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ON COHOMOLOGY FOR PRODUCT SYSTEMS

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ABSTRACT. A cohomology for product systems of Hilbert bimodules is defined via the Ext functor. For the class of product systems corresponding to irreversible algebraic dynamics, relevant resolutions are found explicitly and it is shown how the underlying product system can be twisted by 2-cocycles. In particular, this process gives rise to cohomological deformations of the C^*-algebras associated with the product system. Concrete examples of deformations of the Cuntz's algebra $\mathcal{Q}_{\mathbb{N}}$ arising this way are investigated, and we show that they are simple and purely infinite.

1. Introduction

The application of cohomology to deformations of C^*-algebras and von Neumann algebras has been studied for decades, and yet it remains an active area of research in this field. Among the most recent contributions, we would like to mention the work of Buss and Exel on inverse semigroups in [3] and of Kumjian, Pask, and Sims on higher-rank graphs in [14]. Often the deformation of the C^*-algebra is related to a cohomological perturbation of another underlying object. A typical example of such a process comes from a *twisted* (semi)group action leading to the *twisted* crossed product.

In the present article, we introduce a cohomology theory for product systems of Hilbert bimodules over discrete semigroups as defined by Fowler in [10]. Interestingly, better understanding of twisting of semigroup actions was one of the motivations behind the very introduction of such product systems. In Section 3,

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we take the classical point of view as found in [1], and we define cohomology groups of a product system X via the Ext functor applied to a suitable module $\mathfrak{M}$ of a ring $\mathfrak{R}$ naturally associated with X. First examples include cohomologies of groups, graphs, and certain product systems arising from semigroup actions on abelian groups.

In Section 4, we restrict our attention to a certain class of product systems arising from irreversible algebraic dynamics, corresponding to actions of discrete semigroups P on compact groups. For such product systems, we construct explicitly a free resolution of module $\mathfrak{M}$ and thus obtain working formulas for cocycles and coboundaries. The construction of the resolution takes advantage of the fact that all fibers X_p of these systems $X = \bigsqcup_{p\in P} X_p$ are free modules over the coefficient C^*-algebra A. To each 2-cocycle ξ, we associate a twisted product system X^ξ. The twisting is obtained by perturbing multiplication between the fibers. Then each twisted product system X^ξ gives rise to several C^*-algebras, including the Toeplitz algebra $\mathcal{T}(X^\xi)$ and the Cuntz–Pimsner algebra $\mathcal{O}(X^\xi)$. These algebras may be considered twisted versions of the Toeplitz algebra $\mathcal{T}(X)$ and the Cuntz–Pimsner algebra $\mathcal{O}(X)$, respectively, associated with the original product system X.

In Section 5, we test this deformation procedure on the product system X whose Cuntz–Pimsner algebra $\mathcal{O}(X)$ coincides with Cuntz's algebra $\mathcal{Q}_{\mathbb{N}}$ associated to the $ax+b$-semigroup over $\mathbb{N}$ (see [5], [11]). We look at certain numerical 2-cocycles ξ and we show that the corresponding twisted C^*-algebras $\mathcal{O}(X^\xi)$ are purely infinite and simple.

2. Preliminaries on product systems

Let A be a C^*-algebra and X be a complex vector space with a right action of A. Suppose that there is an A-valued inner product $\langle\cdot,\cdot\rangle_A$ on X which is conjugate linear in the first variable and which satisfies

(1) $\langle\xi,\eta\rangle_A = \langle\eta,\xi\rangle_A^*$,
(2) $\langle\xi,\eta\cdot a\rangle_A = \langle\xi,\eta\rangle_A a$,
(3) $\langle\xi,\xi\rangle_A \geq 0$, and $\langle\xi,\xi\rangle_A = 0 \Longleftrightarrow \xi = 0$,

for $\xi,\eta\in X$ and for $a\in A$. Then X becomes a right Hilbert A-module when it is complete with respect to the norm given by $\|\xi\| := \|\langle\xi,\xi\rangle_A\|^{1/2}$ for $\xi\in X$.

A module map $T : X \to X$ is said to be *adjointable* if there is a map $T^* : X \to X$ such that

$$\langle T\xi,\zeta\rangle_A = \langle\xi,T^*\zeta\rangle_A$$

for all $\xi,\eta\in X$. An adjointable map is automatically norm-bounded, and the set $\mathcal{L}(X)$ of all adjointable operators on X endowed with the operator norm is a C^*-algebra. The rank 1 operator $\theta_{\xi,\eta}$ defined on X as

$$\theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle_A \quad \text{for } \xi,\eta,\zeta\in X$$

is adjointable, and we have $\theta_{\xi,\eta}^* = \theta_{\eta,\xi}$. Then $\mathcal{K}(X) = \overline{\operatorname{span}}\{\theta_{\xi,\eta} \mid \xi,\eta\in X\}$ is the ideal of compact operators in $\mathcal{L}(X)$.

Suppose that X is a right Hilbert A-module. A $*$-homomorphism $\varphi : A \to \mathcal{L}(X)$ induces a left action of A on X by $a\xi := \varphi(a)\xi$, for $a\in A$ and for $\xi\in X$. Then

X becomes a right Hilbert A–A-bimodule (or C^*-correspondence over A). The standard bimodule ${}_AA_A$ is equipped with $\langle a, b\rangle_A = a^*b$, and the right and left actions are simply given by right and left multiplication in A, respectively.

