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LEVEL SETS OF THE CONDITION SPECTRUM

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ABSTRACT. For $0 < \epsilon \leq 1$ and an element a of a complex unital Banach algebra $\mathcal{A}$, we prove the following two topological properties about the level sets of the condition spectrum. (1) If $\epsilon = 1$, then the 1-level set of the condition spectrum of a has an empty interior unless a is a scalar multiple of the unity. (2) If $0 < \epsilon < 1$, then the ϵ-level set of the condition spectrum of a has an empty interior in the unbounded component of the resolvent set of a. Further, we show that, if the Banach space X is complex uniformly convex or if X^* is complex uniformly convex, then, for any operator T acting on X, the level set of the ϵ-condition spectrum of T has an empty interior.

1. INTRODUCTION

Let $\mathcal{A}$ be a complex Banach algebra with unity e, and let Ω be an open subset of $\mathbb{C}$. We will identify $\lambda.e = \lambda$ for any $\lambda \in \mathbb{C}$. As most of our results are trivial for the elements which are a scalar multiple of the unity, we denote the set of all elements in $\mathcal{A}$ which are not a scalar multiple of the unity by $\mathcal{A} \setminus \mathbb{C}e$. A function $f : \Omega \to \mathcal{A}$ is said to be differentiable at the point $\mu \in \Omega$ (see [13, Definition 3.3]) if there exists an element $f'(\mu) \in \mathcal{A}$ such that

$$\lim_{\lambda \to \mu} \left\| \frac{f(\lambda) - f(\mu)}{\lambda - \mu} - f'(\mu) \right\| = 0.$$

If f is differentiable at every point in Ω, then f is regarded as analytic in Ω.

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Consider a nonconstant analytic function $f : \Omega \to \mathcal{A}$. For $M > 0$, we ask the question

$$\text{Does the level set} \coloneqq \big\{\lambda \in \Omega : \big\|f(\lambda)\big\| = M\big\} \text{ have a nonempty interior?} \quad (1.1)$$

The answer to this question depends on the topology of Ω and the Banach algebra $\mathcal{A}$. The following two examples show that, for some $M > 0$ and for a general nonconstant analytic Banach algebra-valued function, the interior of the level set may be empty or may not be empty.

Example 1.1. If Ω is a connected, open subset of $\mathbb{C}, \mathcal{A} = \mathbb{C}$ and if $f : \Omega \to \mathbb{C}$ is a nonconstant analytic map, then by the maximum modulus theorem, for any $M > 0$, the interior of the level set defined in (1.1) is empty.

Example 1.2. Consider $\Omega = \mathbb{C}$, $\mathcal{A} = \mathbb{M}_2(\mathbb{C}) \coloneqq \big\{A : A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right) \text{ where } a_{ij} \in \mathbb{C}\big\}$ with norm $\|A\|_\infty = \max_{1\le i\le 2}\{\sum_{j=1}^2 |a_{ij}|\}$. Define $\psi : \mathbb{C} \to \mathbb{M}_2(\mathbb{C})$ by $\psi(\lambda) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1\end{smallmatrix}\right)$. For any $\mu \in \mathbb{C}$, it is easy to see that

$$\lim_{\lambda\to\mu} \left\| \frac{\psi(\lambda) - \psi(\mu)}{\lambda - \mu} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\|_\infty = 0.$$

Thus ψ is analytic. Moreover,

$$\big\|\psi(\lambda)\big\|_\infty = \begin{cases} 1 & \text{if } |\lambda| \le 1, \\ |\lambda| & \text{if } |\lambda| > 1. \end{cases}$$

The level set of ψ for $M = 1$ is

$$\big\{\lambda \in \mathbb{C} : \big\|\psi(\lambda)\big\|_\infty = 1\big\} = \big\{\lambda \in \mathbb{C} : |\lambda| \le 1\big\}.$$

Clearly, 0 is an interior point to the above set.

For $a \in \mathcal{A}$, the resolvent set of a is defined as $\{\lambda \in \mathbb{C} : (a - \lambda) \in \mathcal{A}^{-1}\}$, where $\mathcal{A}^{-1}$ denotes the set of all invertible elements of $\mathcal{A}$. The resolvent set is denoted as $\rho(a)$, and it is known that $\rho(a)$ is an open subset of $\mathbb{C}$. The complement of $\rho(a)$ is called the *spectrum* of a, and it is denoted by $\sigma(a)$. It is well known that $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$. The spectral radius of a is defined as

$$r(a) \coloneqq \sup\big\{|\lambda| : \lambda \in \sigma(a)\big\}.$$

The map $R : \rho(a) \to \mathcal{A}$ defined by $R(\lambda) = (a - \lambda)^{-1}$ is called the *resolvent map*, and we know that the resolvent map is an analytic Banach algebra-valued map.

For $\epsilon > 0$, Globevnik in [10] raised the following question:

$$\text{Does the level set } \big\{\lambda \in \rho(a) : \big\|(a - \lambda)^{-1}\big\| = \epsilon\big\} \text{ have a nonempty interior?}$$

He was unable to answer this question. He showed that (a) the resolvent norm of an element of a unital Banach algebra cannot be constant on an open subset of the unbounded component of the resolvent set, and that (b) the resolvent norm of a bounded linear operator on a Banach space cannot be constant on an open set if the underlying space is complex uniformly convex (see Definition 4.4). One can find some more answers related to this question in [5], [3], and [4]. In [14, Theorem 3.1], Shargorodsky proved there exists an invertible bounded operator T acting on the Banach space

$$\ell_\infty(\mathbb{Z}) \coloneqq \Big\{x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \;\Big|\; \sup_{-\infty \le i \le \infty} |x_i| < \infty \text{ and } x_i \in \mathbb{C}\Big\}$$

with norm $\|x\|_* = \sup_{k \neq 0} |x_k| + |x_0|$ such that $\|(T-\lambda)^{-1}\|$ is constant in a neighborhood of $\lambda = 0$, which is an affirmative answer to the question of Globevnik. (See [7] for the results related to the level sets of the resolvent norm of a linear operator.)

