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GREEN'S THEOREM FOR CROSSED PRODUCTS BY HILBERT C^*-BIMODULES

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Abstract. Green's theorem gives a Morita equivalence $C_0(G/H, A) \rtimes G \sim A \rtimes H$ for a closed subgroup H of a locally compact group G acting on a C^*-algebra A. We prove an analogue of Green's theorem in the case $G = \mathbb{Z}$, where the automorphism generating the action is replaced by a Hilbert C^*-bimodule.

1. Introduction

The crossed product $A \rtimes X$ of a C^*-algebra A by a Hilbert A-A bimodule X, as defined in [2], is a generalization of the crossed product $A \rtimes_\alpha \mathbb{Z}$ of A by an automorphism α of A. Given an automorphism α of A, one can twist the trivial bimodule ${}_AA_A$ replacing the right structure by defining $x \cdot_\alpha a = x\alpha(a)$ and $\langle x, y\rangle_R^\alpha = \alpha^{-1}(a^*b)$ for $a, x, y \in A$, to get a C^*-bimodule, denoted by A_α, satisfying $A \rtimes_\alpha \mathbb{Z} \cong A \rtimes A_\alpha$ canonically.

Green's theorem, as stated in [5, Theorem 4.22], gives a Morita equivalence $C_0(G/H, A) \rtimes G \sim A \rtimes_{\alpha|_H} H$ for a general locally compact C^*-dynamical system (A, G, α) and a closed subgroup $H \leq G$. In the special case $G = \mathbb{Z}$, $H = n\mathbb{Z}$, for $n \in \mathbb{N}$, we have $G/H = \mathbb{Z}_n$ so that $C_0(G/H, A) = C_0(\mathbb{Z}_n, A) \cong A^n$ (n-fold direct sum) and $(H, \alpha|_H, A) = (n\mathbb{Z}, \alpha|_{n\mathbb{Z}}, A) \cong (\mathbb{Z}, \alpha^n, A)$ so that $A \rtimes_{\alpha|_H} H \cong A \rtimes_{\alpha^n} \mathbb{Z}$, where α also denotes the single automorphism generating the action of $\mathbb{Z}$ on A, and α^n denotes its nth composition power. Then, for this special case, we have the Morita equivalence $A^n \rtimes_\sigma \mathbb{Z} \sim A \rtimes_{\alpha^n} \mathbb{Z}$ for a certain action σ on A^n. Translating

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this into the C^*-bimodule language, we get $A^n \rtimes A^n_\sigma \sim A \rtimes A_{\alpha^n} \cong A \rtimes [A_\alpha]^{\otimes n}$, where we use the isomorphism $A_{\alpha^n} \cong [A_\alpha]^{\otimes n}$ (n-fold tensor product).

In this context, we show that one can replace A_α by a general right full Hilbert A-A bimodule X and establish a Morita equivalence of the form

$$A^n \rtimes X^n_\sigma \sim A \rtimes X^{\otimes n}.$$

We obtain this as a consequence of Theorem 3.1, which states a Morita equivalence of the form

$$(A_1 \oplus \cdots \oplus A_n) \rtimes (X_1 \oplus \cdots \oplus X_n)_\sigma \sim A_1 \rtimes (X_1 \otimes \cdots \otimes X_n),$$

for a "cycle" of bimodules ${}_{A_1}X_{1A_2}, {}_{A_2}X_{2A_3}, \ldots, {}_{A_{n-1}}X_{n-1A_n}, {}_{A_n}X_{nA_1}$, the special case $A_i = A$, $X_i = X$, $i = 1, \ldots, n$, giving the desired result.