For right Hilbert A–A-bimodules X and Y, the balanced tensor product $X \otimes_A Y$ becomes a right Hilbert A–A-bimodule with the right action from Y, the left action implemented by the homomorphism $A \ni a \mapsto \varphi(a) \otimes_A \mathrm{id}_Y \in \mathcal{L}(X \otimes_A Y)$, and the A-valued inner product given by

$$\langle \xi_1 \otimes_A \eta_1, \xi_2 \otimes_A \eta_2\rangle_A = \big\langle \eta_1, \langle \xi_1, \xi_2\rangle_A \cdot \eta_2\big\rangle_A$$

for $\xi_i \in X$ and $\eta_i \in Y$, $i = 1, 2$.

Let P be a multiplicative semigroup with identity e, and let A be a C^*-algebra. Throughout this article, we make a standing assumption that all semigroups we consider are *right cancellative*. For each $p \in P$, let X_p be a complex vector space. Then the disjoint union $X := \bigsqcup_{p\in P} X_p$ is a *product system* over P if the following conditions hold.

(PS1) For each $p \in P \setminus \{e\}$, X_p is a right Hilbert A–A-bimodule.
(PS2) X_e is the standard bimodule ${}_AA_A$.
(PS3) X is a semigroup such that $\xi\eta \in X_{pq}$ for $\xi \in X_p$ and $\eta \in X_q$, $p, q \in P \setminus \{e\}$. It is assumed in this particular case that this product extends to an isomorphism $F^{p,q} : X_p \otimes_A X_q \to X_{pq}$ of right Hilbert A–A-bimodules. If p or q equals e, then the corresponding product in X is induced by the left or the right action of A, respectively.

Remark 2.1. For $p \in P$ there are maps $F^{p,e} : X_p \otimes_A X_e \to X_p$ and $F^{e,p} : X_e \otimes_A X_p \to X_p$ by multiplication (i.e., $F^{p,e}(\xi \otimes a) = \xi a$ and $F^{e,p}(a \otimes \xi) = a\xi$ for $a \in A$ and $\xi \in X_p$). Note that $F^{p,e}$ is always an isomorphism. However, $F^{e,p}$ is an isomorphism only if $\overline{\varphi(A)X_p} = X_p$ or, in the terminology from [10], if X_p is "essential." In all interesting examples we have come across, X_p is essential for all p.

For each $p \in P$, we denote by $\langle \cdot, \cdot\rangle_p$ the A-valued inner product on X_p and by φ_p the $*$-homomorphism from A into $\mathcal{L}(X_p)$. Due to associativity of the multiplication on X, we have $\varphi_{pq}(a)(\xi\eta) = (\varphi_p(a)\xi)\eta$ for all $\xi \in X_p$, $\eta \in X_q$, and $a \in A$. For each pair $p, q \in P \setminus \{e\}$, the isomorphism $F^{p,q} : X_p \otimes_A X_q \to X_{pq}$ allows us to define a $*$-homomorphism $i_p^{pq} : \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$ by $i_p^{pq}(S) = F^{p,q}(S \otimes_A I_q)(F^{p,q})^*$ for $S \in \mathcal{L}(X_p)$. In the case $r \neq pq$, we define $i_p^r : \mathcal{L}(X_p) \to \mathcal{L}(X_r)$ to be the zero map $i_p^r(S) = 0$ for all $S \in \mathcal{L}(X_p)$. Further, we let $i_e^q = \varphi_q$.

Let $X = \bigsqcup_{p\in P} X_p$ be a product system over P of right Hilbert A–A-bimodules. A map ψ from X to a C^*-algebra C is a *Toeplitz representation* of X if the following conditions hold:

(T1) for each $p \in P \setminus \{e\}$, $\psi_p := \psi|_{X_p}$ is linear,
(T2) $\psi_e : A \to C$ is a $*$-homomorphism,
(T3) $\psi_p(\xi)\psi_q(\eta) = \psi_{pq}(\xi\eta)$ for $\xi \in X_p$, $\eta \in X_q$, $p, q \in P$,
(T4) $\psi_p(\xi)^*\psi_p(\eta) = \psi_e(\langle \xi, \eta\rangle_p)$ for $\xi, \eta \in X_p$.

We separated condition (T2) for emphasis only. In fact, assuming (T1) for all $p \in P$, condition (T2) follows from (T1), (T3) and (T4).

For each $p \in P$ there exists a $*$-homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \to C$ such that $\psi^{(p)}(\theta_{\xi,\eta}) = \psi_p(\xi)\psi_p(\eta)^*$, for $\xi, \eta \in X_p$. A Toeplitz representation ψ is *Cuntz–Pimsner covariant* (see [10]) if

(CP) $\psi^{(p)}(\varphi_p(a)) = \psi_e(a)$ for $a \in \varphi_p^{-1}(\mathcal{K}(X_p))$ and all $p \in P$.

The Toeplitz algebra $\mathcal{T}(X)$ associated to the product system X was defined by Fowler as the universal C^*-algebra for Toeplitz representations (see [10]). The Cuntz–Pimsner algebra $\mathcal{O}(X)$ is universal for the Cuntz–Pimsner covariant Toeplitz representations. A number of other related constructions exist in the literature, but we do not discuss them here. However, we would like to mention couniversal algebras studied by Carlsen, Larsen, Sims, and Vittadello in [4], and reduced Cuntz–Pimsner algebras investigated by Kwaśniewski and Szymański in [15].

