the *modulus of smoothness of* X, respectively.

In [5], Banaś defined and studied a new modulus of smoothness; specifically, he defined

$$\delta_X^+(\varepsilon) = \sup\Big\{1 - \frac{\|x+y\|}{2} : x, y \in \mathfrak{B}_1(o), \|x - y\| \leq \varepsilon\Big\}, \quad \varepsilon \in [0,2].$$

Let f and g be two nonnegative functions, each of them defined on a segment $[0,\varepsilon]$. We say that f and g are *equivalent at zero*, denoted by $f(t) \asymp g(t)$ as $t \to 0$, if there exist positive constants a, b, c, d, e such that $af(bt) \leq g(t) \leq cf(dt)$ for $t \in [0, e]$.

3. Right-angled triangles

We will say that a triangle is *right angled* if one of its legs is quasiorthogonal to the other one. (Note that, using also other orthogonality types, there is a large variety of ways to define right-angled triangles in normed planes; see, e.g., [2].) In a Hilbert space, this notion coincides with the common, well-known definition of a right-angled triangle.

Remark 3.1. In a nonsmooth convex Banach space, one leg of a triangle can be quasiorthogonal to the two others.

For a given right-angled triangle abc, where $ac \urcorner bc$, we will say that the legs ac, bc are the *catheti*, and ab the *hypotenuse*, of this triangle. For convenience, we draw a simple figure (see Figure 1), and we introduce related new moduli by explicit geometric construction. Let $x, y \in \partial\mathfrak{B}_1(o)$ be such that $y \urcorner x$. Let $\varepsilon \in (0,1]$, $y_1 = x + \varepsilon y$. Denote by z a point from the unit sphere such that, for the segment zy_1, we have $zy_1 \parallel ox$ and $zy_1 \cap \mathfrak{B}_1(o) = \{z\}$. Let $\{d\} = oy_1 \cap \partial\mathfrak{B}_1(o)$. Write y_2 for the projection of the point d onto the line $\{x + \tau y : \tau \in \mathbb{R}\}$ (in the nonstrictly convex case, we choose y_2 such that $dy_2 \parallel ox$). Let $p \in J_1(x)$ be such

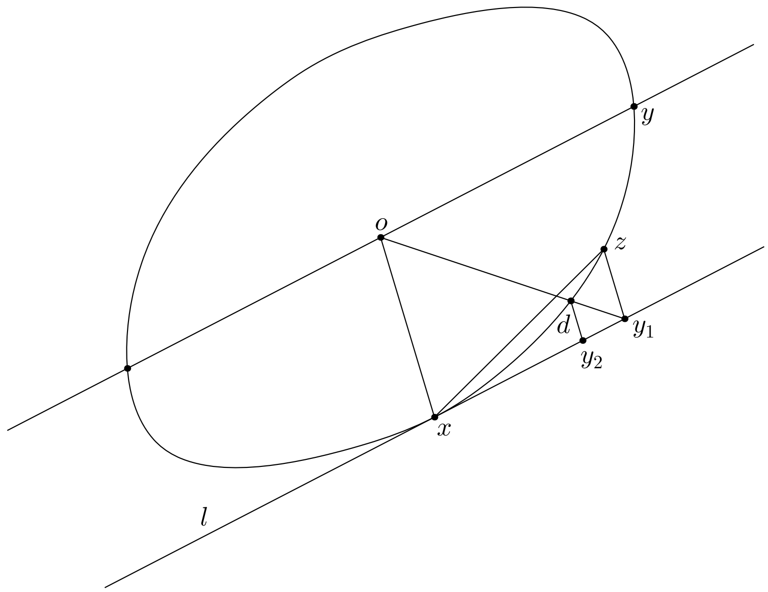

FIGURE 1. Right-angled triangles and the unit sphere.

that $\langle p, y\rangle = 0$; that is, the line $\{x+\tau y : \tau \in \mathbb{R}\}$ lies in the supporting hyperplane $l = \{a \in X : \langle p, a\rangle = 1\}$ of the unit ball at the point x. Then $\|zy_1\| = \langle p, x - z\rangle$.

Consider the right-angled triangle oxy_1 (Figure 1). In a Hilbert space we have $\|oy_1\| = \sqrt{1+\varepsilon^2}$, but in an arbitrary Banach space the length of the hypotenuse oy_1 can vary, and so we introduce moduli that describe the minimal and the maximal length of the hypotenuse in a right-angled triangle in a Banach space. More precisely, we write

$$\zeta_X^-(\varepsilon) := \inf\big\{\|x+\varepsilon y\| : x, y \in \partial\mathfrak{B}_1(o), y\urcorner x\big\}$$

and

$$\zeta_X^+(\varepsilon) := \sup\big\{\|x+\varepsilon y\| : x, y \in \partial\mathfrak{B}_1(o), y\urcorner x\big\},$$

where ε is an arbitrary positive real number. In other words, $\zeta_X^-(\cdot)-1$ and $\zeta_X^+(\cdot)-1$ describe extrema of the *deviation of a point in a supporting hyperplane from the unit ball.*

On the other hand, the length of the segment zy_1 is the *deviation of a point at the unit sphere from the corresponding supporting hyperplane*, and at the same time it is a cathetus in the triangle xzy_1.

