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POISSON SEMIGROUP, AREA FUNCTION, AND THE CHARACTERIZATION OF HARDY SPACE ASSOCIATED TO DEGENERATE SCHRÖDINGER OPERATORS

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Abstract. Let

$$Lf(x) = -\frac{1}{\omega(x)} \sum_{i,j} \partial_i\big(a_{ij}(\cdot)\,\partial_j f\big)(x) + V(x)f(x)$$

be the degenerate Schrödinger operator, where ω is a weight from the Muckenhoupt class A_2 and V is a nonnegative potential that belongs to a certain reverse Hölder class with respect to the measure $\omega(x)\,dx$. Based on some smoothness estimates of the Poisson semigroup $e^{-t\sqrt{L}}$, we introduce the area function S_P^L associated with $e^{-t\sqrt{L}}$ to characterize the Hardy space associated with L.

1. Introduction

As a special class of Calderón–Zygmund singular integrals, the area function is a useful way of building bridges between real analysis and complex analysis. In harmonic analysis, the area function is an important tool to characterize the function spaces. In [7], Fefferman and Stein proved that the area function and the nontangential maximal function are equivalent in the sense of $L^p(\mathbb{R}^n)$; they also established the area function characterization of the Hardy spaces $H^p(\mathbb{R}^n)$. From then on, the Hardy space was extended to other settings. (We refer the reader to [3], [2], [8], [18], [10], and the references therein, which investigate more general

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Hardy space than that found in [7]; however, those texts did not give the area function characterization of these Hardy spaces.) The aim of our present paper is to continue this research in this direction.

Let L be a degenerate Schrödinger operator on $\mathbb{R}^n$ defined as

$$Lf(x) = -\frac{1}{\omega(x)} \sum_{i,j} \partial_i\big(a_{ij}(\cdot)\,\partial_i f\big)(x) + Vf(x), \tag{1.1}$$

where $a_{ij}(x)$ is a real symmetric matrix satisfying

$$C^{-1}\omega(x)|\xi|^2 \le \sum_{i,j} a_{ij}(x)\xi_i\bar{\xi}_j \le C\omega(x)|\xi|^2$$

with ω being a nonnegative weight from the Muckenhoupt class A_2 and $V \ge 0$ belonging to a reverse Hölder class with respect to the measure $d\mu = \omega(x)\,dx$. Denote by $\mathcal{E}(f,g)$ the Dirichlet form associated with L; that is,

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^n} \sum_{i,j} a_{ij}(x)\,\partial_j f(x)\partial_i\bar{g}(x)\,dx + \int_{\mathbb{R}^n} V(x)f(x)\,d\mu(x).$$

In the following, we use the area function generated by the Poisson semigroup $e^{-t\sqrt{L}}$ to characterize the Hardy spaces associated to L. The Hardy spaces are widely used for various fields of analysis and partial differential equations. Let Δ be the Laplace operator on $\mathbb{R}^n$. It is well known that $H^1(\mathbb{R}^n)$ can be characterized by the maximal function $\sup_{t>0}|e^{-t\Delta}f(x)|$ (see Stein [17]). In a sense, $H^1(\mathbb{R}^n)$ can be seen as the Hardy space associated with the operator $-\Delta$. Let L be a general differential operator such as second-order elliptic self-adjoint operators in divergence form, degenerate Schrödinger operators with nonnegative potential, Schrödinger operators with nonnegative potential, and so on. In recent years, the Hardy spaces associated with L became one of the predominant issues in harmonic analysis (see [5], [4], [3], [10], [1], [12], [11], [14], [15], [20], [21], and the references therein). In particular, [4] and [21] deal with the Hardy spaces associated with the degenerate Schrödinger operators.

Let L be a degenerate Schrödinger operator. Denote by $\{T_t\}_{t>0} := \{e^{-tL}\}_{t>0}$ the heat semigroup generated by $-L$. The kernel of $\{T_t\}$ is denoted by $K_t(x,y)$; that is,

$$T_t f(x) = \int_{\mathbb{R}^n} K_t(x,y)f(y)\,d\mu(y).$$

In [4], Dziubański introduced the Hardy space associated with L.

Definition 1.1. An $L^1(d\mu)$ function f belongs to $H^1_L(d\mu)$ if the maximal function $\mathcal{M}f$ is exactly in $L^1(d\mu)$, where

$$\mathcal{M}f(x) = \sup_{t>0}\big|T_t f(x)\big|.$$

The corresponding H^1_L-norm is defined by $\|f\|_{H^1_L} = \|\mathcal{M}f\|_{L^1(d\mu)}$.

In order to characterize the above Hardy space $H^1_L(d\mu)$, Dziubański [4] introduced the following H^1_L-atoms.

Definition 1.2. A function a is an H^1_L-atom associated with a ball $B(x,r)$ if

(1) $r<\rho(x)$, $\operatorname{supp} a\subset B(x,r)$, $\|a\|_{L^\infty}\le \mu(B(x,r))^{-1}$;

(2) if $r\le\rho(x)/4$, then $\int a(y)\,d\mu(y)=0$.

The atomic norm $\|\cdot\|_{H^1_L\text{-atom}}$ is defined by $\|f\|_{H^1_L\text{-atom}}=\inf\sum|\lambda_j|$, where the infimum is taken over all decompositions $f=\sum_j\lambda_j a_j$ and $\{a_j\}$ is a sequence of H^1_L-atoms and $\{\lambda_j\}$ is a sequence of scalars.

One of the main results of [4] is the following proposition.

Proposition 1.3 ([4, Theorem 2.1]). *Assume that $\omega\in(RD)_\nu\cap D_\gamma\cap A_2$ with $2<\nu\le\gamma$. Let $V\in B_{q,\mu}, q>\gamma/2$. Then there exists a constant $C>0$ such that*

$$\frac{1}{C}\|f\|_{H^1_L\text{-atom}}\le\|f\|_{H^1_L}\le C\|f\|_{H^1_L\text{-atom}}.$$

Now we state our main result briefly. Let S^L_P be the area function associated with the Poisson semigroup generated by L (see (3.2) below). Our aim is to establish the area function characterization of $H^1_L(d\mu)$. On the one hand, for any H^1_L-atom a, we obtain that the area function $S^L_P(a)\in L^1(d\mu)$. Proposition 1.3 implies that $\|S^L_P(f)\|_{L^1(d\mu)}\le C\|f\|_{H^1_L}$. Conversely, if $S^L_P(f)\in L^1(d\mu)$, by means of the tent space T^1_2, then we prove that such f can be represented as the linear combination of H^1_L-atoms (see Theorems 3.11 and 3.14 for the details).