3. A cohomology for product systems

Let X be a product system of Hilbert bimodules over a semigroup P and with the coefficient (unital) C^*-algebra A. Then the direct sum of A–A-bimodules

$$\mathfrak{R} := \bigoplus_{p \in P} X_p \tag{3.1}$$

becomes a ring graded over P with the multiplication borrowed from X. We assume that there exists a unital left A-module map $\Psi : \mathfrak{R} \to A$ such that

$$\Psi(xy) = \Psi\big(x\Psi(y)\big), \tag{3.2}$$

for all $x, y \in \mathfrak{R}$. Then $A = X_e$ becomes a left $\mathfrak{R}$-module, with the $\mathfrak{R}$-action $\rightharpoonup$ given by the composition of the multiplication in $\mathfrak{R}$ with Ψ; that is,

$$x \rightharpoonup a := \Psi(xa), \tag{3.3}$$

for $x \in \mathfrak{R}$, $a \in A$. We denote this module $\mathfrak{M}$ and we define the nth-cohomology group of the product system X relative to Ψ as

$$H^n_\Psi(X) := \operatorname{Ext}^n_{\mathfrak{R}}(\mathfrak{M}, \mathfrak{M}) \tag{3.4}$$

(see [1]). Before describing some examples, we want to point out that such a map Ψ always exists—for example, one may take the canonical projection from $\mathfrak{R}$ onto A. However, not every choice of Ψ may lead to interesting cohomology $H^*_\Psi(X)$.

Example 3.1. Let E be a *finite* directed graph, with vertices E^0, edges E^1, and range and source mappings $r : E^1 \to E^0$ and $s : E^1 \to E^0$, respectively. Let X_1 be the standard Hilbert bimodule associated with E (see [13]), with the finite-dimensional coefficient algebra A generated by vertex projections. Let X be the product system over the additive semigroup $\mathbb{N}$ generated by X_1. For each $n \in \mathbb{N}$, X_n is the $\mathbb{C}$-span of paths of length n (paths of length zero being vertices). Multiplication in ring $\mathfrak{R}$ is simply given by concatenation of directed paths. For a path μ, we set $\Psi(\mu) := s(\mu)$. Then for a path μ and a vertex v, we have $\mu \rightharpoonup v = s(\mu)$ if $v = r(\mu)$, and 0 otherwise.

Example 3.2. Let G be a countable group. We set $P = G$ and $X_g = \mathbb{C}g$ for all $g \in G$. Then X is a product system with the usual group algebra multiplication and the inner products $\langle zg, wg\rangle_g = \overline{z}w1$ for $g \in G$ and $z, w \in \mathbb{C}$. We have $\mathfrak{R} = \mathbb{C}G$, the usual complex group algebra. In this case, Ψ is the trivial representation of $\mathbb{C}G$ and $\mathfrak{M}$ is the trivial module.

Example 3.3. Here we consider a product system studied in [16] in connection with Exel's approach to semigroup crossed products via transfer operator in [8], and studied in [18] and [11] in connection with Cuntz's algebra $\mathcal{Q}_{\mathbb{N}}$ in [5]. The product system X is over the multiplicative semigroup $\mathbb{N}^\times$. The coefficient algebra A is $C(\mathbb{T})$, and each fiber X_p is a free left A-module of rank 1 with a basis vector $\mathbb{1}_p$. The right action of A is determined by $\mathbb{1}_p a = \alpha_p(a)\mathbb{1}_p$, where $\alpha_p : A \to A$ is an endomorphism such that $\alpha_p(a)(z) = a(z^p)$ for $a \in A$ and $z \in \mathbb{T}$. The inner product in fiber X_p is given by $\langle a\mathbb{1}_p, b\mathbb{1}_p\rangle_p = L_p(a^*b)$, where $L_p : A \to A$ is a transfer operator for α_p such that $L_p(a)(z) = \frac{1}{p}\sum_{w^p=z} a(w)$. Fibers are multiplied according to the rule $(a\mathbb{1}_p)(b\mathbb{1}_q) = a\alpha_p(b)\mathbb{1}_{pq}$.

It was shown in [11, Lemma 3.1] that the left action of A on each fiber is by compact operators. In fact, this product system belongs to the class of singly generated product systems of finite type, as introduced in [12, Definition 3.5]. We set $\Psi(a\mathbb{1}_p) := a$, for $p \in \mathbb{N}^\times$ and $a \in A$. Then the action of $\mathfrak{R}$ on $\mathfrak{M}$ is determined by $\mathbb{1}_p \rightharpoonup a = \alpha_p(a)$, for $p \in \mathbb{N}^\times$ and $a \in \mathfrak{M}$.

4. Irreversible algebraic dynamics

In this section, we consider irreversible dynamical systems corresponding to injective homomorphisms of abelian groups. We follow the approach of Stammeier, [17] (see also [2]), building on the work of Exel and Vershik [9] and of Cuntz and Vershik [6].