2. Preliminaries

2.1. C^*-modules, C^*-bimodules, equivalence bimodules, and fullness. A *right Hilbert B-module* X_B is defined as a vector space X equipped with a right action of the C^*-algebra B and a B-valued right inner product, which is complete with respect to the induced norm. A left Hilbert A-module ${}_AX$ is defined analogously. A *Hilbert A-B bimodule* ${}_AX_B$ is a vector space X with left and right compatible Hilbert C^*-module structures over C^*-algebras A and B, respectively. Compatibility means that $\langle x, y\rangle_L \cdot z = x \cdot \langle y, z\rangle_R$, for all $x, y, z \in X$. We say that a Hilbert A-B bimodule is *right full* if $\langle X, X\rangle_R = B$, where $\langle X, X\rangle_R = \operatorname{span}(\{\langle x, y\rangle_R : x, y \in X\})$, is the span denoting the closed linear spanned set. Left fullness is defined analogously. Finally, an *equivalence bimodule* is a Hilbert A-B bimodule ${}_AX_B$ which is right full and left full. When an equivalence bimodule ${}_AX_B$ exists, the C^*-algebras A and B are said to be *Morita equivalent*, a situation denoted $A \sim B$. (See [3] for reference.)

2.2. Operations with subspaces. For linear subspaces $X, X_1, \ldots, X_n$ of a fixed normed $*$-algebra C, we define

$$\begin{aligned}
\sum X_i &\equiv X_1 + X_2 + \cdots + X_n \equiv \overline{\{x_1 + x_2 + \cdots + x_n : x_i \in X_i\}}, \\
\prod X_i &\equiv X_1 X_2 \cdots X_n \equiv \overline{\Big\{\sum_k x_{1k} x_{2k} \cdots x_{nk} : x_{ik} \in X_i\Big\}}, \\
X^* &\equiv \{x^* : x \in X\}.
\end{aligned}$$

If $Y_1, \ldots, Y_n$ is another family of subspaces and $\overline{X_i} = \overline{Y_i}$ for $i = 1, \ldots, n$, then $\sum X_i = \sum Y_i$ and $\prod X_i = \prod Y_i$. Consequently, equalities of the form $XY = \overline{X}Y$, $X + Y = \overline{X} + Y$, and so on hold for subspaces X and Y. Also, the following properties are easily checked for subspaces X, Y, Z:

1. $(X + Y) + Z = X + Y + Z = X + (Y + Z)$, 2. $X + Y = Y + X$,
3. $(XY)Z = XYZ = X(YZ)$, 4. $X(Y + Z) = XY + XZ$,
5. $(X + Y)^* = X^* + Y^*$, 6. $(XY)^* = Y^* X^*$, 7. $(X^*)^* = X$.

For a general family of subspaces $\{X_i\}_{i\in I}$, we extend the definition of sum as

$$\sum X_i \equiv \overline{\Big\{\sum_{i\in I_0} x_i : I_0 \subseteq I \text{ finite}, x_i \in X_i\Big\}}.$$

For every such a family and subspace X, we have

$$8.\quad X\Big(\sum X_i\Big) = \sum XX_i, \qquad 9.\quad \Big(\sum X_i\Big)^* = \sum X_i^*.$$

2.2.1. Let C be a fixed normed $*$-algebra, let $A \subseteq C$ be a $*$-subalgebra, and let $X \subseteq C$ be a linear subspace such that

$$1.\quad AX \subseteq X, \qquad 2.\quad XA \subseteq X, \qquad 3.\quad X^*X \subseteq A, \qquad 4.\quad XX^* \subseteq A.$$