The concept of the condition spectrum was first introduced by Kulkarni and Sukumar in [11], and because of the inequality in the definition, it is evident that, in order to understand the condition spectrum geometrically, one has to know more about its boundary set. Since the boundary set is the subset of the level sets of the condition spectrum, one has to concentrate on the level sets. The definition of the level sets of the condition spectrum is the following,

Definition 1.3. Let $0 < \epsilon \le 1$. The ϵ-level set of the condition spectrum of $a \in \mathcal{A}$ is defined as

$$L_\epsilon(a) \coloneqq \Big\{\lambda \in \mathbb{C} : \big\|(a-\lambda)\big\|\big\|(a-\lambda)^{-1}\big\| = \frac{1}{\epsilon}\Big\}.$$

In the computational point of view, if we are sure that the level sets of the condition spectrum do not contain any interior point, then it can help us to trace out the boundary set of the condition spectrum. Because of the reasons discussed so far, in this article we focus on the following question: is the interior of $L_\epsilon(a)$ nonempty?

For $0 < \epsilon < 1$ and $a \in \mathcal{A} \setminus \mathbb{C}e$, the preliminary section of this article discusses the basic facts about $L_\epsilon(a)$. In Section 3, we construct some examples to show the contrast between the topological property of $L_1(a)$ and $L_\epsilon(a)$, and we prove that $L_1(a)$ has an empty interior. For $0 < \epsilon < 1$, in Section 4, we study the interior property of $L_\epsilon(a)$.

Throughout this article, $B(a, r)$ denotes the open ball in $\mathbb{C}$ with center a and radius r, and $B(X)$ denotes the set of all bounded linear operators defined on the complex Banach space X.

2. Preliminaries

In this section, we introduce some definitions and terminology used in this article. We also prove some basic properties of the level sets of the condition spectrum.

Definition 2.1. ([11, Definition 2.5]) Let $0 < \epsilon < 1$. The ϵ-condition spectrum of $a \in \mathcal{A}$ is defined as

$$\sigma_\epsilon(a) = \Big\{\lambda \in \mathbb{C} : \big\|(a-\lambda)\big\|\big\|(a-\lambda)^{-1}\big\| \ge \frac{1}{\epsilon}\Big\}$$

with the convention that $\|(a-\lambda)\|\|(a-\lambda)^{-1}\| = \infty$ if $(a-\lambda)$ is not invertible.

Note 2.2. For $0 < \epsilon < 1$, it is clear that $L_\epsilon(a) \subset \sigma_\epsilon(a)$. If $a = \lambda$ for some $\lambda \in \mathbb{C}$, then $L_\epsilon(a) = \emptyset$, and so the interior of $L_\epsilon(a) = \emptyset$. Further, $L_1(a) = \mathbb{C} \setminus \{\lambda\}$, and so the interior of $L_1(a) \neq \emptyset$.

Consider the Banach algebra $\mathbb{M}_2(\mathbb{C})$. For every $A \in \mathbb{M}_2(\mathbb{C})$, the 2-norm of A is defined as $\|A\| = s_{\max}(A)$, where $s_{\max}(A)$ denotes the maximum singular value of A. For any $0 < \epsilon < 1$, we find out explicitly the ϵ-level set of the condition spectrum of an upper triangular 2×2 matrix.

Proposition 2.3. *Let $0 < \epsilon < 1$, and let $A = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \mathbb{M}_2(\mathbb{C})$. Then*

$$\begin{aligned} L_\epsilon(A) = \Big\{\mu \in \mathbb{C} :& \\ &\frac{\left(\sqrt{(|\mu - a| + |\mu - c|)^2 + |b|^2} + \sqrt{(|\mu - a| - |\mu - c|)^2 + |b|^2}\right)^2}{4|\mu - a||\mu - c|} \\ &= \frac{1}{\epsilon}\Big\}. \end{aligned} \tag{2.1}$$

Proof. For $A \in \mathbb{M}_2(\mathbb{C})$ with the 2-norm, we have the following:

$$\|A - \mu\| = s_{\max}(A - \mu) \qquad \text{and} \qquad \left\|(A - \mu)^{-1}\right\| = \frac{1}{s_{\min}(A - \mu)},$$

where $s_{\min}(A - \mu)$ denotes the minimum singular value of $A - \mu$. Hence

$$L_\epsilon(A) = \Big\{\mu : \frac{s_{\max}(A - \mu)}{s_{\min}(A - \mu)} = \frac{1}{\epsilon}\Big\}.$$

Now,

$$s_{\max}(A - \mu)s_{\min}(A - \mu) = \left|\det(A - \mu)\right| = |a - \mu||c - \mu|, \tag{2.2}$$

$$\begin{aligned} \left[s_{\max}(A - \mu)\right]^2 + \left[s_{\min}(A - \mu)\right]^2 &= \operatorname{trace}\left[(A - \mu)^*(A - \mu)\right] \\ &= |\mu - a|^2 + |\mu - c|^2 + |b|^2. \end{aligned} \tag{2.3}$$

From the above two equations, we get

$$\left[s_{\max}(A - \mu) \pm s_{\min}(A - \mu)\right]^2 = \left(|\mu - a| \pm |\mu - c|\right)^2 + |b|^2. \tag{2.4}$$

After simplification, we see $L_\epsilon(a)$ as given in equation (2.1) $\square$

Note 2.4. We know that any complex matrix is unitarily similar to an upper triangular matrix, and hence the level set of any matrix $A \in \mathbb{M}_2(\mathbb{C})$ is of the form given in Proposition 2.3.