Let G be a countable abelian group, and let P be a semigroup with identity e. Let θ be an action of P on G by injective group homomorphisms. We denote by $A := C^*(G)$ the group C^*-algebra of G. For each $p \in P$, let X_p be a free left A-module of rank 1 with a basis element $\mathbb{1}_p$. The right action of A on X_p is determined by $\mathbb{1}_p a = \theta_p(a)\mathbb{1}_p$, $a \in A$. The inner product in X_p is defined as

$$\langle a\mathbb{1}_p, b\mathbb{1}_p\rangle_p := \theta_p^{-1}E_p(a^*b), \tag{4.1}$$

for $p \in \mathbb{N}^\times$ and $a, b \in A$. Here $E_p : C^*(G) \to C^*(\theta_p(G))$ is the conditional expectation given by restriction. For $a = g$ and $b = h$ with $g, h \in G$, this yields

$$\langle g\mathbb{1}_p, h\mathbb{1}_p\rangle_p = \begin{cases} \theta_p^{-1}(g^{-1}h) & \text{if } g^{-1}h \in \theta_p(G), \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

If index $[G : \theta_p(G)]$ is finite, then in the dual picture, with $\hat{\theta}_p$ acting on $C(\widehat{G})$, this inner product corresponds to the transfer operator given by averaging over the finitely many inverse image points (see [17]). Finally, fibers are multiplied according to the rule

$$(a\mathbb{1}_p)(b\mathbb{1}_q) = a\theta_p(b)\mathbb{1}_{pq}. \tag{4.3}$$

In this case, ring $\mathfrak{R}$ is the skew product $\mathbb{Z}G \rtimes_\theta P$, with multiplication

$$(gp)(hq) = \big(g\theta_p(h)\big)(pq),$$

$g, h \in G$, $p, q \in P$. We take $\Psi(g\mathbb{1}_p) := g$, $g \in G$, $p \in P$. Then the action of $\mathfrak{R}$ on $\mathfrak{M}$ is given by

$$(g\mathbb{1}_p) \rightharpoonup h = g\theta_p(h),$$

$g, h \in G$, $p \in P$. Example 3.3 from Section 3 arises as a special case of this construction.

Now, we describe an acyclic, free resolution of the $\mathfrak{R}$-module $\mathfrak{M}$. To this end, we define a complex of $\mathfrak{R}$-modules and maps

$$\cdots \xrightarrow{\partial_2} \mathfrak{F}_2 \xrightarrow{\partial_1} \mathfrak{F}_1 \xrightarrow{\partial_0} \mathfrak{F}_0 \xrightarrow{\partial_{-1}} \mathfrak{M} \longrightarrow 0, \tag{4.4}$$

as follows. We let $\mathfrak{F}_0$ be a free left $\mathfrak{R}$-module of rank 1 with a basis element $[\,]$. For $n \geq 1$, we let $\mathfrak{F}_n$ be a free left $\mathfrak{R}$-module with a basis

$$\big\{[p_1, \ldots, p_n] \bigm| p_k \in P, k = 1, \ldots, n\big\}. \tag{4.5}$$

The maps ∂_* are defined as $\mathfrak{R}$-module homomorphisms such that

$$\begin{aligned}\partial_{-1}\big([\,]\big) &= \mathbb{1}_1,\\ \partial_0\big([p]\big) &= (\mathbb{1}_p - \mathbb{1}_1)[\,],\end{aligned}$$

and for $n \geq 2$, we set

$$\begin{aligned}\partial_{n-1}\big([p_1, \ldots, p_n]\big) = {}& \mathbb{1}_{p_1}[p_2, \ldots, p_n]\\ &+ \sum_{i=1}^{n-1}(-1)^i[p_1, \ldots, p_{i-1}, p_ip_{i+1}, p_{i+2}, \ldots, p_n]\\ &+ (-1)^n[p_1, \ldots, p_{n-1}].\end{aligned}$$

A routine calculation shows that

$$\partial_n\partial_{n+1} = 0$$

for all $n \geq -1$.

To show that complex (4.4) is acyclic, we construct splitting homotopies. That is, we define abelian group homomorphisms $h_{-1} : \mathfrak{M} \to \mathfrak{F}_0$ and $h_n : \mathfrak{F}_n \to \mathfrak{F}_{n+1}$, $n \geq 0$, such that

$$\begin{aligned}\partial_{-1}h_{-1} &= \mathrm{id}_{\mathfrak{M}},\\ \partial_nh_n + h_{n-1}\partial_{n-1} &= \mathrm{id}_{\mathfrak{F}_n} \quad \text{for } n \geq 0.\end{aligned}$$

For example, we may take

$$\begin{aligned}h_{-1}(a) &= a[\,],\\ h_0\big(a\mathbb{1}_p[\,]\big) &= a[p],\\ h_n\big(a\mathbb{1}_{p_0}[p_1, \ldots, p_n]\big) &= a[p_0, p_1, \ldots, p_n], \quad n \geq 1,\end{aligned}$$

for $a \in C^*(G)$. Now, applying the $\mathrm{Hom}_{\mathfrak{R}}(*, \mathfrak{M})$ functor to chain complex (4.4), with $\mathfrak{M}$ deleted, we obtain the following complex of homogeneous cochains:

$$0 \longrightarrow \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{F}_0, \mathfrak{M}) \xrightarrow{\partial_0^*} \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{F}_1, \mathfrak{M}) \xrightarrow{\partial_1^*} \cdots . \tag{4.6}$$

By definition, we have

$$H^n_\Psi(X) = \frac{\ker(\partial^*_n)}{\mathrm{im}(\partial^*_{n-1})}. \tag{4.7}$$

Restricting in (4.6) elements of $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{F}_n, \mathfrak{M})$ to the basis (4.5) of the free $\mathfrak{R}$-module $\mathfrak{F}_n$, we obtain the following complex of inhomogeneous cochains:

$$0 \longrightarrow C^0(P, \mathfrak{M}) \xrightarrow{\partial^0} C^1(P, \mathfrak{M}) \xrightarrow{\partial^1} C^2(P, \mathfrak{M}) \xrightarrow{\partial^2} \cdots . \tag{4.8}$$