Let $x, y \in \partial\mathfrak{B}_1(o)$ be such that $y\urcorner x$. By definition, put

$$\lambda_X(x, y, \varepsilon) := \min\big\{\lambda \in \mathbb{R} : \|x+\varepsilon y - \lambda x\| = 1\big\}$$

for any $\varepsilon \in [0,1]$. In the notation of Figure 1, we have $\lambda_X(x, y, \varepsilon) = \|zy_1\|$. The minimal and the maximal value of $\lambda_X(x, y, \varepsilon)$ characterize the deviation of the unit sphere from an arbitrary supporting hyperplane. Let us introduce now further moduli.

Define the *modulus of supporting convexity* by

$$\lambda_X^-(\varepsilon) = \inf\big\{\lambda_X(x, y, \varepsilon) : x, y \in \mathfrak{B}_1(o), y\urcorner x\big\},$$

and define the *modulus of supporting smoothness* by

$$\lambda_X^+(\varepsilon) = \sup\bigl\{\lambda_X(x,y,\varepsilon) : x,y \in \mathfrak{B}_1(o), y\urcorner x\bigr\}.$$

The notions of moduli of supporting convexity and supporting smoothness were introduced and studied in [15]. These moduli are very convenient for solving problems concerning the local behavior of the unit ball compared with that of corresponding supporting hyperplanes, and we will use some of their properties here.

In [15, Theorems 4.1, 4.2], the following inequalities were proved:

$$\rho_X\Bigl(\frac{\varepsilon}{2}\Bigr) \le \lambda_X^+(\varepsilon) \le \rho_X(2\varepsilon), \quad \varepsilon \in \Bigl[0,\frac{1}{2}\Bigr], \tag{3.1}$$

$$\delta_X(\varepsilon) \le \lambda_X^-(\varepsilon) \le \delta_X(2\varepsilon), \quad \varepsilon \in [0,1], \tag{3.2}$$

and

$$0 \le \lambda_X^-(\varepsilon) \le \lambda_X^+(\varepsilon) \le \varepsilon. \tag{3.3}$$

In addition, a Day–Nordlander-type result, referring to these moduli, was proved in [15, Corollary 5.2]:

$$\lambda_X^-(\varepsilon) \le \lambda_H^-(\varepsilon) = 1-\sqrt{1-\varepsilon^2} = \lambda_H^+(\varepsilon) \le \lambda_X^+(\varepsilon) \quad \forall \varepsilon \in [0,1].$$

In some sense, moduli of supporting convexity and supporting smoothness are estimates of a possible value referring to tangents in a Banach space (we fix the length of one of the catheti, and calculate then the minimal and maximal length of the corresponding other cathetus, which is quasiorthogonal to the first one).

Remark 3.2. For arbitrary unit vectors x, y satisfying $y\urcorner x$, the convexity of the unit ball implies that the function $\lambda_X(x,y,\cdot)$ is a nonnegative monotone convex function on the interval $[0,1]$.

But what can one say about the length of the segment zy_1 with fixed norm $\|zx\|$ (in the notation of Figure 1)? Let us introduce the following new moduli of a Banach space:

$$\varphi_X^-(\varepsilon) = \inf\bigl\{\langle p, x-z\rangle : x,z \in \partial\mathfrak{B}_1(o), \|x-z\| \ge \varepsilon, p \in J_1(x)\bigr\} \tag{3.4}$$

and

$$\varphi_X^+(\varepsilon) = \sup\bigl\{\langle p, x-z\rangle : x,z \in \partial\mathfrak{B}_1(o), \|x-z\| \le \varepsilon, p \in J_1(x)\bigr\} \tag{3.5}$$

for $\varepsilon \in [0,2]$.

Remark 3.3. Due to the convexity of the unit ball, we can substitute inequalities in the definitions of $\varphi_X^-(\cdot)$ and $\varphi_X^+(\cdot)$ to equalities (i.e., $\|x-y\| \ge \varepsilon$ and $\|x-y\| \le \varepsilon$ to be $\|x-y\| = \varepsilon$).

4. Properties of the catheti

Lemma 4.1. *In the notation of Figure 1, we have* $2\|y_1x\| \geq \|xz\|$.

Proof. By the triangle inequality, it suffices to show that $\|y_1x\| \geq \|zy_1\|$. Let the line ℓ_y be parallel to ox with $y \in \ell_y$. By construction, we have that the points x, y_1, z, o, y and the line ℓ_y lie in the same plane—the linear span of the vectors x and y. Then the lines ℓ_y and xy_1 intersect, and we denote their intersection point by c. Note that $oycx$ is a parallelogram and that $\|yc\| = 1$; the segment yx belongs to the unit ball and does not intersect the interior of the segment zy_1. Let $\{z'\} = zy_1 \cap yx$. By similarity, we have

$$\|zy_1\| \leq \|y_1z'\| = \frac{\|xy_1\|}{\|xc\|}\|yc\| = \|xy_1\|. \qquad \square$$

It is worth noticing that, under the conditions of Lemma 4.1, we have that y_1 is a projection along the vector ox of the point z on some supporting hyperplane of the unit ball at x. Moreover, y_1 belongs to the metric projection of the point y on this hyperplane. In other words, Lemma 4.1 shows us that if one projects the segment xz along the vector ox onto the hyperplane which supports the unit ball at x, then the length of the segment decreases no more than by a factor of 2.

Lemma 4.2. *Let X be an arbitrary Banach space. Then the following inequalities hold:*

$$\lambda_X^-\left(\frac{\varepsilon}{2}\right) \leq \varphi_X^-(\varepsilon) \leq \lambda_X^-(2\varepsilon) \qquad \text{and} \tag{4.1}$$

$$\lambda_X^+\left(\frac{\varepsilon}{2}\right) \leq \varphi_X^+(\varepsilon) \leq \lambda_X^+(2\varepsilon) \tag{4.2}$$

for $\varepsilon \in [0, 1/2]$.