For $0 < \epsilon < 1$, if $a \in \mathcal{A} \setminus \mathbb{C}e$, then the boundary of $\sigma_\epsilon(a)$ is a subset of $L_\epsilon(a)$. Since $\sigma_\epsilon(a)$ is a nonempty compact set, $L_\epsilon(a)$ is also a nonempty set. The following example shows that every element of $L_\epsilon(a)$ need not come from the boundary of $\sigma_\epsilon(a)$.

Example 2.5. Consider the Banach space $\ell_\infty(\mathbb{Z})$ with norm

$$\|x\|_* = |x_0| + \sup_{n \neq 0} |x_n| \quad \text{where } x = (\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \ldots),$$

where the box represents the zeroth coordinate of an element in $\ell_\infty(\mathbb{Z})$. Take an operator $A \in B(\ell_\infty(\mathbb{Z}))$ such that

$$A(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \ldots) = \Big(\ldots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{4}}, x_2, x_3, \ldots\Big).$$

For $\epsilon = \frac{1}{5}$, we prove that the scalar 0 belongs to $L_\epsilon(A)$ but not in the boundary of $\sigma_\epsilon(A)$. By Theorem 3.1 in [14],

$$\big\|(A-\lambda)^{-1}\big\| = 4 \quad \text{for } \lambda \in B\Big(0, \frac{1}{4}\Big). \tag{2.5}$$

For any $x \in \ell_\infty(\mathbb{Z})$,

$$\|Ax\|_* = \Big(\Big|\frac{x_1}{4}\Big| + \sup_{n \neq 1} |x_n|\Big) \leq \frac{5}{4}\|x\|_*. \tag{2.6}$$

Take the unit norm element $y = (y_k)_{k=-\infty}^{\infty}$ such that

$$y_k = \begin{cases} 1 & \text{for } k = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see $\|Ay\|_* = \frac{5}{4}$, and thus $\|A\| = \frac{5}{4}$. Equation (2.5) and the fact $\|A\| = \frac{5}{4}$ together imply that $\|A\|\|A^{-1}\| = 5$, and hence $0 \in L_\epsilon(A)$. Consider the unit norm element $y = (y_k)_{k=-\infty}^{\infty}$ such that

$$y_k = \begin{cases} 1 & \text{for } k = 1, 4, \\ -\overline{\lambda} & \text{for } k = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda \in B(0, \frac{1}{4}) \setminus \{0\}$. Then

$$\begin{aligned} &\big\|(A-\lambda)y\big\|_* \\ &\quad = \Big\|\Big(\ldots, y_{-1} - \lambda y_{-2}, y_0 - \lambda y_{-1}, \boxed{\frac{y_1}{4} - \lambda y_0}, y_2 - \lambda y_1, y_3 - \lambda y_2, \ldots\Big)\Big\|_* \\ &\quad = \Big|\frac{y_1}{4} - \lambda y_0\Big| + \sup_{n \neq 0} |y_{n+1} - \lambda y_n| > \frac{5}{4}. \end{aligned}$$

Hence

$$\|A - \lambda\| > \frac{5}{4} \quad \text{for } \lambda \in B\Big(0, \frac{1}{4}\Big) \Big\backslash \{0\}. \tag{2.7}$$

From equation (2.5) and equation (2.7), we get

$$\|A - \lambda\|\big\|(A-\lambda)^{-1}\big\| > 5 \quad \text{for } \lambda \in B\Big(0, \frac{1}{4}\Big) \Big\backslash \{0\}.$$

Thus $B(0, \frac{1}{4}) \subset \sigma_\epsilon(A)$, and this clearly tells us that 0 is not a boundary point of $\sigma_\epsilon(A)$.

Note 2.6. From Theorem 3.1 in [11], we know that $\sigma_\epsilon(a)$ is a perfect set for any $a \in \mathcal{A} \setminus \mathbb{C}e$. But from the last example, we note that $L_\epsilon(a)$ need not be a perfect set, whereas the following proposition shows that $L_\epsilon(a)$ is a compact set with uncountable cardinality.

Proposition 2.7. *Let $0 < \epsilon < 1$. If $a \in \mathcal{A} \setminus \mathbb{C}e$, then $L_\epsilon(a)$ is a compact subset of $\mathbb{C}$ with an uncountable number of elements.*

Proof. For $a \in \mathcal{A} \setminus \mathbb{C}e$, we know that $L_\epsilon(a)$ is a closed subset of $\sigma_\epsilon(a)$, and hence $L_\epsilon(a)$ is compact. Suppose that $L_\epsilon(a)$ has a countable number of elements. Then we choose an isolated point λ_0 from the boundary of $\sigma_\epsilon(a)$. There exist an $r > 0$ such that

$$B(\lambda_0, r) \cap \sigma(a) = \emptyset, \qquad B(\lambda_0, r) \cap \sigma_\epsilon(a) \neq \emptyset, \qquad B(\lambda_0, r) \cap \sigma_\epsilon(a)^c \neq \emptyset.$$

Take $E := B(\lambda_0, r) \setminus L_\epsilon(a)$, and define the following function:

$$\phi : E \to \mathbb{C} \quad \text{by } \phi(\lambda) = \big\|(a - \lambda)\big\|\big\|(a - \lambda)^{-1}\big\|.$$

Clearly, ϕ is continuous, and

$$E = \Big\{\lambda \in \rho(a) : \phi(\lambda) > \frac{1}{\epsilon}\Big\} \cup \Big\{\lambda \in \rho(a) : \phi(\lambda) < \frac{1}{\epsilon}\Big\}.$$

This is a contradiction to the fact that E is connected. Thus $L_\epsilon(a)$ has an uncountable number of points. $\square$

3. 1-level set of the condition spectrum

This section deals with the 1-level set of the condition spectrum. We mainly prove that the interior of the 1-level set of the condition spectrum is empty, and we also give a better geometric picture of the 1-level set of the condition spectrum (see Lemma 3.4). In fact, excluding the case when the number of elements in $\sigma(a)$ is two, we prove that $L_1(a)$ contains at most one element (see Theorems 3.5, 3.8).