Here we denote

$$\begin{aligned} C^0(P, \mathfrak{M}) &= \mathfrak{M}, \\ C^n(P, \mathfrak{M}) &= \{\xi : P^n \to \mathfrak{M}\}, \quad n \geq 1. \end{aligned}$$

The cochain maps are

$$\begin{aligned} \partial^0(a)(p) &= \theta_p(a) - a, \\ \partial^n(\xi)(p_1, \ldots, p_{n+1}) &= \theta_{p_1}\big(\xi(p_2, \ldots, p_{n+1})\big) \\ &\quad + \sum_{i=1}^{n} (-1)^i \xi(p_1, \ldots, p_{i-1}, p_i p_{i+1}, p_{i+2}, \ldots, p_{n+1}) \\ &\quad + (-1)^{n+1} \xi(p_1, \ldots, p_n), \end{aligned}$$

for $n \geq 1$, $a \in \mathfrak{M}$, $\xi \in C^n(P, \mathfrak{M})$, p and $p_1, \ldots, p_{n+1} \in P$. We have

$$H^n_\Psi(X) \cong \frac{\ker(\partial^n)}{\mathrm{im}(\partial^{n-1})}. \tag{4.9}$$

Now, let $\xi : P \times P \to A_{\mathrm{sa}}$ be a normalized (i.e. $\xi(p,q) = 0$ if $p = 1$ or $q = 1$) 2-cocycle with self-adjoint values. We define a new product system X^ξ over P and with coefficients in A, as follows. For each $p \in P$, fiber X^ξ_p coincides with X_p (but we denote the generator by $\mathbb{1}^\xi_p$ to avoid confusion). However, the multiplication between fibers is twisted by ξ according to the rule

$$(a\mathbb{1}^\xi_p)(b\mathbb{1}^\xi_q) := \exp\big(i\xi(p,q)\big) a \theta_p(b) \mathbb{1}^\xi_{pq}. \tag{4.10}$$

It is not difficult to verify that X^ξ satisfies axioms (PS1)–(PS3) of a product system as given in our preceding Section 2. Consequently, the corresponding Toeplitz and Cuntz–Pimsner algebras $\mathcal{T}(X^\xi)$ and $\mathcal{O}(X^\xi)$, respectively, may be considered as ξ-twisted versions of $\mathcal{T}(X)$ and $\mathcal{O}(X)$, respectively.

Proposition 4.1. *Let ξ, η be normalized, self-adjoint 2-cocycles such that $[\xi] = [\eta]$ in $H^2_\Psi(X)$. Then the corresponding twisted product systems X^ξ and X^η are isomorphic.*

Proof. By hypothesis, there is a $\psi : P \to \mathfrak{M}$ such that $\xi - \eta = \partial^1(\psi)$. Replacing ψ with $\frac{1}{2}(\psi + \psi^*)$ if necessary, we may assume that $\psi(p)$ is self-adjoint for all $p \in P$. Define a map $X^\xi \to X^\eta$ so that $a\mathbb{1}^\xi_p \mapsto \exp(i\psi(p)) a \mathbb{1}^\eta_p$ for all $p \in P$, $a \in A$. One easily verifies that this map yields the required isomorphism between X^ξ and X^η. $\square$

5. Twisted $\mathcal{Q}_{\mathbb{N}}$

In this section, we apply the twisting procedure described in Section 4 to the product system X discussed in Example 3.3 from Section 3. We begin by having a quick look at the 0-, 1-, and 2-cohomology groups.

The 0-cohomology is clear. Indeed, it follows from (4.9) that we simply have

$$\begin{aligned} H^0_\Psi(X) &= \left\{ a \in A \;\middle|\; \alpha_p(a) = a, \forall p \in \mathbb{N}^\times \right\} \\ &= \mathbb{C}1. \end{aligned}$$

To define a normalized 1-cocycle $\xi : \mathbb{N}^\times \to A = C(\mathbb{T})$, let $\xi(1) = 0$ and $\xi(p) \in A$ be arbitrary for each *prime* $p \in \mathbb{N}^\times$. Suppose that $1 \neq q \in \mathbb{N}^\times$ have prime factorization $q = p_1 \cdots p_m$, with $p_1 \leq p_2 \leq \cdots \leq p_m$. Proceeding by induction on m, define $\xi(q) := \alpha_{q/p_m}(\xi(p_m)) + \xi(q/p_m)$. Then $\xi : q \mapsto \xi(q)$, $q \in \mathbb{N}^\times$, is a 1-cocycle, and in fact all normalized 1-cocycles arise this way. For ξ to be a 1-coboundary, there must exist a function $\psi \in C(\mathbb{T})$ such that, for all prime $p \in \mathbb{N}^\times$ and all $z \in \mathbb{T}$, we have

$$\psi(z) = \psi(z^p) - \xi(p)(z).$$

To construct such a ψ, fix a prime p for a moment and define $\psi(z)$ for $z \in \mathbb{T}$ such that $z^{p^k} = 1$, by induction on k, as follows:

$$\begin{aligned} &\psi(1) := 0, \\ &\psi(z) := \psi(z^p) - \xi(p)(z). \end{aligned}$$

In this way, ψ is densely defined on $\mathbb{T}$ at all roots of unity. It follows that ξ is a 1-coboundary if and only if ψ can be extended to a continuous function on the entire circle $\mathbb{T}$.