Proof. In the notation of Figure 1, we assume that, for arbitrary x, y with $y \urcorner x$, the equality $\|zx\| = \varepsilon$ holds. Then $\lambda_X(x, y, \|xy_1\|) = \|y_1z\|$. Let $p \in J_1(x)$ be such that $\langle p, y\rangle = 0$. Hence $\|y_1z\| = \langle p, x - z\rangle$. Since $\|xy_1\| \leq \|y_1z\| + \|zx\| \leq 2\varepsilon$, and taking into account Lemma 4.1, we get

$$\frac{\varepsilon}{2} \leq \|xy_1\| \leq 2\varepsilon \leq 1.$$

Due to this and by Remark 3.2, we have

$$\lambda_X\left(x, y, \frac{\varepsilon}{2}\right) \leq \langle p, x - z\rangle \leq \lambda_X(x, y, 2\varepsilon).$$

Taking the infimum (supremum) on the right-hand side, the left-hand side, or in the middle part of the last inequality, we obtain (4.1) and (4.2). $\square$

From Lemma 4.2 and the inequalities (3.2) and (3.1), we have the following corollary.

Corollary 4.3. *Let X be an arbitrary Banach space. Then $\varphi_X^+(\varepsilon) \asymp \rho_X(\varepsilon)$ and $\varphi_X^-(\varepsilon) \asymp \delta_X(\varepsilon)$ as $\varepsilon \to 0$, and for $\varepsilon \in [0, \frac{1}{2}]$ the following inequalities hold:*

$$\rho_X\Big(\frac{\varepsilon}{4}\Big) \le \varphi_X^+(\varepsilon) \le \rho_X(4\varepsilon) \qquad \text{and}$$
$$\delta_X(\varepsilon) \le \varphi_X^-(\varepsilon) \le \delta_X(4\varepsilon).$$

Now we will prove a Day–Nordlander-type result for $\varphi_X^-(\cdot)$ and $\varphi_X^+(\cdot)$. Let us suitably generalize the notion of the modulus of convexity and the notion of the Banaś modulus. Namely, let

$$\delta_X(\varepsilon, t) = \inf\Big\{1 - \frac{\|tx + (1-t)y\|}{2} : x, y \in \partial\mathfrak{B}_1(o), \|x - y\| = \varepsilon\Big\}$$

and

$$\delta_X^+(\varepsilon, t) = \sup\Big\{1 - \frac{\|tx + (1-t)y\|}{2} : x, y \in \partial\mathfrak{B}_1(o), \|x - y\| = \varepsilon\Big\},$$

respectively. Using the same method as in the classical paper [18, Main Lemma], we get the following lemma.

Lemma 4.4. *Let X be an arbitrary Banach space. Then the following inequalities hold:*

$$\delta_X(\varepsilon, t) \le \delta_H(\varepsilon, t) = 1 - \sqrt{1 - t(1-t)\varepsilon^2} = \delta_H^+(\varepsilon, t) \le \delta_X^+(\varepsilon, t). \tag{4.3}$$

Proof. Since the proof is almost the same as in [18], we present only a short sketch. Clearly, again it is sufficient to prove the lemma in the 2-dimensional case.

If the two unit vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are rotated around the unit circle, while their difference $x - y$ has constantly the norm ε, the endpoint of the vector $tx + (1-t)y$ describes a curve Γ_t.

The following integral expresses the area of the region inside the curve described by the endpoint of the vector $x - y$ if this vector is laid off from a fixed point:

$$\int (y_1 - x_1)\, d(y_2 - x_2).$$

On the other hand, the mentioned curve is a homothet of the unit circle with ratio ε. Hence this integral equals $\varepsilon^2 A$, where A is the area of the unit ball ($A = \int x_1\, dx_2 = \int y_1\, dy_2$). From this we have

$$\int x_1\, dy_2 + \int y_1\, dx_2 = 2A - \varepsilon^2 A.$$

Now it is clear that the area of the region inside Γ_t equals

$$\int \big(tx_1 + (1-t)y_1\big)\, d\big(tx_2 + (1-t)y_2\big) = A\big(1 - t(1-t)\varepsilon^2\big).$$

Hence continuity arguments imply that there exists a point $z \in \Gamma_t$ with the norm $\sqrt{1 - t(1-t)\varepsilon^2}$. □

Theorem 4.5. *Let X be an arbitrary Banach space. Then the following inequalities hold:*

$$\varphi_X^-(\varepsilon) \le \varphi_H^-(\varepsilon) = \frac{\varepsilon^2}{2} = \varphi_H^+(\varepsilon) \le \varphi_X^+(\varepsilon). \tag{4.4}$$

Proof. It is sufficient to prove the theorem in the 2-dimensional case. Let $x \in \partial\mathfrak{B}_1(o)$, and let $p \in J_1(x)$.