The following examples show that the nature of $L_1(a)$ is different from $L_\epsilon(a)$, particularly when $L_1(a)$ may be empty, may be unbounded, or may have a countable number of points.

Example 3.1. The set $\mathcal{D} = \{f \in C([a, b]) \mid f' \in C([a, b])\}$ forms a complex unital Banach algebra with respect to pointwise addition, pointwise multiplication, and with the norm $\|f\|_d = \|f\|_\infty + \|f'\|_\infty$. Since $\|f'\|_\infty \neq 0$ for every nonscalar invertible element $f \in \mathcal{D}$, we have

$$\|f\|_d\|f^{-1}\|_d \geq \|f\|_\infty\|f^{-1}\|_\infty + \|f^{-1}\|_\infty\|f'\|_\infty > 1.$$

Thus $L_1(f) = \emptyset$ for every nonscalar invertible element $f \in \mathcal{D}$.

Example 3.2. Consider the complex Hilbert space

$$\ell^2(\mathbb{N}) := \Big\{x = (x_1, x_2, x_3, x_4, \dots) \;\Big|\; \sum_{i=1}^{\infty} |x_i|^2 < \infty \text{ and } x_i \in \mathbb{C}\Big\}$$

with norm

$$\|x\|_2 = \Big(\sum_{i=1}^{\infty} |x_i|^2\Big)^{1/2}.$$

For some fixed $n \in \mathbb{N}$ with $n \geq 2$, consider an operator T in $B(\ell^2(\mathbb{N}))$ defined as

$$T(e_i) = \begin{cases} e_{(n+1)-i} & \text{for } 1 \leq i \leq n, \\ e_i & \text{for all } i \geq n+1, \end{cases}$$

where the e_i's form the standard orthonormal basis for $\ell^2(\mathbb{N})$. It is easy to see that $T = T^* = T^{-1}$ and $\sigma(T) = \{-1, 1\}$. For any $\lambda \in \rho(T)$, the operators $T - \lambda$ and $(T-\lambda)^{-1}$ are normal, and so their norms are equal to its spectral radius. We have

$$\|T-\lambda\| = \max\big\{|\lambda-1|, |\lambda+1|\big\} \qquad \text{and} \qquad \big\|(T-\lambda)^{-1}\big\| = \max\Big\{\frac{1}{|\lambda-1|}, \frac{1}{|\lambda+1|}\Big\}.$$

Hence,

$$L_1(T) = \Big\{\lambda : \frac{|\lambda-1|}{|\lambda+1|} = 1\Big\} \cup \Big\{\lambda : \frac{|\lambda+1|}{|\lambda-1|} = 1\Big\} = \big\{\lambda : |\lambda-1| = |\lambda+1|\big\}.$$

This shows that $L_1(T)$ is unbounded.

Example 3.3. For $n \geq 2$, consider a Banach space $\mathbb{C}^n$ with infinity norm. Take an operator $S \in B(\mathbb{C}^n)$ such that $S(e_i) = e_{(n+1)-i}$, where e_i is the standard basis of $\mathbb{C}^n$. It is clear that $S = S^{-1}$, $\|S\| = 1$, and $\sigma(S) = \{-1, 1\}$. For any $\lambda \in \rho(S)$, we observe that

$$(S-\lambda)(e_i) = -\lambda e_i + e_{(n+1)-i} \quad \text{with } \big\|(S-\lambda)\big\| = 1 + |\lambda|$$

and

$$(S-\lambda)^{-1}(e_i) = \frac{-1}{\lambda^2-1}(\lambda e_i + e_{(n+1)-i}) \quad \text{with } \big\|(S-\lambda)^{-1}\big\| = \frac{1+|\lambda|}{|\lambda^2-1|}.$$

It is easy to verify that $L_1(S) = \{0\}$.

Lemma 3.4. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $L_1(a)$ is nonempty, then for each $\mu \in L_1(a)$,*

$$\|a-\mu\| = |\mu-\lambda| \qquad \textit{and} \qquad \big\|(a-\mu)^{-1}\big\| = \frac{1}{|\mu-\lambda|} \quad \textit{for all } \lambda \in \sigma(a).$$

Proof. Let $\mu \in L_1(a)$. Then

$$\begin{aligned} \big\|(a-\mu)^{-1}\big\| &\geq \frac{1}{\inf\{|\mu-\lambda| : \lambda \in \sigma(a)\}} \\ &\geq \frac{1}{\sup\{|\mu-\lambda| : \lambda \in \sigma(a)\}} \\ &= \frac{1}{r(a-\mu)} \geq \frac{1}{\|a-\mu\|} = \big\|(a-\mu)^{-1}\big\|. \end{aligned}$$

Hence $|\mu-\lambda| = \|a-\mu\|$ and $\frac{1}{|\mu-\lambda|} = \|(a-\mu)^{-1}\|$ for all $\lambda \in \sigma(a)$. □

Theorem 3.5. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $\sigma(a)$ has more than two elements, then $L_1(a)$ has at most one element.*