For a 2-cocycle $\xi : \mathbb{N}^\times \times \mathbb{N}^\times \to A$, suppose that $\psi : \mathbb{N}^\times \to A$ is such that $\xi = \partial^1(\psi)$. Then for any two primes p, q, we must have

$$\begin{aligned} \psi(pq) &= \alpha_p\big(\psi(q)\big) + \psi(p) - \xi(p,q) \\ &= \alpha_q\big(\psi(p)\big) + \psi(q) - \xi(q,p), \end{aligned}$$

and hence

$$\big(\psi(q)(z^p) - \psi(q)(z)\big) - \big(\psi(p)(z^q) - \psi(p)(z)\big) = \xi(p,q) - \xi(q,p) \tag{5.1}$$

for all $z \in \mathbb{T}$. Thus, for ξ to give a nonzero element in $H^2_\Psi(X)$, it suffices to have $\xi(p,q)(1) \neq \xi(q,p)(1)$ for some primes p and q. For a more specific example, let $\xi : \mathbb{N}^\times \times \mathbb{N}^\times \to \mathbb{C}1$ be a map such that

$$\xi(mn,k) = \xi(m,k) + \xi(n,k) \qquad \text{and} \qquad \xi(m,nk) = \xi(m,n) + \xi(m,k). \tag{5.2}$$

Then ξ is a 2-cocycle. For example, given two distinct primes p and q and complex numbers a, b, c, d, we can set

$$\xi(mp^k q^l, np^r q^j) := (ak + bl)(cr + dj), \tag{5.3}$$

with m, n relatively prime with both p and q. By the above, if $ad \neq bc$, then ξ is not a coboundary.

Let $\xi : \mathbb{N}^\times \times \mathbb{N}^\times \to \mathbb{R}1$ be a 2-cocycle defined in (5.3), with a, b, c, d real numbers. We denote by u the standard unitary generator of $A = C(\mathbb{T})$ and for $m \in \mathbb{N}^\times$ we denote by s_m the canonical image of $\mathbb{1}_m^\xi$ in $\mathcal{Q}_\mathbb{N}^\xi := \mathcal{O}(X^\xi)$. (Of course, s_m depends also on ξ. We do not indicate this explicitly to lighten the notation.) Similarly to [5] and [11], each s_m is an isometry and the following relations hold:
(QX1) $s_m s_n = e^{i(ak+bl)(cr+dj)} s_{mn}$,
(QX2) $s_m u^l = u^{ml} s_m$, for all $l \in \mathbb{Z}$,
(QX3) $\sum_{k=0}^{m-1} u^k s_m s_m^* u^{-k} = 1$,
where k, r are the numbers of p-factors of m and n, respectively, and l, j are the numbers of q-factors of m and n, respectively.

Proposition 5.1. *The C^*-algebra $\mathcal{Q}_\mathbb{N}^\xi$ is simple.*

A proof of simplicity of $\mathcal{Q}_\mathbb{N}^\xi$, claimed in Proposition 5.1 above, may be given as an application of [15, Theorem 5.10]. This requires showing minimality and topological aperiodicity (in the sense of Definition 5.7 and Definition 5.3 of [15], respectively) of the underlying product system X^ξ. Since both proofs are essentially the same as those from [15, Section 6.5] (treating the case of untwisted $\mathcal{Q}_\mathbb{N}$), we omit the details.

We want to investigate the structure of C^*-algebra $\mathcal{Q}_\mathbb{N}^\xi$ a little bit further. To this end, we note that X^ξ is a *regular* product system (i.e., the left action φ_m on each fiber X_m^ξ is injective and by compacts; see [15, Definition 3.1]) over an Ore semigroup $\mathbb{N}^\times$. Thus, it follows from a very general argument (see [15, Lemma 3.7]) that

$$\mathcal{Q}_\mathbb{N}^\xi = \overline{\mathrm{span}}\{a s_m s_n^* b \mid m, n \in \mathbb{N}^\times, a, b \in A\}.$$

Furthermore,

$$\mathcal{F}_\mathbb{N}^\xi := \overline{\mathrm{span}}\{a s_m s_m^* b \mid m \in \mathbb{N}^\times, a, b \in A\}$$

is a unital $*$-subalgebra of $\mathcal{Q}_\mathbb{N}^\xi$. Since the ξ-twist does not affect $\mathcal{F}_\mathbb{N}^\xi$, this algebra is unchanged by introduction of the cocycle. In fact, as shown by Cuntz in [5, Section 3], it is a simple Bunce–Deddens algebra with a unique trace.

In the present situation, since the enveloping group $\mathbb{Q}_+^\times$ of $\mathbb{N}^\times$ is amenable, $\mathcal{Q}_\mathbb{N}^\xi = \mathcal{O}(X^\xi)$ coincides with the reduced algebra $\mathcal{O}((X^\xi)^r)$ (see [7] and [15]) and with the couniversal algebra $\mathcal{NO}_{X^\xi}^r$ (see [4]). Thus, there exists a faithful conditional expectation $E : \mathcal{Q}_\mathbb{N}^\xi \to \mathcal{F}_\mathbb{N}^\xi$ onto $\mathcal{F}_\mathbb{N}^\xi$ such that for all $m, n \in \mathbb{N}^\times$, $a, b \in A$, we have

$$E(a s_m s_n^* b) = 0 \quad \text{if } m \neq n.$$

Let $\mathcal{D}_\mathbb{N}^\xi$ be the C^*-subalgebra of $\mathcal{F}_\mathbb{N}^\xi$ generated by all projections $u^k s_m s_m^* u^{-k}$; that is

$$\mathcal{D}_\mathbb{N}^\xi := \overline{\mathrm{span}}\{u^k s_m s_m^* u^{-k} \mid m \in \mathbb{N}^\times, k \in \mathbb{Z}\}.$$