In the proof of Theorem 3.14, one of main tools is a reproducing formula (3.4) related to L in the distributional sense. We point out that our reproducing formula (3.4) holds for the elements in $(\mathrm{BMO}_L(d\mu))^*$, which is a subclass of the Schwartz tempered distribution spaces $\mathcal{S}'$. The reason lies in the fact that the kernel $K_t(\cdot,\cdot)$ only satisfies some Lipschitz condition for a general potential V (see Proposition 3.1). If $\partial^k K_t/\partial t^k(\cdot,\cdot)$ still has a Gaussian upper bound, then the reproducing formula can be extended to all tempered distributions under this assumption.

Remark 1.4.

(i) Yang and Zhou [21] developed a theory of localized Hardy spaces $H^1_\rho(\mathcal{X})$ associated with the admissible function ρ, where $\mathcal{X}$ is a RD-space. The Hardy space $H^1_L(d\mu)$ in this paper is a special case of $H^1_\rho(\mathcal{X})$. In [21], the authors also give several maximal function characterizations of $H^1_\rho(\mathcal{X})$ without the area function characterization. We will focus on the latter in this article.

(ii) Our main results can be seen as the generalization of the classical case. It is easy to see that, for the special case $\omega(x)\,dx=dx$ and $L=\Delta=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}$, the space $H^1_\triangle$ is exactly the classical space $H^1(\mathbb{R}^n)$. It is well known that the Hardy space $H^1(\mathbb{R}^n)$ has the area integral characterization associated with the heat semigroup $e^{-t\triangle}$.

Throughout this article, we will use c and C to denote the positive constants which are independent of main parameters and may be different at each occurrence. By $B_1\sim B_2$, we mean that there exists a constant $C>1$ such that $\frac{1}{C}\le\frac{B_1}{B_2}\le C$.

2. Preliminaries

A nonnegative function ω is an element of the Muckenhoupt class A_2 if there exists a constant $C > 0$ such that, for every ball B,

$$\Big(\frac{1}{|B|}\int_B \omega(x)\,dx\Big)\Big(\frac{1}{|B|}\int_B \omega^{-1}(x)\,dx\Big) \le C. \tag{2.1}$$

Here and subsequently, $|B|$ denotes the volume of the ball B with respect to the Lebesgue measure dx. It is well known that (2.1) implies that the measure $d\mu(x) = \omega(x)\,dx$ satisfies the doubling condition; that is, there exists a constant $C_0 > 0$ such that, for every $x \in \mathbb{R}^n$, $r > 0$,

$$\mu\big(B(x,2r)\big) \le C_0\mu\big(B(x,r)\big). \tag{2.2}$$

Using the notation from [13], we say that $\omega \in D_\gamma$, $\gamma > 0$ if there is a constant $C > 0$ such that, for every $t > 1$,

$$\mu\big(B(x,tr)\big) \le Ct^\gamma\mu\big(B(x,r)\big).$$

Let us note that (2.2) guarantees the existence of such a γ.

Similarly, $\omega \in (RD)_\nu$ if, for every $t > 1$,

$$t^\nu\mu\big(B(x,r)\big) \le C\mu\big(B(x,tr)\big).$$

A nonnegative potential V belongs to the reverse Hölder class $B_{q,\mu}$, $q > 1$ with respect to the measure $d\mu$ if there exists a constant $C > 0$ such that, for every Euclidean ball B, one has

$$\Big(\frac{1}{\mu(B)}\int_B V^q(y)\,d\mu(y)\Big)^{1/q} \le C\Big(\frac{1}{\mu(B)}\int_B V(y)\,d\mu(y)\Big).$$

From now on we will assume that $\omega \in A_2 \cap D_\gamma \cap (RD)_\nu$, $2 < \nu < \gamma$, $d\mu(x) = \omega(x)\,dx$, and $V \in B_{q,\mu}$, $q > \frac{\gamma}{2}$. We set $\delta = 2 - \frac{\gamma}{2}$.

In order to establish a reproducing formula (3.4) in Section 3, we need the following bounded mean oscillation space associated with L, which was introduced by Yang, Yang, and Zhou in [19]. For any ball B, let f_B denote the mean of f on B; that is,

$$f_B = \frac{1}{\mu(B)}\int_B f(y)\,d\mu(y).$$

Definition 2.1. A function $f \in L^1_{loc}(d\mu)$ is said to be in the space $\mathrm{BMO}_L(d\mu)$ if

$$\begin{aligned}\|f\|_{\mathrm{BMO}_L(d\mu)} := &\sup_{B(x,r):r<\rho(x)} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} \big|f(y) - f_{B(x,r)}\big|\,d\mu(y)\\ &+ \sup_{B(x,r):r\ge\rho(x)} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} \big|f(y)\big|\,d\mu(y) < \infty.\end{aligned}$$

(We refer the reader to [19] for further information on the space $\mathrm{BMO}_L(d\mu)$.) In Definitions 1.2 and 2.1, we have used the auxiliary function $m(x,V)$ defined by

$$\rho(x) = m(x,V)^{-1} = \sup_{r>0}\Big\{\frac{r^2}{\mu(B(x,r))}\int_{B(x,r)} V(y)\,d\mu(y) \le 1\Big\},$$

which plays an important role in the estimate of the fundamental solution of Schrödinger operators L (see [16]). The Hardy space $H^1_L(d\mu)$ has been given in Section 1. The dual space of $H^1_L(d\mu)$ is exactly the BMO-type space $\mathrm{BMO}_L(d\mu)$ (see [19]).

It is easy to see that, via a perturbation formula,

$$0 \le K_t(x,y) \le h_t(x,y),$$

where $h_t(x,y)$ are the integral kernels of the semigroup $\{S_t\}_{t>0}$ on $L^2(d\mu)$ generated by $-L_0$, where

$$L_0 f(x) = -\frac{1}{\omega(x)}\sum_{i,j}\partial_i(a_{ij}\,\partial_j f)(x).$$

It is known that the kernels $h_t(x,y)$ satisfy the Gaussian estimates

$$\frac{c_1}{\mu(B(x,\sqrt{t}))}\exp\Big(-\frac{|x-y|^2}{c_2 t}\Big) \le h_t(x,y) \le \frac{C_1}{\mu(B(x,\sqrt{t}))}\exp\Big(-\frac{|x-y|^2}{C_2 t}\Big) \quad (2.3)$$

for all $x, y \in \mathbb{R}^n$. Then we conclude that the kernels $K_t(x,y)$ have a Gaussian upper bound. Furthermore, Dziubański in [4] proves the following pointwise estimate.

Lemma 2.2 ([4, Theorem 2.2])**.** *There exists a constant $C > 0$ such that, for every $N > 0$, there exists a constant C_N such that*

$$K_t(x,y) \le \frac{C_N}{\mu(B(x,\sqrt{t}))}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N}\exp\big(-C|x-y|^2/t\big) \quad (2.4)$$

for all $x, y \in \mathbb{R}^n$.