Proof. Let $\lambda_1, \lambda_2, \lambda_3 \in \sigma(a)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Suppose that $L_1(a)$ has two distinct elements z_1 and z_2. Then, by Lemma 3.4, we have

$$|z_1 - \lambda_1| = |z_1 - \lambda_2| = |z_1 - \lambda_3| = \|a - z_1\|$$

and

$$|z_2 - \lambda_1| = |z_2 - \lambda_2| = |z_2 - \lambda_3| = \|a - z_2\|.$$

The above two equations imply that two circles with distinct centers intersect in three distinct points. This is a contradiction. □

Theorem 3.6. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a)$ has more than one element. Then the interior of $L_1(a)$ is empty.*

Proof. If $\sigma(a)$ has more than two elements, then, by Theorem 3.5, the interior of $L_1(a)$ is empty. Next, we assume that $\sigma(a) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 \neq \lambda_2$. Suppose that there exists an $r > 0$ such that $B(\eta_0, r) \subseteq L_1(a)$ for some $\eta_0 \in \mathbb{C}$. By Lemma 3.4, we have $|\lambda_1 - \mu| = |\lambda_2 - \mu|$ for all $\mu \in B(\eta_0, r)$. This is a contradiction. □

A well-known problem in operator theory is that, if $T \in B(X)$ with $\sigma(T) = \{1\}$, then under what additional conditions can we conclude that $T = I$? In connection with this problem, Theorem 3.7 gives a sufficient condition. A survey article contains details of many classical results related to this problem (see [15]). Another sufficient condition is also given in [11, Corollary, 3.5] in terms of the condition spectrum.

Theorem 3.7 ([1, Theorem 1.1])**.** *Let $a \in \mathcal{A}$. If $\sigma(a) = \{1\}$ and a is a doubly power bound element of $\mathcal{A}$, which means that* $\sup\{\|a^n\| : n \in \mathbb{Z}\} < \infty$, *then $a = e$.*

We prove the following result with the help of Theorem 3.7.

Theorem 3.8. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $\sigma(a) = \{\lambda\}$, then $L_1(a)$ is empty, and in particular the interior of $L_1(a)$ is also empty.*

Proof. Suppose that $L_1(a) \neq \emptyset$, and suppose that $\mu \in L_1(a)$. Then, by Lemma 3.4, $\|a - \mu\| = |\mu - \lambda|$. Consider the element $b := \frac{(a-\mu)}{\lambda-\mu}$. It is clear that $\sigma(b) = \{1\}$. Since $\|b\| = 1$, we have $\|b^n\| \leq 1$ for all positive integers n. By Lemma 3.4, $\|b^{-1}\| = 1$, and hence $\|b^n\| \leq 1$ for all negative integers n, and hence b is doubly power bound. By Theorem 3.7, we conclude that $b = e$; this implies that $a = \lambda$, which is a contradiction. □

From Theorem 3.8, we observe that if $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $L_1(a)$ is nonempty, then $\sigma(a)$ contains more than one element, and that if $a \in \mathcal{A}$ with $L_1(a) = \mathbb{C} \setminus \{\mu\}$ for some $\mu \in \mathbb{C}$, then $a = \mu$. From Example 3.1, we understand that $L_1(a)$ may be empty for $a \in \mathcal{A} \setminus \mathbb{C}e$ with $\sigma(a)$ containing more than one element.

The following theorem and Example 3.1 prove that $L_1(a) = \emptyset$ for some elements of every Banach algebra and every element of some Banach algebra.

Theorem 3.9. *For any complex unital Banach algebra $\mathcal{A}$ there always exists an $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $L_1(a) = \emptyset$.*

Proof. Suppose that there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a) = \{\lambda\}$. Then, by Theorem 3.6, $L_1(a) = \emptyset$. If there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 \neq \lambda_2$, then, by Proposition 9 in Section 7 of [2], there exists idempotents e_1 and e_2 such that $\sigma(ae_1) = \{\lambda_1\}, \sigma(ae_2) = \{\lambda_2\}$, and $a = ae_1 + ae_2$. We must have either $ae_1 \in \mathcal{A} \setminus \mathbb{C}e$ or $ae_2 \in \mathcal{A} \setminus \mathbb{C}e$; otherwise $a \notin \mathcal{A} \setminus \mathbb{C}e$. Hence, by Theorem 3.6, we get $L_1(ae_1) = \emptyset$ or $L_1(ae_2) = \emptyset$. If there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\{\lambda_1, \lambda_2, \lambda_3\} \subseteq \sigma(a)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then consider the following polynomial:

$$p(z) = \frac{(z-\lambda_2)(z-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{(z-\lambda_1)(z-\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}.$$

Clearly, $\{-1, 0, 1\} \subseteq \sigma(p(a))$, and so, by Lemma 3.4, $L_1(p(a)) = \emptyset$. □

To get Theorem 3.8, we need a doubly power bound element $a \in \mathcal{A} \setminus \mathbb{C}e$. We now ask the following question: for $a \in \mathcal{A} \setminus \mathbb{C}e$ with $L_1(a)$ empty, is it necessary that a be doubly power bound? We get a negative answer from the following example.

Example 3.10. Consider the Banach algebra $C[0,2]$ and element $f \in C[0,2]$ such that $f(x) = x$. By Lemma 3.4, $L_1(f) = \emptyset$, but $\|f^n\|_\infty \to \infty$ as $n \to \infty$. Hence f is not a doubly power bound element.

4. ϵ-LEVEL SET OF THE CONDITION SPECTRUM

For $0 < \epsilon < 1$ and $a \in \mathcal{A} \setminus \mathbb{C}e$, our first main result in this section proves that $L_\epsilon(a)$ has an empty interior in the unbounded component of the resolvent set of a. For that, we prove a version of the maximum modulus theorem for the product of n analytic vector-valued functions (where $n \in \mathbb{N}$.) This proof is similar to the proof of Theorem 2.1 in [14].