Then, as in [5, Section 3], $\mathcal{D}_\mathbb{N}^\xi$ is commutative and there exists a faithful conditional expectation $F : \mathcal{F}_\mathbb{N}^\xi \to \mathcal{D}_\mathbb{N}^\xi$ onto $\mathcal{D}_\mathbb{N}^\xi$ such that for all $m \in \mathbb{N}^\times$, $k, l \in \mathbb{Z}$, we have

$$F(u^k s_m s_m^* u^{-l}) = 0 \quad \text{if } k \neq l.$$

The composition $G := F \circ E$ yields a faithful conditional expectation from $\mathcal{Q}^{\xi}_{\mathbb{N}}$ onto $\mathcal{D}^{\xi}_{\mathbb{N}}$. We also recall from [5, Lemma 3.2(a)] that for all $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}^{\times}$, we have

$$u^k s_m s_m^* u^{-k} = \sum_{j=0}^{n-1} u^{k+jm} s_{mn} s_{mn}^* u^{-k-jm}. \tag{5.4}$$

One immediate consequence of this identity is that

$$s_r^* u^t s_r = 0 \text{ unless } t \text{ is divisible by } r. \tag{5.5}$$

Another one is the identity

$$s_m s_m^* s_n s_n^* = s_{m\vee n} s_{m\vee n}^*, \tag{5.6}$$

where symbol $\vee$ denotes the least common multiple of two positive integers.

Lemma 5.2. *Let $k, l \in \mathbb{Z}$ and $m, n \in \mathbb{N}^{\times}$. Then*

$$u^k s_m s_m^* u^{-k} \leq u^l s_n s_n^* u^{-l}$$

if and only if both m and $k - l$ are divisible by n.

Proof. By (5.4), we have

$$u^k s_m s_m^* u^{-k} = \sum_{j=0}^{n-1} u^{k+jm} s_{mn} s_{mn}^* u^{-k-jm},$$

$$u^l s_n s_n^* u^{-l} = \sum_{j=0}^{m-1} u^{l+jn} s_{mn} s_{mn}^* u^{-l-jn}.$$

Thus, $u^k s_m s_m^* u^{-k} \leq u^l s_n s_n^* u^{-l}$ if and only if for each $j \in \{0, \ldots, n-1\}$ there is a $j' \in \{0, \ldots, m-1\}$ such that $k + jm = l + j'n$ in $\mathbb{Z}_{mn}$. This clearly implies the claim. $\square$

Now, we will show that C^*-algebra $\mathcal{Q}^{\xi}_{\mathbb{N}}$ is purely infinite, as in the untwisted case (see [5, Theorem 3.4]). Our proof imitates the classical argument of Cuntz, which was employed also in [6, Theorem 2.6], and it relies on the following technical lemma.

Lemma 5.3. *Let Q be a nonzero projection in $\mathcal{D}^{\xi}_{\mathbb{N}}$, and let $k_0, l_0 \in \mathbb{Z}$, $m_0, n_0 \in \mathbb{N}^{\times}$ be such that either $k_0 \neq l_0$ or $m_0 \neq n_0$. Then there exist $k \in \mathbb{Z}$ and $m \in \mathbb{N}^{\times}$ such that*

(1) $u^k s_m s_m^* u^{-k} \leq Q$, *and*
(2) $(u^k s_m s_m^* u^{-k})(u^{k_0} s_{m_0} s_{n_0}^* u^{-l_0})(u^k s_m s_m^* u^{-k}) = 0$.

Proof. By the definition of $\mathcal{D}^{\xi}_{\mathbb{N}}$, there exist $k' \in \mathbb{Z}$ and $m' \in \mathbb{N}^{\times}$ such that $u^{k'} s_{m'} s_{m'}^* u^{-k'} \leq Q$. Thus, it suffices to work with $u^{k'} s_{m'} s_{m'}^* u^{-k'}$ instead of Q.

If $u^{k'} s_{m'} s_{m'}^* u^{-k'}$ is not a subprojection of either $u^{k_0} s_{m_0} s_{m_0}^* u^{-k_0}$ or $u^{l_0} s_{n_0} s_{n_0}^* u^{-l_0}$, then to have (i) and (ii) satisfied it suffices to take $k \in \mathbb{Z}$ and $m \in \mathbb{N}^{\times}$ such that either $u^k s_m s_m^* u^{-k} \leq u^{k'} s_{m'} s_{m'}^* u^{-k'}(1 - u^{k_0} s_{m_0} s_{m_0}^* u^{-k_0})$ or $u^k s_m s_m^* u^{-k} \leq u^{k'} s_{m'} s_{m'}^* u^{-k'}(1 - u^{l_0} s_{n_0} s_{n_0}^* u^{-l_0})$, respectively.