In [9], Hebisch and Saloff-Coste proved the following estimates for the heat kernels of L_0:

$$\big|h_t(x,y)-h_t(x,z)\big| \le \frac{C}{\mu(B(x,\sqrt{t}))}\Big(\frac{|y-z|}{\sqrt{t}}\Big)^{\alpha}\exp\big(-\big(|x-y|-2|y-z|\big)_+^2/ct\big) \quad (2.5)$$

for all $x, y, z \in \mathbb{R}^n$ with constants $\alpha > 0$, $c > 0$, $C > 0$, and

$$\big|\partial_t^k h_t(x,y)\big| \le \frac{C_k}{t^k \mu(B(x,\sqrt{t}))}\exp\big(-|x-y|^2/ct\big)$$

for all $x, y \in \mathbb{R}^n$. In the rest of this section, we state some properties of the function $m(x,V)$ which will be used in the proofs of the main results.

Lemma 2.3 ([13, Lemma 2])**.** *Assume that $\omega \in D_\gamma$, $V \in B_{q,\mu}$ with $q > \gamma/2$. Then there exists a constant $C > 0$ such that, for every $0 < r < R < \infty$ and $y \in \mathbb{R}^n$, we have*

$$\frac{r^2}{\mu(B(y,r))}\int_{B(y,r)} V(x)\,d\mu(x) \le C\Big(\frac{r}{R}\Big)^{\delta}\frac{R^2}{\mu(B(y,R))}\int_{B(y,R)} V(x)\,d\mu(x).$$

Lemma 2.4 ([13, Lemma 3]). *Under the assumptions of Lemma 2.3, for every constant* $C_1 > 1$, *there exists a constant* $C_2 > 1$ *such that if*

$$\frac{1}{C_1} \leq \frac{r^2}{\mu(B(x,r))} \int_{B(x,r)} V(y)\, d\mu(y) \leq C_1,$$

then $C_2^{-1} \leq rm(x,V) \leq C_2$.

Lemma 2.5 ([13, Lemma 4]). *Under the assumptions of Lemma 2.3, for every constant* $C_1 \geq 1$ *there is a constant* $C_2 \geq 1$ *such that* $\frac{1}{C_2} \leq \frac{m(x,V)}{m(y,V)} \leq C_2$ *for* $|x-y| \leq C_1\rho(x)$. *Moreover, there exist constants* $k_0, C, c > 0$ *such that*

$$m(y,V) \leq C\big(1+|x-y|m(x,V)\big)^{k_0} m(x,V)$$

and such that

$$m(y,V) \geq cm(x,V)\big(1+|x-y|m(x,V)\big)^{-k_0/(1+k_0)}.$$

Lemma 2.6 ([4, Lemma 4.4]). *There exist constants* $l, C > 0$ *such that*

$$\frac{R^2}{\mu(B(x,R))} \int_{B(x,R)} V(y)\, d\mu(y) \leq C\big(Rm(x,V)\big)^l \quad \textit{provided } R \geq m(x,V)^{-1}.$$

Lemma 2.7 ([4, Corollary 4.5]). *For any constants* $c, C' > 0$ *there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^n} \frac{e^{-c|x-y|^2/t} V(y)}{\mu(B(x,\sqrt{t}))}\, d\mu(y) \leq Ct^{-1}\big(\sqrt{t}m(x,V)\big)^\delta \quad \textit{for } \sqrt{t} \leq C'm(x,V)^{-1}.$$

Lemma 2.8. *For* $V \in B_{q,\mu}$ *and* $l > 0$ *there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^n} \frac{1}{\mu(B(x,\sqrt{t}))} V(z) e^{-\frac{|x-z|^2}{t}}\, d\mu(z) \leq \frac{C}{t}\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^l, \quad t \geq \rho(x)^2.$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^n} V(z) \frac{e^{-c|x-z|^2/t}}{\mu(B(x,\sqrt{t}))}\, d\mu(z) &\leq \Big(\int_{|x-z|<\sqrt{t}} + \int_{|x-z|\geq\sqrt{t}}\Big) V(z) \frac{e^{-c|x-z|^2/t}}{\mu(B(x,\sqrt{t}))}\, d\mu(z) \\ &=: I_1 + I_2.\end{aligned}$$

For I_1, using Lemma 2.6, we have

$$I_1 \leq \frac{(\sqrt{t})^2}{t\mu(B(x,\sqrt{t}))} \int_{B(x,\sqrt{t})} V(z)\, d\mu(z) \leq \frac{C}{t}\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^l.$$

Similarly, for I_2, we have

$$\begin{aligned} I_2 &\leq \sum_{j=0}^\infty \frac{1}{\mu(B(x,\sqrt{t}))} \int_{2^j\sqrt{t}\leq|x-z|<2^{j+1}\sqrt{t}} V(z)\Big(1+\frac{|x-z|^2}{t}\Big)^{-N} d\mu(z) \\ &\leq \sum_{j=0}^\infty \frac{1}{\mu(B(x,\sqrt{t}))} \frac{1}{(1+2^{2j})^N} \int_{|x-z|<2^{j+1}\sqrt{t}} V(z)\, d\mu(z) \\ &\leq \sum_{j=0}^\infty 2^{-j(N-\gamma-2)}\big(2^{j+1}\sqrt{t}m(x,V)\big)^l \leq \frac{C}{t}\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^l. \end{aligned}$$

□

3. Area function characterization associated to the Poisson semigroup

3.1. **Smoothness estimates of the semigroup $\{e^{-t\sqrt{L}}\}$.** Let L be the degenerate Schrödinger operator defined by (1.1). In this section, we give some smoothness estimates for Poisson semigroup associated with L. At first, we state several smoothness results about the heat semigroup e^{-tL}. In a manner similar to Dziubański and Zienkiewicz's steps in [5] and [6], we can prove the following two propositions.

Proposition 3.1. *For every $0<\delta'<\delta_0=\min\{\alpha,\delta,\nu\}$ there exists a constant C_M such that, for every $M>0$ and $|h|<\sqrt{t}$,*

$$\begin{aligned}&\big|K_t(x,y+h)-K_t(x,y)\big|\\&\quad\le C_M\Big(\frac{|h|}{\sqrt{t}}\Big)^{\delta'}\frac{1}{\mu(B(x,\sqrt{t}))}e^{-c|x-y|^2/t}\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-M}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-M}.\end{aligned}$$

Proposition 3.2. *Let $Q_t(x,y)=t^2\frac{\partial}{\partial s}K_s(x,y)|_{s=t^2}$ and let $Q_s(x,y)=s\,\partial_s K_s(x,y)$.*

(a) *For $N>0$ there exists a constant $C_N>0$ such that*

$$\big|Q_s(x,y)\big|\le\frac{C_N}{\mu(B(x,\sqrt{s}))}e^{-|x-y|^2/s}\Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N}.$$