Assume that X is a uniformly smooth space. Notice that p is a Fréchet derivative of the norm at the point x. Taking into account that $\mathfrak{B}_1(o)$ is convex, for an arbitrary y we have

$$\langle p, x-y\rangle = \lim_{t\searrow 0} \frac{\|x\| - \|x+t(y-x)\|}{t} = \liminf_{t>0} \frac{1-\|x+t(y-x)\|}{t}.$$

Fix an arbitrary $\gamma > 0$. Since X is uniformly smooth, there exists a $t_0 < \gamma$ such that, for arbitrary $x, y \in \partial\mathfrak{B}_1(o)$, $\|y-x\| = \varepsilon$, and $t \in (0, t_0)$, we have

$$\frac{1-\|x+t(y-x)\|}{t} - \gamma \le \langle p, x-y\rangle \le \frac{1-\|x+t(y-x)\|}{t}. \tag{4.5}$$

Taking the infimum (supremum) in the last line, we get

$$\begin{gathered} \frac{\delta_X(\varepsilon,t)}{t} - \gamma \le \varphi_X^-(\varepsilon) \le \frac{\delta_X(\varepsilon,t)}{t} \\ \Big(\frac{\delta_X^+(\varepsilon,t)}{t} - \gamma \le \varphi_X^+(\varepsilon) \le \frac{\delta_X^+(\varepsilon,t)}{t}\Big). \end{gathered}$$

Passing to the limit as $\gamma \to 0$, we have

$$\begin{gathered} \varphi_X^-(\varepsilon) = \lim_{t\to 0} \frac{\delta_X(\varepsilon,t)}{t} \le \lim_{t\to 0} \frac{\delta_H(\varepsilon,t)}{t} = \frac{\varepsilon^2}{2} \\ \Big(\varphi_X^+(\varepsilon) = \lim_{t\to 0} \frac{\delta_X^+(\varepsilon,t)}{t} \ge \lim_{t\to 0} \frac{\delta_H^+(\varepsilon,t)}{t} = \frac{\varepsilon^2}{2}\Big). \end{gathered}$$

Let us now consider the case of a nonsmooth space X. Let SP be the set of all points of smoothness at the unit circle. We know that the unit circle is compact. Then there exists $t_0 < \gamma$ such that, for arbitrary $x \in SP$, $y \in \partial\mathfrak{B}_1(o)$, $\|y-x\| = \varepsilon$, and $t \in (0, t_0)$, we can write the inequality (4.5).

Moreover, the set $\partial\mathfrak{B}_1(o)\backslash SP$ has measure zero. Thus, the infimum (supremum) of $1 - \|x + t(y-x)\|$ taken over all $x \in SP$ coincides with $\delta_X(\varepsilon,t)$ ($\delta_X^+(\varepsilon,t)$). Then we have

$$\varphi_X^-(\varepsilon) \le \limsup_{t\to 0} \frac{\delta_X(\varepsilon,t)}{t} \le \frac{\varepsilon^2}{2}$$

and

$$\varphi_X^+(\varepsilon) \ge \liminf_{t\to 0} \frac{\delta_X^+(\varepsilon,t)}{t} \ge \frac{\varepsilon^2}{2}. \qquad \square$$

5. Properties of the hypotenuse

Lemma 5.1. *Let X be an arbitrary Banach space. Then for $\varepsilon \in [0,1]$ the following inequalities hold:*

$$\lambda_X^-\Big(\frac{\varepsilon}{1+\varepsilon}\Big) \le \zeta_X^-(\varepsilon) - 1 \le \lambda_X^-(\varepsilon), \tag{5.1}$$

$$\lambda_X^+\Big(\frac{\varepsilon}{1+\varepsilon}\Big) \le \zeta_X^+(\varepsilon) - 1 \le \lambda_X^+(\varepsilon). \tag{5.2}$$

Proof. From the triangle inequality, we have that $\|y_1 d\|$ equals the distance from the point y_1 to the unit ball. Hence

$$\|y_1 d\| \le \|y_1 z\| = \lambda_X(x,y,\varepsilon) \le \varepsilon. \tag{5.3}$$

By similarity arguments and (5.3), we have

$$\|xy_2\| = \frac{\|od\|}{\|od\| + \|dy_1\|}\|xy_1\| = \frac{1}{1+\|dy_1\|}\varepsilon \ge \frac{\varepsilon}{1+\varepsilon}.$$

Then, by construction and by the convexity of the unit ball, we get the inequality

$$\|y_2 d\| = \lambda_X\big(x,y,\|xy_2\|\big) \ge \lambda_X\Big(x,y,\frac{\varepsilon}{1+\varepsilon}\Big). \tag{5.4}$$

Since y_2 is a projection of the point d onto the line $\{x+\tau y : \tau \in \mathbb{R}\}$, we have $\|y_2 d\| \le \|dy_1\|$. Combining the previous inequality with (5.3) and (5.4), we obtain the inequalities

$$\lambda_X\Big(x,y,\frac{\varepsilon}{1+\varepsilon}\Big) \le \|dy_1\| \le \lambda_X(x,y,\varepsilon).$$

Taking the infimum (supremum) on the right-hand side, the left-hand side, or in the middle part of the last line, we obtain (5.1) and (5.2). □

Corollary 5.2. *Let X be an arbitrary Banach space. Then $\zeta_X^+(\varepsilon) - 1 \asymp \rho_X(\varepsilon)$ and $\zeta_X^-(\varepsilon) - 1 \asymp \delta_X(\varepsilon)$ as $\varepsilon \to 0$, and the following inequalities hold:*

$$\rho_X\Big(\frac{\varepsilon}{2(1+\varepsilon)}\Big) \le \zeta_X^+(\varepsilon) \le \rho_X(2\varepsilon), \quad \varepsilon \in \Big[0,\frac{1}{2}\Big], \qquad \textit{and}$$

$$\delta_X\Big(\frac{\varepsilon}{1+\varepsilon}\Big) \le \zeta_X^-(\varepsilon) \le \delta_X(2\varepsilon), \quad \varepsilon \in [0,1].$$

Now we will prove results of the Day–Nordlander type for $\zeta_X^-(\cdot)$ and $\zeta_X^+(\cdot)$. Suppose that we have an orientation ω in $\mathbb{R}^2$. We will say that a curve C in the plane is a *good curve* if it is a closed rectifiable simple Jordan curve, which is enclosed by a star-shaped set S with the center at the origin and a continuous radial function.