Lemma 4.1. *Let Ω_0 be a connected, open subset of $\mathbb{C}$, let Ω be an open subset of Ω_0, and let X be a complex Banach space. For $i = 1, \dots, n$, suppose that we have the following:*

(1) *$\psi_i : \Omega_0 \to X$ are analytic vector-valued functions,*
(2) *$\prod_{i=1}^n \|\psi_i(\lambda)\| \leq M$ for all $\lambda \in \Omega$,*
(3) *$\prod_{i=1}^n \|\psi_i(\mu)\| < M$ for some $\mu \in \Omega_0$.*

Then, $\prod_{i=1}^n \|\psi_i(\lambda)\| < M$ for all $\lambda \in \Omega$.

Proof. Suppose that there exists $\lambda_0 \in \Omega$ such that

$$\prod_{i=1}^n \big\|\psi_i(\lambda_0)\big\| = M.$$

Then, by the Hahn–Banach theorem, for each $\psi_i(\lambda_0)$ there exists $g_i \in X^*$ such that $\|g_i\| = 1$ and

$$g_i\big(\psi_i(\lambda_0)\big) = \big\|\psi_i(\lambda_0)\big\|. \tag{4.1}$$

Consider the function

$$\phi : \Omega_0 \to \mathbb{C} \quad \text{defined by } \phi(\lambda) = \prod_{i=1}^{n} g_i\big(\psi_i(\lambda)\big).$$

Here ϕ is analytic because $g_i(\psi_i)$ is analytic on Ω_0 for each i. By assumption (2),

$$\big|\phi(\lambda)\big| = \Big|\prod_{i=1}^{n} g_i\big(\psi_i(\lambda)\big)\Big| \le \prod_{i=1}^{n} \|g_i\|\big\|\psi_i(\lambda)\big\| \le M \quad \text{for all } \lambda \in \Omega.$$

Particularly for $\lambda_0 \in \Omega$ and from equation (4.1), we get

$$\big|\phi(\lambda_0)\big| = \Big|\prod_{i=1}^{n} g_i\big(\psi_i(\lambda_0)\big)\Big| = \prod_{i=1}^{n}\big|g_i\big(\psi_i(\lambda_0)\big)\big| = \prod_{i=1}^{n}\big\|\psi_i(\lambda_0)\big\| = M.$$

Thus $|\phi|$ attains the local maximum at Ω_0. Since Ω_0 is connected by the maximum modulus theorem, ϕ is constant and $\phi \equiv M$. On the other hand, by assumption (3) and by the definition of all g_i, we have

$$M = \big|\phi(\mu)\big| = \Big|\prod_{i=1}^{n} g_i\big(\psi_i(\mu)\big)\Big| \le \prod_{i=1}^{n} \|g_i\|\big\|\psi_i(\mu)\big\| = \prod_{i=1}^{n}\big\|\psi_i(\mu)\big\| < M.$$

This is a contradiction. □

Theorem 4.2. *Let $M > 1$, let $a \in \mathcal{A} \setminus \mathbb{C}e$, and let Ω be an open subset in the unbounded component of $\rho(a)$. If*

$$\big\|(a-\lambda)\big\|\big\|(a-\lambda)^{-1}\big\| \le M \quad \textit{for all } \lambda \in \Omega,$$

then

$$\big\|(a-\lambda)\big\|\big\|(a-\lambda)^{-1}\big\| < M \quad \textit{for all } \lambda \in \Omega.$$

Proof. Let Ω_0 be the unbounded component of $\rho(a)$. By our assumption, $\Omega \subset \Omega_0$, and

$$\big\|(a-\lambda)\big\|\big\|(a-\lambda)^{-1}\big\| \le M \quad \text{for all } \lambda \in \Omega.$$

Since $\sigma_{\frac{1}{M}}(a)$ is compact, we must have

$$\big\{\lambda \in \mathbb{C} : \big\|(a-\lambda)\big\|\big\|(a-\lambda)^{-1}\big\| < M\big\} \cap \Omega_0 \neq \emptyset.$$

Take $\mu \in \{\lambda \in \mathbb{C} : \|(a-\lambda)\|\|(a-\lambda)^{-1}\| < M\} \cap \Omega_0$. Apply Lemma 4.1 to the analytic functions $\lambda \mapsto (a-\lambda)$ and $\lambda \mapsto (a-\lambda)^{-1}$ which are defined from Ω_0 to $\mathcal{A}$ and to the scalar $\mu \in \Omega_0$. Then we get $\|(a-\lambda)\|\|(a-\lambda)^{-1}\| < M$ for all $\lambda \in \Omega$. □

Corollary 4.3. *Let $a \in \mathcal{A} \setminus \mathbb{C}e$, and let $0 < \epsilon < 1$. Then $L_\epsilon(a)$ has an empty interior in the unbounded component of $\rho(a)$. In particular, the interior of $L_\epsilon(a)$ is empty if $\rho(a)$ is connected.*

Proof. The proof follows immediately from Theorem 4.2. □

Our next result shows that if $T \in B(X)$ where X is a complex uniformly convex Banach space, then the interior of $L_\epsilon(T)$ is empty for $0 < \epsilon < 1$. We first see the notion of complex uniformly convex Banach space and some important remarks related to them.