For $k \in \mathbb{Z}$, we define $X^k = XX\cdots X$ (k times) if $k \geq 1$, $X^0 = A$ and $X^k = (X^*)^{-k}$ if $k \leq -1$. We have $X^kX^l \subseteq X^{k+l}$ for all $k,l \in \mathbb{Z}$, and $X^kX^l = X^{k+l}$ if $kl > 0$. Denote by $A[X]$ the closed $*$-subalgebra of C generated by $A \cup X$. That is,

$$A[X] = C^*(A\cup X) = \sum_{k\in\mathbb{Z}} X^k.$$

2.2.2. With $A, X \subseteq C$ as before, let $B \subseteq C$ be a $*$-subalgebra. Note that

$$\text{if } BA = AB \text{ and } BX = XB, \quad \text{then } BA[X] = A[X]B.$$

Indeed, in this case $BX^k = X^kB$ for all $k \in \mathbb{Z}$; then

$$BA[X] = B\sum_k X^k = \sum_k BX^k = \sum_k X^kB = A[X]B.$$

In a similar fashion, we can prove that

$$\text{if } BA = A \text{ and } BX = X, \quad \text{then } BA[X] = A[X].$$

2.2.3. If in the context of Section 2.2.1 C is a C^*-algebra and A and X are closed, then A is a C^*-algebra and X is a Hilbert A-A bimodule with the operations given by the restriction of the trivial Hilbert C-C bimodule structure of C. Then we have $AX = X$ and $XA = X$ because both actions are automatically nondegenerate. Moreover, if we assume that X is right full, that is, $X^*X = A$, then we have $X^{-k}X^l = X^{l-k}$ for $k,l \geq 0$.

2.3. **Crossed product by a Hilbert bimodule.** Crossed products of C^*-algebras by Hilbert bimodules are introduced in [2]. We summarize here their definition and principal properties.

2.3.1. Covariant pairs. Given a Hilbert A-A bimodule X and a C^*-algebra C, a *covariant pair* from ${}_AX_A$ to C is a pair of maps (φ,ψ) where $\varphi\colon A \to C$ is a $*$-morphism and $\psi\colon X \to C$ a linear map satisfying

$$1.\quad \psi(a\cdot x) = \varphi(a)\psi(x), \qquad 2.\quad \varphi\big(\langle x,y\rangle_L\big) = \psi(x)\psi(y)^*,$$
$$3.\quad \psi(x\cdot a) = \psi(x)\varphi(a), \qquad 4.\quad \varphi\big(\langle x,y\rangle_R\big) = \psi(x)^*\psi(y)$$

for all $a \in A$, $x,y \in X$. That is, the pair preserves the Hilbert bimodule structure considering on C the trivial Hilbert C-C bimodule structure.

2.3.2. The crossed product. A *crossed product* of a C^*-algebra A by a Hilbert A-A bimodule X is a C^*-algebra $A \rtimes X$ (denoted $A \rtimes_X \mathbb{Z}$ in [2]) together with a covariant pair (ι_A, ι_X) from ${}_AX_A$ to $A \rtimes X$ satisfying the following universal property: *for any covariant pair (φ, ψ) from ${}_AX_A$ to a C^*-algebra C there exists a unique $*$-morphism $\varphi \rtimes \psi \colon A \rtimes X \to C$ such that $\varphi = (\varphi \rtimes \psi) \circ \iota_A$ and $\psi = (\varphi \rtimes \psi) \circ \iota_X$.*

2.3.3. Basic properties. The crossed product exists and is unique up to isomorphism. The maps ι_A and ι_X are injective, so that we may consider $A, X \subseteq A \rtimes X$ and the induced $*$-morphism $\varphi \rtimes \psi$ as an extension of the covariant pair (φ, ψ). Moreover, for any covariant pair (φ, ψ) we have that $\operatorname{Im} \varphi \rtimes \psi = C^*(\operatorname{Im} \varphi \cup \operatorname{Im} \psi)$ and that $\varphi \rtimes \psi$ is injective if φ is injective.

3. The main theorem

3.1. **Twisting Hilbert modules.** If X_B is a right Hilbert B-module, C a C^*-algebra, and $\sigma \colon C \to B$ a $*$-isomorphism, then we denote by X_σ the right Hilbert module over C obtained by considering on the vector space X the operations

$$x \cdot_\sigma c = x \cdot \sigma(c) \quad \text{and} \quad \langle x, y \rangle^\sigma = \sigma^{-1}\big(\langle x, y \rangle\big) \quad \text{for } c \in C, x, y \in X.$$

If in addition X is a Hilbert A-B bimodule, then X_σ is a Hilbert A-C bimodule with the original left structure. The module X is right full if and only if X_σ is right full.