Lemma 5.3. *Let C_1 be a closed simple Jordan curve enclosing the convex set S_1 with area $A_1 > 0$ and $0 \in \operatorname{int} S_1$. Let C_2 be a good curve, which is enclosing an area of measure A_2. Then we have the following:*

(1) *We can parameterize C_i by a function $f^i(\cdot) : [0,1) \to C_i$ ($i = 1,2$) in such a way that*
   (a) *$f^2(\tau)$ is a direction vector of the supporting line of the set S_1 at the point $f^1(\tau)$ for all $\tau \in [0,1)$,*

(b) $[f^1(\tau), f^2(\tau)] = \omega$ *for all* $\tau \in [0,1)$,

(c) *the functions* $f^i(\cdot)$ $(i = 1,2)$ *are angle monotone.*

(2) *The curve* $C_3 = \{f^1(\tau) + f^2(\tau) : \tau \in [0,1)\}$ *encloses an area of measure* $A_1 + A_2$.

Proof. (1) First of all, due to the continuity of the radial function of the curve C_2, we can assume that C_2 and C_1 are coincident. Let C_1 be a smooth curve. Let $f^1 : [0,1) \to C_1$ be a parameterization given by clockwise rotation. Then at every point $f^1(\tau)$, we have a unique supporting line to S_1, and we can choose $f^2(\tau)$ in a proper way. In this case the problem is quite easy, and one can see its geometric interpretation. The general case (when C_1 has nonsmooth points) yields additional difficulties. At a point of nonsmoothness we have continuously many supporting lines; hence we cannot give a parameterization depending only on this point of C_1. However, in [16, Section 2] Joly gives a suitable parameterization.

(2) Let A_3 be the measure of the area enclosed by C_3. Let $f^i(\cdot)$ be the parameterization of C_i $(i = 1,2)$ constructed above. Fix $\mu \in \mathbb{R}$. Denote by $S(\mu)$ and $A(\mu)$ the set and the area enclosed by the curve $C(\mu) = \{f^1(\tau) + \mu f^2(\tau) : \tau \in [0,1)\}$, respectively. Since for all $\tau \in [0,1)$ we have that $f^2(\tau)$ is a direction vector of the supporting line of the set S_1 at the point $f^1(\tau)$, then we have $S_1 \subset S(\mu)$. Hence $A(\mu) \geq A_1$. Using consequences of Green's formula and properties of the Stieltjes integral, we have

$$\int_{\tau\in[0,1)} f_1^1\, df_2^1 \leq \int_{\tau\in[0,1)} \big(f_1^1(\tau) + \mu f_1^2(\tau)\big)\, d\big(f_2^1(\tau) + \mu f_2^2(\tau)\big).$$

Therefore, for all $\mu \in \mathbb{R}$ the following inequality holds:

$$\mu^2 \int_{\tau\in[0,1)} f_1^2\, df_2^2 + \mu\Big(\int_{\tau\in[0,1)} f_1^1\, df_2^2 + \int_{\tau\in[0,1)} f_1^2\, df_2^1\Big) \geq 0.$$

This implies that

$$\Big(\int_{\tau\in[0,1)} f_1^1\, df_2^2 + \int_{\tau\in[0,1)} f_1^2\, df_2^1\Big) = 0.$$

Then we have

$$\begin{aligned} A_3 = A(1) &= \int_{\tau\in[0,1)} \big(f_1^1(\tau) + f_1^2(\tau)\big)\, d\big(f_2^1(\tau) + f_2^2(\tau)\big) \\ &= \int_{\tau\in[0,1)} f_1^1\, df_2^1 + \int_{\tau\in[0,1)} f_1^2\, df_2^2 = A_1 + A_2. \end{aligned}$$

□

Theorem 5.4. *Let X be an arbitrary Banach space. Then the following inequalities hold:*

$$\zeta_X^-(\varepsilon) \leq \zeta_H^-(\varepsilon) = \sqrt{1+\varepsilon^2} = \zeta_H^+(\varepsilon) \leq \zeta_X^+(\varepsilon). \tag{5.5}$$

Proof. Again it is sufficient to prove the theorem in the 2-dimensional case. Applying Lemma 5.3 for $C_1 = \partial\mathfrak{B}_1(o)$, $C_2 = \partial\mathfrak{B}_\varepsilon(o)$ and using continuity arguments, we obtain (5.5). □

Remark 5.5. In [16, Proposition in Section 2], inequality (5.5) was proved for the subcase $\varepsilon = 1$.

6. Some notes about monotonicity properties of the dual mapping

The notion of the *monotone operator* is well known, and has a lot of applications and useful generalizations. Let us recall some related notions, and, based on them, explain their relations to the geometry of the unit sphere.

Let X be a Banach space, let $T : X \to 2^{X^*}$ be a point-to-set operator, and let $G(T)$ be its graph. Suppose that the following inequality holds:

$$\langle p_x - p_y, x - y\rangle \geq \alpha \|x - y\|^2 \quad \text{for all } (x, p_x), (y, p_y) \in G(T). \tag{6.1}$$

(1) If $\alpha = 0$, then T is a monotone operator. For example, the subdifferential of a convex function is a monotone operator.
(2) If $\alpha > 0$, then T is a strongly monotone operator. For example, the subdifferential of a strongly convex function on a Hilbert space is a strongly monotone operator.
(3) If $\alpha < 0$, then T is a hypomonotone operator. For example, the subdifferential of a prox-regular function on a Hilbert space is a hypomonotone operator (see [20]).