(b) *Let $0<\delta'\le\delta_0$, and let $|h|<\sqrt{s}$, where δ_0 appears in Proposition 3.1. For any $N>0$, there exists a constant $C_N>0$ such that*

$$\big|Q_s(x+h,y)-Q_s(x,y)\big|\le\frac{C_Ne^{-c|x-y|^2/s}}{\mu(B(x,\sqrt{s}))}\Big(\frac{|h|}{\sqrt{s}}\Big)^{\delta'}\Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N}\Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N}.$$

(c) *For any $N>0$, there exists a constant $C_N>0$ such that*

$$\Big|\int_{\mathbb{R}^n}Q_s(x,y)\,d\mu(y)\Big|\le C_N\Big(\frac{\sqrt{s}}{\rho(x)}\Big)^{\delta}\Big(1+\frac{\sqrt{s}}{\rho(x)}\Big)^{-N}.$$

Proposition 3.3. *Let $\{e^{-t\sqrt{L}}\}_{t>0}$ be the semigroup of linear operators generated by $-\sqrt{L}$. Denote by $P_t^L(x,y)$ the integral kernel of $e^{-t\sqrt{t}}$. We have the following estimate:*

$$\big|P_t^L(x,y)\big|\le\frac{C_\gamma t}{(t^2+4|x-y|^2)^{1/2}}\frac{1}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}.$$

Proof. By the functional calculation, we have

$$e^{-t\sqrt{L}}f(x)=c\int_0^\infty\frac{e^{-u}}{\sqrt{u}}e^{-\frac{t^2L}{4u}}f(x)\,du.$$

By (2.4), we can see that

$$K_{\frac{t^2}{4u}}(x,y)\le\frac{C_N}{\mu(B(x,t/\sqrt{4u}))}e^{-4Cu|x-y|^2/t^2}.$$

Setting $v = (1 + \frac{4C|x-y|^2}{t^2})u$, we have

$$\begin{aligned} P_t^L(x,y) &= c\int_0^\infty \frac{e^{-u}}{\sqrt{u}} K_{t^2/4u}(x,y)\,du \\ &\le C_N c\int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{1}{\mu(B(x,t/\sqrt{4u}))} e^{-\frac{4Cu|x-y|^2}{t^2}}\,du \\ &\le \frac{C't}{(t^2+4C|x-y|^2)^{1/2}} \\ &\quad \times \Big(\int_0^1 + \int_1^\infty\Big) \frac{1}{\sqrt{v}} \frac{e^{-v}}{\mu(B(x,\sqrt{t^2+4C|x-y|^2}/\sqrt{4v}))}\,dv \\ &:= I_1 + I_2. \end{aligned}$$

For I_2, using $v \ge 1$ and the doubling condition of the measure μ, we have

$$\begin{aligned} I_2 &\le \frac{t}{(t^2+4|x-y|^2)^{1/2}} \frac{C'}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))} \int_1^\infty \frac{e^{-v}}{\sqrt{v}} (\sqrt{4v})^\gamma\,dv \\ &\le \frac{t}{(t^2+4|x-y|^2)^{1/2}} \frac{C_\gamma}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}. \end{aligned}$$

Next we estimate the term I_1. Because $0 < v \le 1$ and the measure $\mu \in (RD)_\nu$, for $t > 1$ we have $t^\nu \mu(B(x,r)) \le C\mu(B(x,tr))$ and

$$(1/\sqrt{4v})^\gamma \mu\big(B(x,\sqrt{t^2+4|x-y|^2})\big) \le \mu\big(B(x,\sqrt{t^2+4|x-y|^2}/\sqrt{4v})\big).$$

Then we could get

$$\begin{aligned} I_1 &\le \frac{t}{(t^2+4|x-y|^2)^{1/2}} \frac{C'}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))} \int_0^1 \frac{e^{-v}}{\sqrt{v}} (\sqrt{4v})^\gamma\,dv \\ &\le \frac{t}{(t^2+4|x-y|^2)^{1/2}} \frac{C_\gamma}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}. \end{aligned}$$

□

Proposition 3.4. *There exists a constant C_γ such that, for every $M > 0$,*

$$\begin{aligned} &\big|P_t^L(x,y)\big| \\ &\quad \le \frac{C_\gamma t}{(t^2+4|x-y|^2)^{1/2}} \frac{1}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))} \Big(1+\frac{t}{\rho(x)}\Big)^{-M} \Big(1+\frac{t}{\rho(y)}\Big)^{-M}. \end{aligned}$$

Proof. The estimate (2.4) and the functional calculation imply that

$$\begin{aligned} \big|P_t^L(x,y)\big| &= \Big|\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2L/(4u)}\,du\Big| \\ &\le \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{e^{-4Cu|x-y|^2/t^2}}{\mu(B(x,t/\sqrt{4u}))} \Big(1+\frac{t}{\sqrt{4u}\rho(x)}\Big)^{-M} \Big(1+\frac{t}{\sqrt{4u}\rho(y)}\Big)^{-M}\,du \\ &\le \int_0^\infty e^{-u(1+4C|x-y|^2/t^2)} u^{M-1/2} \Big(\frac{t}{\rho(x)}\Big)^{-M} \Big(\frac{t}{\rho(y)}\Big)^{-M} \\ &\qquad \times \frac{1}{\mu(B(x,t/\sqrt{4u}))}\,du. \end{aligned}$$

Taking the change of variables, we can get

$$\begin{aligned}\left|P_t^L(x,y)\right| &\le C_N \frac{(\frac{t}{\rho(x)})^{-M}(\frac{t}{\rho(y)})^{-M}t}{\sqrt{t^2+4C|x-y|^2}}\Big(\int_0^1+\int_1^\infty\Big)\frac{v^{M-1/2}e^{-v}}{\mu(B(x,\frac{\sqrt{t^2+4C|x-y|^2}}{\sqrt{4v}}))}\,dv\\ &:= I_3+I_4.\end{aligned}$$

For I_3, because $0\le v\le 1$ and $\mu\in(RD)_\nu$, we have

$$I_3\le \frac{C}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}\int_0^1 v^{M-\frac12+\frac{\nu}{2}}e^{-v}\,dv\le \frac{C_\nu}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}.$$

For I_4, it can be deduced from $v>1$ and $\mu\in D_\gamma$ that

$$I_4\le \frac{C}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}\int_1^\infty e^{-v}v^{M-\frac12+\frac{\gamma}{2}}\,dv\le \frac{C_\gamma}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}.$$

Therefore, we get

$$\left|P_t^L(x,y)\right|\le \frac{C}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}\frac{t}{\sqrt{t^2+4|x-y|^2}}\Big(\frac{t}{\rho(x)}\Big)^{-M}\Big(\frac{t}{\rho(x)}\Big)^{-M}.$$