3.2. **The twisted sum of a cycle of Hilbert bimodules.** Given Hilbert bimodules ${}_{A_i}X_{i B_i}$ for $i = 1, \dots, n$, we have that $\bigoplus X_i$ is a Hilbert $\bigoplus A_i$-$\bigoplus B_i$ bimodule with pointwise operations. The bimodule $\bigoplus X_i$ is right full if and only if X_i is right full for all $i = 1, \dots, n$.

Now, given a "cycle" of Hilbert bimodules ${}_{A_1}X_{1 A_2}, {}_{A_2}X_{2 A_3}, \dots, {}_{A_n}X_{n A_1}$, we can make $\bigoplus X_i$ into a Hilbert bimodule over $\bigoplus A_i$ twisting the right action in the previous constriction with the isomorphism $\sigma \colon A_1 \oplus A_2 \oplus \cdots \oplus A_n \to A_2 \oplus \cdots \oplus A_n \oplus A_1$ given by

$$\sigma(a_1, a_2, \dots, a_n) = (a_2, \dots, a_n, a_1) \quad \text{for } a_k \in A_k.$$

Theorem 3.1. *Let ${}_{A_1}X_{1 A_2}, {}_{A_2}X_{2 A_3}, \dots, {}_{A_n}X_{n A_1}$ be right full Hilbert bimodules and consider their twisted sum $(X_1 \oplus \cdots \oplus X_n)_\sigma$ as in 3.2. Then we have the following Morita equivalence*

$$A_1 \rtimes (X_1 \otimes \cdots \otimes X_n) \sim (A_1 \oplus \cdots \oplus A_n) \rtimes (X_1 \oplus \cdots \oplus X_n)_\sigma.$$

Proof. Let $A = A_1 \oplus \cdots \oplus A_n$, let $X = (X_1 \oplus \cdots \oplus X_n)_\sigma$, and let $C = A \rtimes X$. We may suppose that $A_k \subseteq A \subseteq C$ and $X_k \subseteq X \subseteq C$, for $k = 1, \dots, n$, so that the module operations of each bimodule X_k and also the ones of the bimodule X are given by the operations of the C^*-algebra C, that is, by the restriction of the trivial Hilbert C-C bimodule structure of C. Note that as the spaces $A, X \subseteq C$ verify the conditions in Section 2.2.1, then we can define X^k for $k \in \mathbb{Z}$ as done there. Moreover, as X is a Hilbert A-A bimodule (hence, nondegenerate for both actions) and is right full, because each X_k is, we have that $A, X \subseteq C$ verify the conditions of Section 2.2.3.

We extend the families $\{A_k\}_{k=1}^n$ and $\{X_k\}_{k=1}^n$ to families $\{A_k\}_{k\in\mathbb{Z}}$ and $\{X_k\}_{k\in\mathbb{Z}}$ letting $A_k = A_l$ and $X_k = X_l$ if $k = l \bmod n$. For all $k \in \mathbb{Z}$, we have

$$A_k X_k = X_k = X_k A_{k+1}, \qquad X_k X_k^* \subseteq A_k, \quad \text{and} \quad X_k^* X_k = A_{k+1},$$

because each X_k is a Hilbert A_k-A_{k+1} bimodule (hence nondegenerate for both actions) and right full. We also have

$$A_k A_l = A_k X_l = X_k A_{l+1} = 0 \quad \text{for } k,l \in \mathbb{Z}, k \neq l \bmod n;$$

therefore, as $A = \sum_{k=1}^n A_k$ and $X = \sum_{k=1}^n X_k$,

$$A_k = A_k A = A A_k, \qquad X_k = A_k X = X A_{k+1} \quad \text{for } k \in \mathbb{Z},$$