Inequality (6.1) is often called the *variational inequality.* Usually, the operator T is a derivative or subderivative of a convex function. Then we can speak about the variational inequality for a convex function.

As usual in convex analysis, we can reformulate inequality (6.1) for convex (or prox-regular) sets and their normal cone (or Fréchet normal cone) (see [19]); in this case, $T(x)$ is a intersection of the $\partial\mathfrak{B}_1^*(o)$ and the normal cone to the set at point x. In a Hilbert space there are some characterizations of strongly convex and prox-regular functions (or strongly convex and prox-regular sets) via the variational inequality (see [7], [20], and [19]).

But in a Banach space the situation is much more complicated, and it is getting obvious that the right-hand side of the variational inequality cannot always be a quadratic function. Then in many applications we have to substitute $\alpha\|x - y\|^2$ in (6.1) by some proper convex function $\alpha(\|x - y\|)$.

For example, what can we say about the most simple convex function in a Banach space—its norm (in this case, T is a dual mapping)? Even in a Hilbert space, for arbitrary x, y we can only put zero in the right-hand side of the variational inequality. Nevertheless, there exist variational inequalities for norms depending on $\|x\|, \|y\|$, and $\|x - y\|$. For example, in [23, Theorems 1, 2] characterizations of uniformly smooth and uniformly convex Banach spaces were given in terms of monotonicity properties of the dual mapping.

In this paragraph, we investigate monotonicity properties of the dual mapping onto the unit sphere. In fact, we study monotonicity properties of the convex function on its Lebesgue level. Hence this result can be generalized to an arbitrary convex function. We are interested in asymptotically tight lower and upper bounds for the value of $\langle p_1 - p_2, x_1 - x_2\rangle$, where $x_1, x_2 \in \partial\mathfrak{B}_1(o), p_1 \in J_1(x_1), p_2 \in J_1(x_2)$. For the sake of convenience, we introduce new moduli:

$$\gamma_X^+(\varepsilon) = \sup\{\langle p_1 - p_2, x_1 - x_2\rangle\} \qquad \text{and} \qquad \gamma_X^-(\varepsilon) = \inf\{\langle p_1 - p_2, x_1 - x_2\rangle\},$$

where we choose $x_1, x_2 \in \partial\mathfrak{B}_1(o), \|x_1 - x_2\| = \varepsilon, p_1 \in J_1(x_1), p_2 \in J_1(x_2)$ for each $\varepsilon \in [0, 2]$.

Lemma 6.1. *Let X be an arbitrary Banach space. Then the functions $\gamma_X^+(\cdot)$ and $\gamma_X^-(\cdot)$ are monotonically increasing functions on* $[0, 2]$.

Proof. In the notation of Figure 1, let z_1, z_2 be points in the arc $-xyx$ of the unit circle such that z_1 belongs to the arc xz_2 (here and in the sequel all arcs lie in the plane of xoy). Let $p \in J_1(x)$, let $q_1 \in J_1(z_1)$, and let $q_2 \in J_1(z_2)$. It is worth mentioning that $\|xz_1\| \le \|xz_2\|$ (see [1, Lemma 1]). Then, to prove our lemma, it is sufficient to show that

$$\langle p - q_1, x - z_1\rangle \le \langle p - q_2, x - z_2\rangle. \tag{6.2}$$

From the convexity of the unit ball we have that $\langle p, x - z_1\rangle \le \langle p, x - z_2\rangle$. To prove inequality (6.2), let us show that $\langle q_1, z_1 - x\rangle \le \langle q_2, z_2 - x\rangle$.

We can assume that X is the plane of xoy. By definition, put $l = \{a \in X : \langle a, p\rangle = 1\}$, $l_1 = \{a \in X : \langle a, q_1\rangle = 1\}$, $l_2 = \{a \in X : \langle a, q_2\rangle = 1\}$, and $H^+ = \{p \in X : \langle a, p\rangle \ge 1\}$.

First case: Let z_2 be in the arc xy of the unit circle (see Figure 2). All three cases $l = l_1$, $l = l_2$, or $l_1 = l_2$ are trivial. Let $l \cap l_1 = \{b_1\}$, $l \cap l_2 = \{b_2\}$. Again, all three cases $x = b_1$, $x = b_2$, or $b_1 = b_2$ are trivial. By convexity arguments, b_1 belongs to the relative interior of the segment xb_2 and $l_1 \cap l_2 \notin H^+$. Hence l_1 separates point x and the ray $l_2 \cap H^+$ in the half-plane H^+. Let x_2 be a projection of the point x onto l_2 (in the nonstrictly convex case, we choose x_2 such that $xx_2 \parallel oz_2$). Then the segment xx_2 is parallel to oz_2, and therefore $xx_2 \subset H^+$. Now we can say that the segment x_2x and the line l_1 have an intersection point; let it be x_1. Since the values $\langle q_1, z_1 - x\rangle$ and $\langle q_2, z_2 - x\rangle$ are equal to the distances from the point x to