Definition 4.4 ([9, Definition 2.4(ii)]). A complex Banach space X is said to be *complex uniformly convex* (uniformly convex) if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in X,\ \|y\| \geq \epsilon \text{ and } \|x + \zeta y\| \leq 1, \forall \zeta \in \mathbb{C}(\zeta \in \mathbb{R}), \text{with } |\zeta| \leq 1 \Rightarrow \|x\| \leq 1 - \delta.$$

It is so obvious that every uniformly convex Banach space is a complex uniformly convex space. It is proved in [6] that Hilbert spaces and L_p (with $1 < p < \infty$) spaces are uniformly convex Banach spaces, and hence that they are all complex uniformly convex Banach spaces. In [9, Theorem 1], Globevnik showed that the L_1 space is complex uniformly convex. The Banach space L_∞ is not a complex uniformly convex Banach space. The dual space L_∞^* is isometrically isomorphic to a space of bounded finitely additive set functions (see [8, Chapter IV, Section 8, Theorem 16] and [8, Chapter III, Section 1, Lemma 5]). The space of bounded finitely additive set functions is a complex uniformly convex space proved in Proposition 1.1 in [12], and so L_∞^* is complex uniformly convex.

Definition 4.5 ([9, Remark]). Consider a complex Banach space X and $\delta > 0$. We define $\omega_c(\delta)$ as follows:

$$\omega_c(\delta) = \sup\big\{\|y\| : x, y \in X \text{ with } \|x\| = 1, \|x + \zeta y\| \leq 1 + \delta, \big(\zeta \in B(0,1)\big)\big\}.$$

Remark 4.6 ([9, Remark]). Let X be a complex Banach space. Then X is complex uniformly convex if and only if $\lim_{\delta \to 0} \omega_c(\delta) = 0$.

The proof of the following Theorem is similar to the proof of Proposition 2 in [10].

Theorem 4.7. *Let X be a complex uniformly convex Banach space, and let $M > 1$. If $T \in B(X)$ with*

$$\|T - \lambda\| \big\|(T - \lambda)^{-1}\big\| \leq M \quad \textit{for all } \lambda \in B(0,1),$$

then

$$\|T - \lambda\| \big\|(T - \lambda)^{-1}\big\| < M \quad \textit{for all } \lambda \in B(0,1).$$

Proof. We claim that there exists $\mu \in B(0,1)$ such that

$$\|T - \mu\| \big\|(T - \mu)^{-1}\big\| < M.$$

Suppose that

$$\big\|(T - \lambda)\big\| \big\|(T - \lambda)^{-1}\big\| = M \quad \text{for all } \lambda \in B(0,1). \tag{4.2}$$

We arrive at a contradiction in the following three steps.

Step 1: In this step, we define a sequence of function ψ_n from $B(0,1)$ to X for each $n \in \mathbb{N}$, and we prove that each ψ_n is a bounded analytic vector-valued function. We know that there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that

$$\lim_{n \to \infty} \big\|T^{-1}(x_n)\big\| = \|T^{-1}\|.$$

By the Hahn–Banach theorem, there exists $g \in B(X)^*$ such that $g(T) = \|T\|$ with $\|g\| = 1$. For each x_n, we define the following function:

$$\psi_n : B(0,1) \to X \quad \text{by } \psi_n(\lambda) = \frac{[g(T-\lambda)](T-\lambda)^{-1}x_n}{\|T\|\|T^{-1}\|}.$$

Now,

$$\begin{aligned} \big\|\psi_n(\lambda)\big\| &= \Big\|\frac{[g(T-\lambda)](T-\lambda)^{-1}x_n}{\|T\|\|T^{-1}\|}\Big\| \\ &\le \frac{\|g\|\|T-\lambda\|\|(T-\lambda)^{-1}\|\|x_n\|}{\|T\|\|T^{-1}\|}. \end{aligned} \tag{4.3}$$

Since $\|g\| = 1$, and by equations (4.2) and (4.3), we get

$$\big\|\psi_n(\lambda)\big\| \le 1. \tag{4.4}$$

Each ψ_n is analytic because the maps $\lambda \mapsto g(T-\lambda)$ and $\lambda \mapsto (T-\lambda)^{-1}$ are analytic.

Step 2: In this step we apply Theorem 2 in [9] to the functions ψ_n, and we see the consequence. Applying Theorem 2 in [9] to the function ψ_n, we get

$$\big\|\psi_n(\lambda) - \psi_n(0)\big\| \le \Big(\frac{2|\lambda|}{1-|\lambda|}\Big) w_c\big(1 - \big\|\psi_n(0)\big\|\big) \quad \text{for all } \lambda \in B(0,1).$$

Substituting the corresponding values of ψ_n in the above equation,

$$\Big\|\frac{[g(T-\lambda)](T-\lambda)^{-1}x_n}{\|T\|\|T^{-1}\|} - \frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|}\Big\| \le \Big(\frac{2|\lambda|}{1-|\lambda|}\Big) w_c\Big(1 - \Big\|\frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|}\Big\|\Big).$$

Applying $g(T) = \|T\|$ to the right-hand side of the above inequality, we get

$$\Big\|\frac{[g(T-\lambda)](T-\lambda)^{-1}x_n}{\|T\|\|T^{-1}\|} - \frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|}\Big\| \le \Big(\frac{2|\lambda|}{1-|\lambda|}\Big) w_c\Big(1 - \frac{\|T^{-1}x_n\|}{\|T^{-1}\|}\Big).$$

Using Remark 4.6 and the fact that $1 - \frac{\|T^{-1}x_n\|}{\|T^{-1}\|} \to 0$, we note that

$$\lim_{n\to\infty} \big\|\big[g(T-\lambda)\big](T-\lambda)^{-1}x_n - g(T)T^{-1}x_n\big\| = 0. \tag{4.5}$$

Substitute $(T-\lambda)^{-1} = T^{-1} + \lambda T^{-1}(T-\lambda)^{-1}$. We get

$$\begin{aligned} &\big[g(T-\lambda)\big](T-\lambda)^{-1} - g(T)T^{-1} \\ &\quad = g(T)\big[T^{-1} + \lambda T^{-1}(T-\lambda)^{-1}\big] - \lambda g(I)(T-\lambda)^{-1} - g(T)T^{-1} \\ &\quad = \lambda\big[g(T)T^{-1} - g(I)\big](T-\lambda)^{-1}. \end{aligned} \tag{4.6}$$

Equation (4.5) and equation (4.6) together yield

$$\lim_{n\to\infty} \big\|\big[g(T)T^{-1} - g(I)\big](T-\lambda)^{-1}x_n\big\| = 0 \tag{4.7}$$

and

$$\lim_{n\to\infty} \big\|\big[g(T) - g(I)T\big](T-\lambda)^{-1}x_n\big\| = 0 \tag{4.8}$$

for all $\lambda \in B(0,1)$.