and then

$$A_k X^l = X^l A_{k+l} \quad \text{for } k,l \in \mathbb{Z}.$$

In particular, for $k \in \mathbb{Z}$ we have that $A_k X^n = X^n A_k$ so that the pairs A_k, $A_k X^n$ satisfy the conditions of Sections 2.2.1 and 2.2.3. Following the notation in Section 2.2.1, we define the C^*-subalgebra

$$B = A_1[A_1 X^n] = C^*(A_1 \cup A_1 X^n) \subseteq C$$

and the closed subspace

$$Z = BX^0 \oplus BX \oplus \cdots \oplus BX^{n-1} \subseteq M_{1\times n}(C),$$

where $M_{1\times n}(C)$ is considered as a Hilbert C-$M_n(C)$ bimodule with the usual matrix operations. To prove the theorem, it is enough to show that $ZZ^* \subseteq C$ and $Z^*Z \subseteq M_n(C)$ are C^*-subalgebras, that Z is an equivalence ZZ^*-Z^*Z bimodule with the restricted (matrix) operations, and that we have isomorphisms $ZZ^* \cong A_1 \rtimes (X_1 \otimes \cdots \otimes X_n)$ and $Z^*Z \cong C$.

Let us consider the issues concerning the left side first. Note that the equalities $AA_1 = A_1 = A_1 A$ and $A(A_1 X^n) = A_1 X^n = (A_1 X^n)A$ imply, by Section 2.2.2, that $AB = B = BA$, because $B = A_1[A_1 X^n]$ by definition. Then we can calculate

$$ZZ^* = \sum_{k=0}^{n-1} (BX^k)(BX^k)^* = B\Big(\sum_{k=0}^{n-1} X^k X^{-k}\Big)B = BAB = B,$$

where we use the fact that $\sum_{k=0}^{n-1} X^k X^{-k} = A$, which is a consequence of the relations $X^k X^{-k} \subseteq A$ and $X^0 = A$, and that $AB = B$. Besides, from the definition of Z it is apparent that Z is B-invariant for the left action. Then we have shown that Z is a full left Hilbert B-module. Finally, to see that $B \cong A_1 \rtimes (X_1 \otimes \cdots \otimes X_n)$, consider the covariant pair (φ, ψ) given by

$$\varphi\colon A_1 \to C, \quad \varphi(a) = a \quad \text{for } a \in A_1,$$
$$\psi\colon X_1 \otimes \cdots \otimes X_n \to C, \quad \psi(x_1 \otimes \cdots \otimes x_n) = x_1 \cdots x_n \quad \text{for } x_k \in X_k.$$

By the universal property of the crossed product (see Section 2.3.2), this covariant pair extends to a $*$-morphism $\varphi \rtimes \psi\colon A_1 \rtimes (X_1 \otimes \cdots \otimes X_n) \to C$, which is injective because φ is injective (see Section 2.3.3). Moreover, $\operatorname{Im} \varphi \rtimes \psi = C^*(\operatorname{Im}\varphi \cup \operatorname{Im}\psi) = B$ because $\operatorname{Im}\varphi = A_1$, $\operatorname{Im}\psi = X_1 \cdots X_n = A_1 X \cdots A_n X = A_1 X^n$, and $B = C^*(A_1 \cup A_1 X^n)$ by definition. Hence we have the desired isomorphism.

Now, turning to the right side, note that Z^*Z is clearly a closed self-adjoint subspace of $M_n(C)$. From the equalities $ZZ^* = B$ and $BZ = Z$ we also deduce that $(Z^*Z)(Z^*Z) = Z^*BZ = Z^*Z$, so that Z^*Z is a C^*-subalgebra of $M_n(C)$ and Z is a full right Hilbert Z^*Z-module.