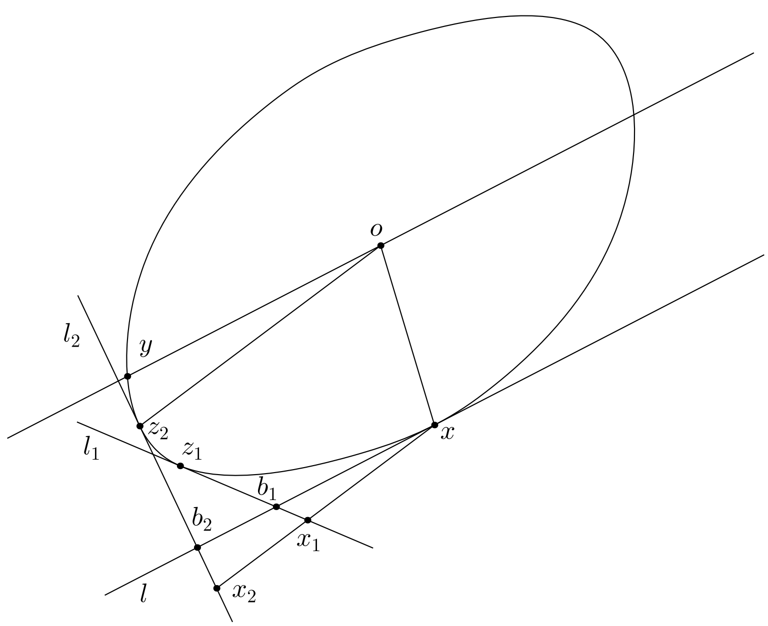

FIGURE 2. Illustration of the proof of Lemma 6.1.

the lines l_1 and l_2, respectively, we have

$$\langle q_1, z_1 - x\rangle \le \|xx_1\| < \|xx_2\| = \langle q_2, z_2 - x\rangle.$$

Second case: Let z_2 be in the arc $-xy$ of the unit circle. We can assume that z_1 lies on the arc $-xy$ of the unit circle, too (if z_1 lies on the arc xy of the unit circle, then by the first case, we can substitute z_1 to y). We have that $\langle -q_i, -z_i - x\rangle = 2 - \langle q_1, z_i - x\rangle$ for $i = 1, 2$. Therefore, applying the first case to the points $-z_1, -z_2, x$ and to the functionals $p, -q_1, -q_2$, we have proved the second case. □

Remark 6.2. It is worth mentioning that in the first case of Lemma 6.1, the lines l_1 and ox can have no common point in H^+.

Remark 6.3. Using Lemma 6.1, we can modify the definitions of $\gamma_X^+(\cdot)$ and $\gamma_X^-(\cdot)$ by

$$\gamma_X^+(\varepsilon) = \sup\bigl\{\langle p_1 - p_2, x_1 - x_2\rangle\bigr\},$$

where we choose $x_1, x_2 \in \partial\mathfrak{B}_1(o), \|x_1 - x_2\| \le \varepsilon, p_1 \in J_1(x_1), p_2 \in J_1(x_2)$, and by

$$\gamma_X^-(\varepsilon) = \inf\bigl\{\langle p_1 - p_2, x_1 - x_2\rangle\bigr\},$$

where we choose $x_1, x_2 \in \partial\mathfrak{B}_1(o), \|x_1 - x_2\| \ge \varepsilon, p_1 \in J_1(x_1), p_2 \in J_1(x_2)$ for each $\varepsilon \in [0, 2]$.

Lemma 6.4. *Let X be an arbitrary Banach space. Then the following inequalities hold:*

$$\varphi_X^+(\varepsilon) \le \gamma_X^+(\varepsilon) \le 2\varphi_X^+(\varepsilon) \quad \textit{for } \varepsilon \in [0, 2], \tag{6.3}$$

$$2\varphi_X^-\left(\frac{e}{4}\right) \le \gamma_X^-\left(\frac{\varepsilon}{4}\right) \le \varphi_X^-(\varepsilon) \quad \textit{for } \varepsilon \in [0, 1]. \tag{6.4}$$

Proof. All inequalities, except for the right-hand side of (6.4), are obvious.

Let us prove that $\gamma_X^-(\frac{\varepsilon}{4}) \le 2\varphi_X^-(\varepsilon)$. It is sufficient to prove the lemma in the 2-dimensional case. In this case and in the notation of Figure 1, we can put $\|zx\| = \varepsilon$ and $\|y_1 z\| = \varphi_X^-(\varepsilon)$. Let y_b be a bisecting point of the segment xy_1. Denote by z_b a point from the unit sphere such that $z_b y_b \parallel ox$ and $z_b y_b \cap \mathfrak{B}_1(o) = \{z_b\}$. Let $p_b \in J_1(z_b)$. Denote by l_b the line $\{a \in X : \langle p_b, a\rangle = 1\}$. By convexity the line l_b intersects the segment zy_1, and we denote the intersection point as a_1. By definition, put $\{a_2\} = l_1 \cap \{\tau x : \tau \in \mathbb{R}\}$. From the trapezoid $a_2 x a_1 y_1$, we have

$$\|y_b z_b\| + \|xa_2\| \le \|y_1 a_1\| \le \|zy_1\| = \varphi_X^-(\varepsilon). \tag{6.5}$$

Since $\langle p_b, z_b - x\rangle$ equals the distance from the point x to the line l_b, we have $\langle p_b, z_b - x\rangle \le \|xa_2\|$. From here, since $\langle p, x - z_b\rangle = \|y_b z_b\|$, and from inequality (6.5), we obtain

$$\langle p - p_b, x - z_b\rangle \le \varphi_X^-(\varepsilon).$$

From Lemma 6.1, it is sufficient to show that $\|xz_b\| \ge \frac{\varepsilon}{4}$. By definition, put $\{z'\} = y_b z_b \cap xz$. Obviously, we have

$$\|xz_b\| \ge \|xz'\| - \|z' z_b\| \ge \|xz'\| - \|z' y_b\| = \frac{\varepsilon - \varphi_X^-(\varepsilon)}{2}.$$