Step 3: In this step, we get the required contradiction by applying the appropriate value for $g(I)$ to the equation (4.7) and equation (4.8).

Case 1: If $g(I) = 0$, then equation (4.8) becomes

$$\lim_{n\to\infty} \left\| g(T)(T-\lambda)^{-1} x_n \right\| = 0.$$

Since the operator $T - \lambda$ is continuous for any $\lambda \in B(0,1)$,

$$\lim_{n\to\infty} \|x_n\| = 0,$$

which is a contradiction to $\|x_n\| = 1$.

Case 2: If $|g(I)| \leq 1$, then, from equation (4.7), we get

$$\lim_{n\to\infty} \left\| (T-\lambda)\left[g(T)T^{-1} - g(I)\right](T-\lambda)^{-1} x_n \right\| = 0. \tag{4.9}$$

Since the operators $(T-\lambda)$ and T^{-1} commute, the above equation becomes

$$\lim_{n\to\infty} \left\| \left(g(T)T^{-1} - g(I)\right) x_n \right\| = 0.$$

By the triangle inequality,

$$\lim_{n\to\infty} \left(\left\| g(T)T^{-1} x_n \right\| - \left\| g(I) x_n \right\| \right) = 0.$$

The above equation implies that

$$\lim_{n\to\infty} \|T\| \|T^{-1} x_n\| = 1.$$

Thus

$$\lim_{n\to\infty} \|T^{-1} x_n\| = \frac{1}{\|T\|}.$$

We also know that $\lim_{n\to\infty} \|T^{-1}x_n\| = \|T^{-1}\|$. Hence, $\|T\|\|T^{-1}\| = 1$. But we assumed that $\|T\|\|T^{-1}\| = M$. This is a contradiction to $M > 1$. Hence there exists $\mu \in B(0,1)$ such that $\|T - \mu\|\|(T-\mu)^{-1}\| < M$. Apply Lemma 4.1 to the function $\lambda \mapsto (T-\lambda)$ and $\lambda \mapsto (T-\lambda)^{-1}$, defined from $B(0,1)$ to $B(X)$ and to the point μ, to get the required conclusion. □

Note 4.8. The above result holds for any open ball in the resolvent set of T. Suppose that $\|T-\lambda\|\|(T-\lambda)^{-1}\| \leq M$ for all $\lambda \in B(\mu, r)$ and $M > 1$. If we define an operator $S := \frac{T-\mu}{r} \in B(X)$, then $S \in B(X)$ and $\|S-\lambda\|\|(S-\lambda)^{-1}\| \leq M$ for all $\lambda \in B(0,1)$. In order to prove $\|T-\lambda\|\|(T-\lambda)^{-1}\| < M$ for all $\lambda \in B(\mu, r)$, we apply Theorem 4.7 to the operator S.

Corollary 4.9. *Let X be a complex Banach space such that the dual space X^* is complex uniformly convex and $M > 1$. Suppose also that $T \in B(X)$ with*

$$\|T-\lambda\| \left\| (T-\lambda)^{-1} \right\| \leq M \quad \textit{for all } \lambda \in B(0,1).$$

Then

$$\|T-\lambda\| \left\| (T-\lambda)^{-1} \right\| < M \quad \textit{for all } \lambda \in B(0,1).$$

Proof. Consider the transpose linear map $T^* \in B(X^*)$. For any $\lambda \in B(0,1)$, we have

$$\big\|(T-\lambda)\big\|\big\|(T-\lambda)^{-1}\big\| = \big\|(T^*-\lambda)\big\|\big\|(T^*-\lambda)^{-1}\big\|.$$

Apply Theorem 4.7 to the operator T^* and the Banach space X^*. This completes the proof. □

Corollary 4.10. *Let X be a complex Banach space, let $T \in B(X)$, and let $0 < \epsilon < 1$. If either X or X^* is complex uniformly convex, then $L_\epsilon(T)$ has an empty interior in the resolvent set of T.*

Proof. The proof is an immediate consequence of Theorem 4.7 and Corollary 4.9. □

Corollary 4.11. *Let $0 < \epsilon < 1$, and let $\mathcal{A}$ be a unital C^* algebra. If $a \in \mathcal{A} \setminus \mathbb{C}e$, then the interior of $L_\epsilon(a)$ is empty.*

Proof. We know that there exists a C^* isomorphism ψ form $\mathcal{A}$ to C^* subalgebra of $B(H)$ for some Hilbert space H. For any $a \in \mathcal{A}$, we have $\sigma(a) = \sigma(\psi(a))$ and $\|a\| = \|\psi(a)\|$. These imply that

$$L_\epsilon(a) = L_\epsilon\big(\psi(a)\big).$$

Since the Hilbert space H is complex uniformly convex, and applying Theorem 4.7, we get that the interior of $L_\epsilon(\psi(a))$ is empty. Hence, the interior of $L_\epsilon(a)$ is empty. □

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