To show that $Z^*Z \cong C$, consider the pair of maps $\varphi\colon A \to M_n(C)$ and $\psi\colon X \to M_n(C)$ given by

$$\varphi(a_1,\dots,a_n) = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix}, \quad \psi(x_1,\dots,x_n) = \begin{bmatrix} 0 & x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \\ x_n & \cdots & 0 & 0 \end{bmatrix},$$

for $(a_1,\dots,a_n) \in A = A_1 \oplus \cdots \oplus A_n$ and $(x_1,\dots,x_n) \in X = (X_1 \oplus \cdots \oplus X_n)_\sigma$.

The following calculations show that this pair is a covariant pair. For every $a_k \in A_k$, $x_k, y_k \in X_k$, $k = 1,\dots,n$, we have

$$\begin{aligned}
&\psi\big((a_1,\dots,a_n)\cdot(x_1,\dots,x_n)\big) \\
&\quad = \psi(a_1\cdot x_1,\dots,a_n\cdot x_n) \\
&\quad = \begin{bmatrix} 0 & a_1\cdot x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1}\cdot x_{n-1} \\ a_n\cdot x_n & \cdots & 0 & 0 \end{bmatrix} \\
&\quad = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} \begin{bmatrix} 0 & x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \\ x_n & \cdots & 0 & 0 \end{bmatrix} \\
&\quad = \varphi(a_1,\dots,a_n)\psi(x_1,\dots,x_n); \\
&\varphi\big(\big\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\big\rangle_L\big) \\
&\quad = \varphi\big(\langle x_1,y_1\rangle_L,\dots,\langle x_n,y_n\rangle_L\big) \\
&\quad = \begin{bmatrix} \langle x_1,y_1\rangle_L & 0 & \cdots & 0 \\ 0 & \langle x_2,y_2\rangle_L & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \langle x_n,y_n\rangle_L \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \\ x_n & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & y_n^* \\ y_1^* & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & y_{n-1}^* & 0 \end{bmatrix} \\
&\quad = \psi(x_1,\dots,x_n)\psi(y_1,\dots,y_n)^*; \\
&\psi\big((x_1,\dots,x_n)\cdot_\sigma(a_1,\dots,a_n)\big) \\
&\quad = \psi\big((x_1,\dots,x_n)\cdot\sigma(a_1,\dots,a_n)\big)
\end{aligned}$$

$$= \psi\big((x_1, \dots, x_n) \cdot (a_2, \dots, a_n, a_1)\big) = \psi(x_1 \cdot a_2, \dots, x_{n-1} \cdot a_n, x_n \cdot a_1)$$

$$= \begin{bmatrix} 0 & x_1 \cdot a_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \cdot a_n \\ x_n \cdot a_1 & \cdots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & x_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1} \\ x_n & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix}$$

$$= \psi(x_1, \dots, x_n)\varphi(a_1, \dots, a_n);$$

$$\varphi\big(\big\langle (x_1, \dots, x_n), (y_1, \dots, y_n)\big\rangle_R^\sigma\big)$$

$$= \varphi\big(\sigma^{-1}\big(\big\langle (x_1, \dots, x_n), (y_1, \dots, y_n)\big\rangle_R\big)\big)$$

$$= \varphi\big(\sigma^{-1}\big(\langle x_1, y_1\rangle_R, \dots, \langle x_n, y_n\rangle_R\big)\big) = \varphi\big(\langle x_n, y_n\rangle_R, \langle x_1, y_1\rangle_R, \dots, \langle x_{n-1}, y_{n-1}\rangle_R\big)$$

$$= \begin{bmatrix} \langle x_n, y_n\rangle_R & 0 & \cdots & 0 \\ 0 & \langle x_1, y_1\rangle_R & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \langle x_{n-1}, y_{n-1}\rangle_R \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & x_n^* \\ x_1^* & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & x_{n-1}^* & 0 \end{bmatrix} \begin{bmatrix} 0 & y_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_{n-1} \\ y_n & \cdots & 0 & 0 \end{bmatrix}$$

$$= \psi(x_1, \dots, x_n)^*\psi(y_1, \dots, y_n).$$

By the universal property of the crossed product, the covariant pair (φ, ψ) extends to a $*$-morphism $\varphi \rtimes \psi \colon A \rtimes X \to M_n(C)$, which is injective because φ is injective. Then, to end the proof, it suffices to show that $\operatorname{Im} \varphi \rtimes \psi = Z^*Z$.