Using Theorem 4.5, we see that

$$\|xz_b\| \geq \frac{\varepsilon}{2} - \frac{\varepsilon^2}{4} \geq \frac{\varepsilon}{4}. \qquad \square$$

Corollary 6.5. *Let X be an arbitrary Banach space. Then $\gamma_X^+(\varepsilon) \asymp \rho_X(\varepsilon)$ and $\gamma_X^-(\varepsilon) \asymp \delta_X(\varepsilon)$ as $\varepsilon \to 0$, and for $\varepsilon \in [0, \frac{1}{2}]$ the following inequalities hold:*

$$\rho_X\left(\frac{\varepsilon}{4}\right) \leq \gamma_X^+(\varepsilon) \leq 2\rho_X(4\varepsilon) \qquad \textit{and}$$
$$2\delta_X\left(\frac{\varepsilon}{4}\right) \leq \gamma_X^-\left(\frac{\varepsilon}{4}\right) \leq \delta_X(\varepsilon).$$

Remark 6.6. Combining results from [23, Theorems 1, 2] for some constant c_1, c_2, c_3, c_4 (depending on X), one can get the following inequality:

$$c_1\rho_X(c_2\varepsilon) \geq \gamma_X^+(\varepsilon) \geq \gamma_X^-(\varepsilon) \geq c_3\delta_X(c_4\varepsilon).$$

7. Some open questions

Although there are no difficulties preventing us from proving an analogue of the Day–Nordlander theorem for the moduli $\gamma_X^+(\cdot)$ and $\gamma_X^-(\cdot)$ in the infinite-dimensional case using Dvoretzky's theorem (see [12, Theorem 1]), we have no proof for the following conjecture in the finite-dimensional case.

Conjecture 7.1. *Let X be an arbitrary Banach space. Then the following inequalities hold:*

$$\gamma_X^-(\varepsilon) \leq \gamma_H^-(\varepsilon) = \varepsilon^2 = \gamma_H^+(\varepsilon) \leq \gamma_X^+(\varepsilon). \tag{7.1}$$

All moduli mentioned above characterize certain geometrical properties of the unit ball. Obviously, the geometry of the unit ball totally describes the geometry of the unit ball in the dual space. Nevertheless, we know a few results about coincidences of values of some moduli or other characteristics of a Banach space and its dual space. We are interested in properties of the dual mapping (i.e., $x \to J_1(x)$). The following conjecture seems to be very essential. By definition, put

$$d_X^-(\varepsilon) = \inf\big\{\|p_1 - p_2\| \;\big|\; p_1 \in J_1(x_1), p_2 \in J_1(x_2), \|x_1 - x_2\| = \varepsilon, x_1, x_2 \in \partial\mathfrak{B}_1(o)\big\}$$

and

$$d_X^+(\varepsilon) = \sup\big\{\|p_1 - p_2\| \;\big|\; |p_1 \in J_1(x_1), p_2 \in J_1(x_2), \|x_1 - x_2\| = \varepsilon, x_1, x_2 \in \partial\mathfrak{B}_1(o)\big\}.$$

Conjecture 7.2. *Let X be an arbitrary Banach space. Then the following inequalities hold:*

$$d_X^-(\varepsilon) \leq d_H^-(\varepsilon) = \varepsilon = d_H^+(\varepsilon) \leq d_X^+(\varepsilon). \tag{7.2}$$

It is well known that the equality $\delta_X(\varepsilon) = \delta_H(\varepsilon)$ for $\varepsilon \in [0, 2)$ implies that X is an inner product space (see [10]). There exist such results for some other moduli (see [1] and [4]). We are interested in the following question.

Question 7.3. For what modulus $f_X(\cdot)$ (where $f_X(\cdot)$ can be $\varphi_X^-(\cdot)$, $\varphi_X^+(\cdot)$, $\zeta_X^-(\cdot)$, $\zeta_X^+(\cdot)$, $\lambda_X^-(\cdot)$, $\lambda_X^+(\cdot)$) does the equality $f_X(\varepsilon) = f_H(\varepsilon)$, holding for all ε in the domain of the function $f_X(\cdot)$ (or even for fixed ε), imply that X is an inner product space?

The definitions of the moduli $\zeta_X^-(\cdot)-1$ and $\zeta_X^+(\cdot)-1$ are similar to the definitions of Milman's moduli, which were introduced in [17] as

$$\beta_X^-(\varepsilon) = \inf_{x,y\in\partial\mathfrak{B}_1(o)} \bigl\{\max\bigl\{\|x+\varepsilon y\|, \|x-\varepsilon y\|\bigr\} - 1\bigr\}$$

and

$$\beta_X^+(\varepsilon) = \sup_{x,y\in\partial\mathfrak{B}_1(o)} \bigl\{\min\bigl\{\|x+\varepsilon y\|, \|x-\varepsilon y\|\bigr\} - 1\bigr\}.$$

We think that in the definitions of Milman's moduli it is sufficient to take only $y \urcorner x$. Hence we get the following.

Conjecture 7.4. *Let X be an arbitrary Banach space. Then for positive ε we have*

$$\zeta_X^-(\varepsilon) - 1 = \beta_X^-(\varepsilon) \qquad \textit{and} \qquad \zeta_X^+(\varepsilon) - 1 = \beta_X^+(\varepsilon).$$

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