Now we have proved the following two estimates:

$$\begin{cases}(\frac{t}{\rho(x)})^M(\frac{t}{\rho(x)})^M|P_t^L(x,y)|\le \frac{C}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}\frac{t}{\sqrt{t^2+4|x-y|^2}},\\ |P_t^L(x,y)|\le \frac{C}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}\frac{t}{\sqrt{t^2+4|x-y|^2}}.\end{cases}\tag{3.1}$$

Because the choice of M is arbitrary, we have

$$\Big(1+\frac{t}{\rho(x)}\Big)^M\Big(1+\frac{t}{\rho(y)}\Big)^M\left|P_t^L(x,y)\right|\le \frac{C}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}\frac{t}{\sqrt{t^2+4|x-y|^2}}.$$

This completes the proof of Proposition 3.4. □

Proposition 3.5. *For every $0<\delta'<\delta_0=\min\{\alpha,\delta\}$, there exists a constant C such that, for every $N>0$, there exists a constant $C>0$ such that, for $|h|<\sqrt{t}$,*

$$\begin{aligned}&\left|P_t^L(x,y+h)-P_t^L(x,y)\right|\\ &\quad\le \frac{C_M(|h|/t)^{\delta'}}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}\frac{t}{(t^2+|x-y|^2)^{1/2}}\Big(1+\frac{t}{\rho(x)}\Big)^{-N}\Big(1+\frac{t}{\rho(y)}\Big)^{-N}.\end{aligned}$$

Proof. Recall that $e^{-t\sqrt{L}}=c\int_0^\infty\frac{e^{-u}}{\sqrt{u}}e^{-t^2L/(4u)}\,du$. By Proposition 3.1, we have

$$\begin{aligned}&\left|P_t^L(x,y+h)-P_t^L(x,y)\right|\\ &\le C\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\Big(\frac{|h|}{t/\sqrt{4u}}\Big)^{\delta'}\frac{e^{-4|x-y|^2/t^2}}{\mu(B(x,\frac{t}{\sqrt{4u}}))}\Big(1+\frac{t}{\sqrt{4u}\rho(x)}\Big)^{-M}\Big(1+\frac{t}{\sqrt{4u}\rho(y)}\Big)^{-M}du\\ &\le C\Big(\frac{|h|}{t}\Big)^{\delta'}\Big(\frac{t}{\rho(x)}\Big)^{-M}\Big(\frac{t}{\rho(y)}\Big)^{-M}\int_0^\infty e^{-(1+\frac{4|x-y|^2}{t^2})u}u^{\frac{\delta'}{2}-\frac12+M}\frac{1}{\mu(B(x,t/\sqrt{4u}))}\,du\\ &\le C(|h|/t)^{\delta'}\big(t/\rho(x)\big)^{-M}\big(t/\rho(y)\big)^{-M}\frac{t}{\sqrt{t^2+4|x-y|^2}}\end{aligned}$$

$$\times \Big(\int_0^1 + \int_1^\infty\Big) \frac{e^{-v} v^{\delta'/2-1/2+M}\,dv}{\mu(B(x,\sqrt{t^2+4|x-y|^2}/\sqrt{4v}))}$$
$$:= I_5 + I_6,$$

where in the last inequality we have used the change of variable $(1 + 4|x-y|^2/t^2)u = v$. A direct computation gives

$$I_5 \le C \frac{1}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))} \int_0^1 e^{-v} v^{\frac{\nu}{2}+\frac{\delta'}{2}-\frac{1}{2}+M}\,dv$$
$$\le \frac{C_\nu}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))}$$

and

$$I_6 \le C \frac{1}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))} \int_1^\infty e^{-v} v^{\frac{\gamma}{2}+\frac{\delta'}{2}-\frac{1}{2}+M}\,dv \le \frac{C_\gamma}{\mu(B(t^2+4|x-y|^2))}.$$

Now we have proved

$$\big|P_t^L(x,y+h) - P_t^L(x,y)\big|$$
$$\le \frac{C(|h|/t)^{\delta'}}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))} \frac{t}{\sqrt{t^2+4|x-y|^2}} \big(t/\rho(x)\big)^{-M} \big(t/\rho(y)\big)^{-M}.$$

Similarly, we can also get

$$\big|P_t^L(x,y+h) - P_t^L(x,y)\big| \le C \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \big|\big[K_{t^2/4u}(x,y+h) - K_{t^2/4u}(x,y)\big]\big|\,du$$
$$\le C\Big(\frac{|h|}{t}\Big)^{\delta'} \int_0^\infty e^{-(1+\frac{4|x-y|^2}{t^2})u} u^{\frac{\delta'}{2}-\frac{1}{2}+M} \frac{1}{\mu(B(x,\frac{t}{\sqrt{4u}}))}\,du$$
$$\le \frac{C(|h|/t)^{\delta'}}{\mu(B(x,\sqrt{t^2+4|x-y|^2}))} \frac{t}{\sqrt{t^2+4|x-y|^2}}.$$

We could complete the proof of Proposition 3.5 in a manner similar to Proposition 3.4. □

As in [6, Corollary 6.2], we can use (2.3) to obtain the following lemma.

Lemma 3.6. *The semigroup has the (unique) extension to a holomorphic semigroup T_ξ on $L^2(e^{\eta|x-y|}\,dx)$ in the sector $\triangle_{\pi/4} = \{\xi : |\arg \xi| < \pi/4\}$. Moreover, there exist constants $C, c' > 0$ such that, for every $\eta > 0$, we have*

$$\|T_\xi\|_{L^2(e^{\eta|x-y|}\,dx)\to L^2(e^{\eta|x-y|}\,dx)} \le C e^{c'\eta^2 \operatorname{Re}\xi}.$$

Lemma 3.7. *There exists a constant $c > 0$ such that, for every $M > 0$, there exists a constant $C > 0$ such that, for every $\eta > 0$ and $y \in \mathbb{R}^n$, we have*

$$\int \big|K_\xi(x,y)\big|^2 e^{\eta|x-y|}\,d\mu(x) \le C e^{c\eta^2 \operatorname{Re}\xi} \frac{1}{\mu(B(y,\sqrt{\operatorname{Re}\xi}))} \big(1 + \sqrt{\operatorname{Re}\xi}/\rho(y)\big)^{-M}.$$

Proof. Let $t = \operatorname{Re}\xi$. We have $K_\xi(x,y) = [T_{\xi-t/10}K_{t/10}(\cdot,y)](x)$. Using Lemma 2.2, we have

$$\begin{aligned}
&\int \big|K_\xi(x,y)\big|^2 e^{\eta|x-y|}\,d\mu(x)\\
&\quad\le C\int_{\mathbb{R}^n}\frac{e^{-c|u-y|^2/t}}{\mu(B(u,\sqrt{t}))^2}\frac{e^{c'\eta^2 t}}{(1+\sqrt{t}/\rho(u))^{2M}}\frac{e^{\eta|u-y|}\,d\mu(u)}{(1+\sqrt{t}/\rho(y))^{2M}}\\
&\quad\le Ce^{c'\eta^2 t}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2M}\int_{\mathbb{R}^n} e^{-c|u-y|^2/t+\eta|u-y|}\,d\mu(u).
\end{aligned}$$