We calculate that $Z^*Z \subseteq M_n(C)$ by adopting the following matrix notation:

$$Z^*Z = [E_{ij}]_{i,j=1}^n, \quad \text{where } E_{ij} = (BX^{i-1})^*(BX^{j-1}) \subseteq C, i, j = 1, \dots, n.$$

Simplifying the expressions of the E_{ij}'s, we get

$$E_{ij} = (BX^{i-1})^*(BX^{j-1}) = X^{1-i}BX^{j-1}.$$

Note that

$$E_{ii} = X^{1-i}BX^{i-1} \supseteq X^{1-i}A_1X^{i-1} = X^{1-i}X^{i-1}A_i = A_i \quad \text{for } i = 1, \dots, n,$$
$$E_{ii+1} = E_{ii}X \supseteq A_iX = X_i \quad \text{for } i = 1, \dots, n-1$$

and that

$$E_{n1} = X^{1-n}BX^0 \supseteq X^{1-n}(A_1X^n)A = A_nX^{1-n}X^n = A_nX = X_n.$$

Then we see that $\operatorname{Im}\varphi \subseteq Z^*Z$ and $\operatorname{Im}\psi \subseteq Z^*Z$. As $\operatorname{Im}\varphi \rtimes \psi = C^*(\operatorname{Im}\varphi \cup \operatorname{Im}\psi)$, we conclude that $\operatorname{Im}\varphi \rtimes \psi \subseteq Z^*Z$.

To prove the reverse inclusion, denote $D = \operatorname{Im}\varphi \rtimes \psi = C^*(\operatorname{Im}\varphi \cup \operatorname{Im}\psi)$,

$$\operatorname{Im}\varphi = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{bmatrix} \subseteq D$$

and

$$\widetilde{X} = \operatorname{Im}\psi = \begin{bmatrix} 0 & X_1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & X_{n-1} \\ X_n & \cdots & 0 & 0 \end{bmatrix} \subseteq D.$$

Note that

$$\begin{bmatrix} 0 & X_1 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & X_2 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & X_{n-1} \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ X_n & \cdots & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} X_1 \cdots X_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1X^n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \subseteq D.$$

Then, with $\left[\begin{smallmatrix} A_1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{smallmatrix}\right] \subseteq D$ and $\left[\begin{smallmatrix} A_1X^n & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{smallmatrix}\right] \subseteq D$, we can generate $\widetilde{E}_{11} = \left[\begin{smallmatrix} E_{11} & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{smallmatrix}\right] \subseteq D$ because $E_{11} = B = A_1[A_1X^n]$. Now, for $k, l = 0, \dots, n-1$ we have that the product $\widetilde{X}^{-k}\widetilde{E}_{11}\widetilde{X}^{l} \subseteq D$ is the space that, with the matrix notation, has $X_k^* \cdots X_1^* B X_1 \cdots X_l = X^{-k}BX^l = E_{1+k\,1+l}$ at the $(l+1, k+1)$-entry and 0 elsewhere. As $(k+1, l+1)$ ranges over all entries when $k, l = 0, \dots, n-1$, we conclude that $Z^*Z = [E_{ij}]_{i,j=1}^n \subseteq D$, as desired. □