Now, we may assume that $u^{k'}s_{m'}s_{m'}^*u^{-k'}$ is a subprojection of both $u^{k_0}s_{m_0}s_{m_0}^*u^{-k_0}$ and $u^{l_0}s_{n_0}s_{n_0}^*u^{-l_0}$. Thus, by virtue of Lemma 5.2, both m' and $k'-k_0$ are divisible by m_0, while both m' and $k'-l_0$ are divisible by n_0. Hence

$$\begin{aligned}&(u^{k'}s_{m'}s_{m'}^*u^{-k'})(u^{k_0}s_{m_0}s_{n_0}^*u^{-l_0})(u^{k'}s_{m'}s_{m'}^*u^{-k'})\\ &\quad= u^{k'}s_{m'}s_{m'}^*s_{m_0}u^{(k_0-k')/m_0-(l_0-k')/n_0}s_{n_0}^*s_{m'}s_{m'}^*u^{-k'}\end{aligned}$$

is a partial isometry with the domain projection

$$\begin{aligned}g = (u^{k'}s_{m'}s_{m'}^*u^{-k'})&u^{l_0-n_0(k_0-k')/m_0}s_{n_0(m'/m_0\vee m'/n_0)}\\ &\times s_{n_0(m'/m_0\vee m'/n_0)}^*u^{-(l_0-n_0(k_0-k')/m_0)}\end{aligned}$$

and the range projection

$$\begin{aligned}f = (u^{k'}s_{m'}s_{m'}^*u^{-k'})&u^{k_0-m_0(l_0-k')/n_0}s_{m_0(m'/m_0\vee m'/n_0)}\\ &\times s_{m_0(m'/m_0\vee m'/n_0)}^*u^{-(k_0-m_0(l_0-k')/n_0)}.\end{aligned}$$

Clearly, both g and f are subprojections of $u^{k'}s_{m'}s_{m'}^*u^{-k'}$. If either $g \neq u^{k'}s_{m'}s_{m'}^*u^{-k'}$ or $f \neq u^{k'}s_{m'}s_{m'}^*u^{-k'}$, then we can argue as above. So suppose that both $g = u^{k'}s_{m'}s_{m'}^*u^{-k'}$ and $f = u^{k'}s_{m'}s_{m'}^*u^{-k'}$. Then by Lemma 5.2, m' is divisible by both $n_0(m'/m_0 \vee m'/n_0)$ and $m_0(m'/m_0 \vee m'/n_0)$. This can only happen if $m_0 = n_0$.

Now, since $m_0 = n_0$, we have that $0 \neq k_0 - l_0$ is divisible by m_0. If we take $r \in \mathbb{Z}$ relatively prime with $k_0 - l_0$, then

$$(u^{k'}s_rs_r^*u^{-k'})(u^{k_0}s_{m_0}s_{m_0}^*u^{-l_0})(u^{k'}s_rs_r^*u^{-k'}) = u^{k'}s_rs_r^*u^{k_0-l_0}s_rs_r^*s_{m_0}s_{m_0}^*u^{-k'} = 0$$

by (5.5). Thus, in this case, it suffices to put $k = k'$ and $m = r \vee m'$. □

Theorem 5.4. *The C^*-algebra $\mathcal{Q}_{\mathbb{N}}^{\xi}$ is purely infinite.*

Proof. Let $0 \neq x \in \mathcal{Q}_{\mathbb{N}}^{\xi}$. Since $\mathcal{Q}_{\mathbb{N}}^{\xi}$ is simple, to show it is purely infinite as well we must find elements T, R such that TxR is invertible. We have $0 \neq G(xx^*) \geq 0$. Thus there exists a projection $Q \in \mathcal{D}_{\mathbb{N}}^{\xi}$ such that $G(xx^*)$ is invertible in $Q\mathcal{D}_{\mathbb{N}}^{\xi}$. So let d be a positive element of $\mathcal{D}_{\mathbb{N}}^{\xi}$ such that $G(dxx^*d) = d^2G(xx^*) = Q$.

Now, take a small $\epsilon > 0$. There exists a finite collection $m_j, n_j \in \mathbb{N}^\times$, $k_j, l_j \in \mathbb{Z}$, $\lambda_j \in \mathbb{C}$ such that

$$\Big\| dxx^*d - \sum_j \lambda_j u^{k_j}s_{m_j}s_{n_j}^*u^{-l_j}\Big\| < \epsilon.$$

Applying conditional expectation G, we get

$$\Big\| Q - \sum_{j:m_j=n_j,k_j=l_j} \lambda_j u^{k_j}s_{m_j}s_{m_j}^*u^{-k_j}\Big\| < \epsilon.$$

Combining the two preceding inequalities, we see that

$$\Big\| dxx^*d - Q - \sum_{j:m_j\neq n_j \text{ or } k_j\neq l_j} \lambda_j u^{k_j}s_{m_j}s_{n_j}^*u^{-l_j}\Big\| < 2\epsilon. \tag{5.7}$$

Now, applying repeatedly Lemma 5.3, we find a $k \in \mathbb{Z}$ and an $m \in \mathbb{N}^\times$ such that $u^k s_m s_m^* u^{-k} \leq Q$ and $(u^k s_m s_m^* u^{-k})(u^{k_j} s_{m_j} s_{n_j}^* u^{-l_j})(u^k s_m s_m^* u^{-k}) = 0$ for all j with $m_j \neq n_j$ or $k_j \neq l_j$. Thus inequality (5.7) yields

$$\left\|(u^k s_m s_m^* u^{-k}) d x x^* d (u^k s_m s_m^* u^{-k}) - u^k s_m s_m^* u^{-k}\right\| < 2\epsilon.$$

Setting $T := s_m^* u^{-k} d$ and $R := x^* d u^k s_m$, we then have

$$\|TxR - 1\| < 2\epsilon,$$

and TxR is invertible if $\epsilon \leq 1/2$. This proves that $\mathcal{Q}_{\mathbb{N}}^{\xi}$ is purely infinite. □

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