For every $\omega \in B(y,\sqrt{t})$, $B(y,\sqrt{t}) \subset B(u,|y-u|+\sqrt{t})$. Set

$$\begin{cases} B_0 = \{u : |u-y| < 2\sqrt{t}+\eta t\};\\ B_k = \{u : 2^k\sqrt{t}+\eta t \le |u-y| < 2^{k+1}\sqrt{t}+\eta t\}, \quad k = 1,2,\ldots. \end{cases}$$

We can get

$$\begin{aligned}
&\int \big|K_\xi(x,y)\big|^2 e^{\eta|x-y|}\,d\mu(x)\\
&\le Ce^{c'\eta^2 t}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2M}\frac{1}{\mu(B(y,\sqrt{t}))^2}\sum_{k=0}^{\infty} e^{-c_1(2^k\sqrt{t}+\eta t)^2/t}\frac{\mu(B(y,2^{k+1}\sqrt{t}+\eta t))}{(1+2^k)^l}\\
&\le C\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2M}\frac{e^{c'\eta^2 t}}{\mu(B(y,\sqrt{t}))}\sum_{k=0}^{\infty}(1+2^{k+1}+\eta\sqrt{t})^{\gamma}\frac{e^{-c_1(2^k+\eta\sqrt{t})^2}}{(1+2^k)^l}\\
&\le Ce^{c'\eta^2 t}\frac{1}{\mu(B(y,\sqrt{t}))}\Big(1+\frac{\sqrt{t}}{\rho(y)}\Big)^{-2M}.
\end{aligned}$$

□

Lemma 3.8. *There exists a constant $c > 0$ such that, for every $M > 0$, there is a constant $C_M > 0$ such that, for any $\xi \in \triangle_{\pi/5}$,*

$$\big|K_\xi(x,y)\big| \le \frac{C_M}{\mu(B(y,\sqrt{\operatorname{Re}\xi}))}\Big(1+\frac{\sqrt{\operatorname{Re}\xi}}{\rho(x)}\Big)^{-M}\Big(1+\frac{\sqrt{\operatorname{Re}\xi}}{\rho(y)}\Big)^{-M} e^{-c|x-y|^2/\operatorname{Re}\xi}.$$

Proof. We have

$$\begin{aligned}
&\big|K_\xi(x,y)\big|e^{\eta|x-y|}\\
&\quad= \Big|\int K_{\xi/2}(x,u)K_{\xi/2}(u,y)\,d\mu(u)\Big|e^{\eta|x-y|}\\
&\quad\le \Big(\int \big|K_{\xi/2}(x,u)\big|^2 e^{2\eta|x-u|}\,d\mu(u)\Big)^{1/2}\Big(\int \big|K_{\xi/2}(u,y)\big|^2 e^{2\eta|y-u|}\,d\mu(u)\Big)^{1/2}\\
&\quad\le \frac{1}{\mu(B(x,\sqrt{\operatorname{Re}\xi}))^{1/2}}\frac{e^{c\eta^2\operatorname{Re}\xi}}{\mu(B(y,\sqrt{\operatorname{Re}\xi}))^{1/2}}\Big(1+\frac{\sqrt{\operatorname{Re}\xi}}{\rho(x)}\Big)^{-M}\Big(1+\frac{\sqrt{\operatorname{Re}\xi}}{\rho(y)}\Big)^{-M}.
\end{aligned}$$

Set $\eta = c''|x-y|(\operatorname{Re}\xi)^{-1}$, where c'' is a sufficiently small constant. Then we have

$$\begin{aligned}&\big|K_\xi(x,y)\big|e^{c''|x-y|^2/\operatorname{Re}\xi}\\&\quad\le\frac{C_M}{\mu(B(x,\sqrt{\operatorname{Re}\xi}))^{1/2}}\frac{e^{-c|x-y|^2/\operatorname{Re}\xi}}{\mu(B(y,\sqrt{\operatorname{Re}\xi}))^{1/2}}\Big(1+\frac{\sqrt{\xi}}{\rho(x)}\Big)^{-M}\Big(1+\frac{\sqrt{\xi}}{\rho(y)}\Big)^{-M}\\&\quad\le\frac{C_M}{\mu(B(x,\sqrt{\operatorname{Re}\xi}))}e^{-c'|x-y|^2/\operatorname{Re}\xi}\Big(1+\frac{\sqrt{\xi}}{\rho(x)}\Big)^{-M}\Big(1+\frac{\sqrt{\xi}}{\rho(y)}\Big)^{-M}.\end{aligned}$$

Similarly, we can prove

$$\big|K_\xi(x,y)\big|\le\frac{C_M}{\mu(B(y,\sqrt{\operatorname{Re}\xi}))}e^{-c'|x-y|^2/\sqrt{\operatorname{Re}\xi}}\Big(1+\frac{\sqrt{\xi}}{\rho(x)}\Big)^{-M}\Big(1+\frac{\sqrt{\xi}}{\rho(y)}\Big)^{-M}.$$

This completes the proof of Lemma 3.8. □

We have the following proposition.

Proposition 3.9. *Set $D_t^L f(x)=t^2\,\partial_t^2 e^{-t\sqrt{L}}f(x)=t^2Le^{-t\sqrt{L}}$. Denote by $D_t^L(x,y)$ the kernel of D_t^L.*

(1) *For $N>0$ there exists a constant $C_N>0$ such that*

$$\begin{aligned}&\big|D_t^L(x,y)\big|\\&\quad\le\frac{C_Nt}{(t^2+|x-y|^2)^{1/2}}\frac{1}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\Big(1+\frac{t}{\rho(x)}\Big)^{-N}\Big(1+\frac{t}{\rho(y)}\Big)^{-N}.\end{aligned}$$

(2) *Let $0<\delta'\le\delta_0$ and let $|h|<t$. For any $N>0$, there exists a constant $C_N>0$ such that*

$$\begin{aligned}&\big|D_t^L(x+h,y)-D_t^L(x,y)\big|\\&\quad\le C_N\frac{(|h|/t)^{\delta'}t}{(t^2+|x-y|^2)^{1/2}}\frac{1}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\Big(1+\frac{t}{\rho(x)}\Big)^{-N}\Big(1+\frac{t}{\rho(y)}\Big)^{-N}.\end{aligned}$$

(3) *For any $N>0$ there exists a constant $C_N>0$ such that*

$$\Big|\int_{\mathbb{R}^n}D_t^L(x,y)\,d\mu(y)\Big|\le C_N\big(t/\rho(x)\big)^{\delta}\frac{1}{(1+t/\rho(x))^N}.$$