Remark 3.2. For the special case of Theorem 3.1 in which $A_i = A$ and $X_i = X$ for $i = 1, \dots, n$, we obtain $A^n \rtimes X_\sigma^n \sim A \rtimes X^{\otimes n}$. As pointed out in the Introduction, this can be viewed as a generalization to the C^*-bimodule context of Green's theorem (see [5, Theorem 4.22]) for the cases in which $G = \mathbb{Z}$ and $H = n\mathbb{Z}$. That is, if $\alpha\colon A \to A$ is a $*$-automorphism, taking X as the trivial Hilbert bimodule ${}_AA_A$ twisted by α, the equivalence $A^n \rtimes X_\sigma^n \sim A \rtimes X^{\otimes n}$ becomes $C_0(G/H, A) \rtimes G \sim A \rtimes_{\alpha|_H} H$, where α also denotes the action of $\mathbb{Z}$ generated by the automorphism.

Corollary 3.3. *Let ${}_AX_B$ and ${}_BY_A$ be full right Hilbert bimodules. Then*

$$A \rtimes (X \otimes Y) \sim B \rtimes (Y \otimes X).$$

Proof. Note that the twisted sums $(X\oplus Y)_\sigma$ and $(Y\oplus X)_\sigma$ are isomorphic Hilbert bimodules. Indeed, the pair (φ,ψ) where $\varphi\colon A\oplus B\to B\oplus A$, $\varphi(a,b)=(b,a)$, and $\psi\colon X\oplus Y\to Y\oplus X$, $\psi(x,y)=(y,x)$, is an isomorphism. Then, the corresponding crossed products are isomorphic as well. Therefore

$$A\rtimes(X\otimes Y)\sim(A\oplus B)\rtimes(X\oplus Y)_\sigma\cong(B\oplus A)\rtimes(Y\oplus X)_\sigma\sim B\rtimes(Y\otimes X),$$

where we applied Theorem 3.1 twice for $n=2$. □

Corollary 3.4 ([2, Theorem 4.1])**.** *Let ${}_AX_A$ and ${}_BY_B$ be full right Hilbert bimodules, and let ${}_AM_B$ be an equivalence bimodule such that $X\otimes M\cong M\otimes Y$. Then*

$$A\rtimes X\sim B\rtimes Y.$$

Proof. As M is an equivalence, we have $A\cong M\otimes M^*$, where A is considered as the trivial Hilbert A-A bimodule and M^* denotes the conjugated bimodule of M. Then $X\cong A\otimes X\cong M\otimes M^*\otimes X$. Besides, $M^*\otimes X\otimes M\cong Y$ by hypothesis. Then, as all these isomorphisms give isomorphic crossed products, we have

$$A\rtimes X\cong A\rtimes(A\otimes X)\cong A\rtimes(M\otimes M^*\otimes X)\sim B\rtimes(M^*\otimes X\otimes M)\cong B\rtimes Y,$$

where we applied Corollary 3.3 to commute M and $M^*\otimes X$. □

Remark 3.5. In [1] the augmented Cuntz–Pimsner C^*-algebra $\widetilde{\mathcal{O}}_X$ associated to an A-A correspondence X (see [4]) is described as a crossed product $A_\infty\rtimes X_\infty$, where X_∞ is a Hilbert A_∞-A_∞ bimodule constructed out of the A-A correspondence X. Then, by combining this description with [2, Theorem 4.1] (Corollary 3.4 here), we can show an analogue of this theorem in the context of augmented Cuntz–Pimsner C^*-algebras (see [1, Theorem 4.7]). We believe that using similar techniques to those of [1], it is possible to obtain versions of Theorem 3.1 and Corollary 3.3 for augmented Cuntz–Pimsner algebras. For example, the corresponding version of Corollary 3.3 should establish that $\widetilde{\mathcal{O}}_{X\otimes Y}\sim\widetilde{\mathcal{O}}_{Y\otimes X}$ for full correspondences ${}_AX_B$ and ${}_BY_A$.

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