Proof. (1) Because of Lemma 3.8, we have

$$\big|K_\xi(x,y)\big|\le\frac{C_M}{\mu(B(x,\sqrt{\operatorname{Re}\xi}))}e^{-c|x-y|^2/\operatorname{Re}\xi}\Big(1+\frac{\sqrt{\operatorname{Re}\xi}}{\rho(x)}\Big)^{-M}\Big(1+\frac{\sqrt{\operatorname{Re}\xi}}{\rho(y)}\Big)^{-M}.$$

By use of a direct computation, we can see that

$$\begin{aligned}\big|P_\xi(x,y)\big|&=\Big|\int_0^\infty\frac{e^{-u}}{\sqrt{u}}K_{\xi^2/(4u)}(x,y)\,du\Big|\\&\le C_M\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\frac{1}{\mu(B(x,\frac{\sqrt{\operatorname{Re}\xi^2}}{\sqrt{4u}}))}e^{-4|x-y|^2u/\operatorname{Re}\xi^2}\\&\qquad\times\Big(1+\frac{\sqrt{\operatorname{Re}\xi^2}}{\sqrt{4u}\rho(x)}\Big)^{-M}\Big(1+\frac{\sqrt{\operatorname{Re}\xi^2}}{\sqrt{4u}\rho(x)}\Big)^{-M}\end{aligned}$$

$$\leq \frac{C_M}{\mu(B(x, \sqrt{\operatorname{Re}\xi^2 + |x-y|^2}))} \frac{\sqrt{\operatorname{Re}\xi^2}}{\sqrt{\operatorname{Re}\xi^2 + 4|x-y|^2}} \times \Big(1 + \frac{\sqrt{\operatorname{Re}\xi^2}}{\rho(x)}\Big)^{-M} \Big(1 + \frac{\sqrt{\operatorname{Re}\xi^2}}{\rho(y)}\Big)^{-M}.$$

The above estimate gives

$$\begin{aligned} \big|D_t^L(x,y)\big| &= \Big|\frac{1}{2\pi i}\int_{|\xi - t| = \frac{t}{2}} \frac{t^2 P_\xi^L(x,y)}{(\xi - t)^3}\, d\xi\Big| \\ &\leq \frac{C_M}{\mu(B(x, \sqrt{t^2 + 4|x-y|^2}))} \frac{t}{\sqrt{t^2 + 4|x-y|^2}} \\ &\quad \times \Big(1 + \frac{t}{\rho(x)}\Big)^{-M} \Big(1 + \frac{t}{\rho(y)}\Big)^{-M}. \end{aligned}$$

Now we prove (2). We can see that

$$\begin{aligned} &\big|D_t^L(x+h,y) - D_t^L(x,y)\big| \\ &\leq C_N \int_{\mathbb{R}^n} \big|P_{t/2}^L(x+h,w) - P_{t/2}^L(x,w)\big|\big|D_{t/2}^L(w,y)\big|\, d\mu(w) \\ &\leq C\Big(\frac{|h|}{t}\Big)^{\delta'} \Big(\int_{|x-w|>|y-w|} + \int_{|y-w|\geq|x-w|}\Big) \frac{(1+t/\rho(x))^{-M}}{\mu(B(x, \sqrt{t^2 + |x-w|^2}))} \\ &\quad \times \frac{(1+t/\rho(y))^{-M}}{\mu(B(w, \sqrt{t^2 + |w-y|^2}))} \frac{t}{\sqrt{t^2 + 4|x-w|^2}} \frac{t\, d\mu(w)}{\sqrt{t^2 + 4|w-y|^2}} \\ &:= C\Big(\frac{|h|}{t}\Big)^{\delta'} \Big(1 + \frac{t}{\rho(x)}\Big)^{-M} \Big(1 + \frac{t}{\rho(y)}\Big)^{-M} [I_7 + I_8]. \end{aligned}$$

We first estimate the term I_7, which can be divided into the following two parts:

$$\begin{aligned} I_7 &= \Big(\int_{|x-w|>|y-w|, |y-w|>t} + \int_{|x-w|>|y-w|, |y-w|<t}\Big) \frac{1}{\mu(B(x, \sqrt{t^2 + |x-w|^2}))} \\ &\quad \times \frac{1}{\mu(B(w, \sqrt{t^2 + |w-y|^2}))} \frac{t}{\sqrt{t^2 + 4|x-w|^2}} \frac{t}{\sqrt{t^2 + 4|w-y|^2}}\, d\mu(w) \\ &= I_7^1 + I_7^2. \end{aligned}$$

For I_7^2, we can see that $|x-w| > |y-w|$ implies here that $2|x-w| > |x-y|$ and $c\mu(B(w,t)) \leq \mu(B(w, \sqrt{t^2 + 4|y-w|^2}))$. On the other hand, because $u \in B(y,t)$ implies that $|w-u| \leq 2t$, we have $\mu(B(y,t)) \leq \mu(B(w,2t))$. Then $\mu(B(y,t)) \leq 2^\gamma \mu(B(w,t))$. We can get

$$\begin{aligned} I_7^2 &\leq \frac{ct/(t^2 + |x-y|^2)^{1/2}}{\mu(B(x, \sqrt{t^2 + |x-y|^2}))} \int_{|y-w|<t} \frac{t}{\sqrt{t^2 + |y-w|^2}} \frac{d\mu(w)}{\mu(B(w,t))} \\ &\leq \frac{C_\gamma}{\mu(B(x, \sqrt{t^2 + |x-y|^2}))} \frac{t}{(t^2 + |x-y|^2)^{1/2}}. \end{aligned}$$

Now we estimate I_7^1. Because $|x-w|>|y-w|$, we have $|x-w|>\frac{1}{2}|x-y|$. By the doubling property of the measure μ, we obtain

$$\begin{aligned} I_7^1 &\leq \frac{1}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\frac{t}{(t^2+|x-y|^2)^{1/2}}\sum_{k=0}^{\infty}\int_{2^kt<|y-w|\leq 2^{k+1}t}\frac{2^{-k}\,d\mu(w)}{\mu(B(w,2^kt))} \\ &\leq \frac{C_\gamma}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\frac{t}{(t^2+|x-y|^2)^{1/2}}. \end{aligned}$$

Finally, we have

$$I_7 \leq \frac{C_\gamma}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\frac{t}{(t^2+|x-y|^2)^{1/2}}.$$

Now we estimate I_8. Comparably to I_7, we write

$$\begin{aligned} I_8 &= \Big(\int_{|y-w|\geq|x-w|,|x-w|>t}+\int_{|y-w|\geq|x-w|,|x-w|\leq t}\Big)\frac{1}{\mu(B(x,\sqrt{t^2+|x-w|^2}))} \\ &\quad\times\frac{1}{\mu(B(w,\sqrt{t^2+|w-y|^2}))}\frac{t}{\sqrt{t^2+4|x-w|^2}}\frac{t}{\sqrt{t^2+4|w-y|^2}}\,d\mu(w) \\ &:= I_8^1+I_8^2. \end{aligned}$$

It is easy to see that $|y-w|\geq|x-w|$ implies that $|y-w|>\frac{1}{2}|x-y|$. Also, by the doubling property of μ, we can get $\mu(B(x,\sqrt{t^2+|x-y|^2}))\leq C_\gamma\mu(B(w,\sqrt{t^2+|y-w|^2}))$. These estimates imply that

$$\begin{aligned} I_8^1 &\leq \frac{Ct/\sqrt{t^2+|x-y|^2}}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\int_{|x-w|>t}\frac{1}{\mu(B(x,|x-w|))}\frac{t\,d\mu(w)}{\sqrt{t^2+|x-w|^2}} \\ &\leq \frac{Ct/\sqrt{t^2+|x-y|^2}}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\sum_{k=0}^{\infty}\frac{t}{\sqrt{t^2+2^{2k}t}}\frac{1}{\mu(B(x,2^kt))}\int_{|x-w|<2^{k+1}t}d\mu(w) \\ &\leq \frac{Ct}{\sqrt{t^2+|x-y|^2}}\frac{1}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}. \end{aligned}$$

The estimate for I_8^2 is similar. Since $|y-w|\geq|x-w|$ implies that $|y-w|\geq\frac{1}{2}|x-y|$, we have $I_8^2\leq\frac{t}{\sqrt{t^2+|x-y|^2}}\times I_{8,x}^2$, where

$$I_{8,x}^2=\int_{|y-w|\geq|x-w|,|x-w|\leq t}\frac{t/\sqrt{t^2+4|y-w|^2}}{\mu(B(x,\sqrt{t^2+4|x-w|^2}))}\frac{t\,d\mu(w)}{\sqrt{t^2+4|x-w|^2}}.$$

It is easy to see that

$$\begin{cases}\mu(B(x,t+|x-y|))\leq C\mu(B(w,\sqrt{t^2+|y-w|^2})), & |y-w|\geq t\geq|x-w|;\\ \mu(B(x,t+|x-y|))\leq C\mu(B(w,\sqrt{t^2+|y-w|^2})), & |x-w|\leq|y-w|\leq t.\end{cases}$$

Then we conclude that

$$I_{8,x}^2\leq\frac{1}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\int_{|x-w|<t}\frac{d\mu(w)}{\mu(B(x,t))}\leq\frac{C}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}.$$

Finally, we get

$$\left|D_t^L(x+h,y)-D_t^L(x,y)\right|\le C_M\frac{(1+t/\rho(x))^{-M}(1+t/\rho(y))^{-M}}{\mu(B(x,\sqrt{t^2+|x-y|^2}))}\frac{(|h|/t)^{\delta'}t}{\sqrt{t^2+|x-y|^2}}.$$

At last, we prove (3). Notice that

$$D_t^L(x,y)=\frac{1}{\sqrt{\pi}}\int_0^\infty \partial_t^2 K^L_{\frac{t^2}{4u}}(x,y)\frac{t^4}{4u^2}e^{-u}u^{-1/2}\,du.$$

If $\rho(x)<t$, by condition (c) of Proposition 3.2, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D_t^L(x,y)\,d\mu(y)\Big| &\le \frac{1}{\sqrt{\pi}}\int_0^\infty\Big|\int_{\mathbb{R}^n} Q^L_{2,t/\sqrt{4u}}(x,y)\,d\mu(y)\Big|e^{-u}u^{-1/2}\,du\\ &\le C_N\big(t/\rho(x)\big)^\delta\frac{1}{(1+t/\rho(x))^N}.\end{aligned}$$

If $\rho(x)>t$, then

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D_t^L(x,y)\,d\mu(y)\Big| &\le \frac{1}{\sqrt{\pi}}\int_0^\infty\Big|\int_{\mathbb{R}^n} Q^L_{t/\sqrt{4u}}(x,y)\,d\mu(y)\Big|e^{-u}u^{-1/2}\,du\\ &\le \frac{C_N}{\sqrt{\pi}}\int_0^\infty\Big(\frac{t}{\rho(x)}\Big)^\delta u^{-(\delta+1)/2}e^{-u}\,du\\ &\le \frac{C_N}{(1+t/\rho(x))^N}\big(t/\rho(x)\big)^\delta,\end{aligned}$$

where in the last inequality we have used the fact that $0<(\delta+1)/2<1$. This completes the proof of Proposition 3.9. □

3.2. **The area function characterization via the Poisson semigroup.** Let S_P^L and g_P^L be the Lusin function and Littlewood–Paley function generated by the Poisson semigroup $e^{-t\sqrt{L}}$, respectively. Precisely,

$$S_P^Lf(x)=\Big(\int_0^\infty\int_{|x-y|<t}|D_t^Lf(y)|^2\frac{d\mu(y)\,dt}{t\mu(B(x,t))}\Big)^{1/2}, \tag{3.2}$$

$$g_P^Lf(x)=\Big(\int_0^\infty|D_t^Lf(y)|^2\frac{dt}{t}\Big)^{1/2}. \tag{3.3}$$

We first prove the L^2-boundedness of S_P^L and g_P^L.

Theorem 3.10. *Suppose that $V\in B_{q,\mu}$, $q>1$. Let L be the degenerate Schrödinger operator defined by (1.1). Then the operators S_P^L and g_P^L are bounded on $L^2(\mathbb{R}^n,d\mu)$; that is,*

$$\big\|g_P^Lf\big\|_{L^2}=c\|f\|_{L^2}\qquad\text{and}\qquad\big\|S_P^Lf\big\|_{L^2}\le C\|f\|_{L^2}.$$

Proof. By functional calculus, we have

$$\begin{aligned}\big\|g_P^Lf\big\|^2_{L^2}&=\int_0^\infty\Big\langle t^4\frac{d^2P_s^L}{ds^2}\Big|_{s=t^2}f,t^4\frac{d^2P_s^L}{ds^2}\Big|_{s=t^2}f\Big\rangle\frac{dt}{t}\\ &=\int_0^\infty\Big[\int_0^\infty t^8\lambda^2e^{-2t^2\sqrt{\lambda}}\frac{dt}{t}\Big]\,dE_{f,f}(\lambda)=c\|f\|_{L^2}.\end{aligned}$$

Next we estimate the norm of $\|S_P^L f\|_{L^2}$. Notice that if $|x-y|<t$, $\mu(B(y,t)) \le \mu(B(x,2t))$, then we get $\frac{1}{\mu(B(x,t))} \le \frac{2^\gamma}{\mu(B(y,t))}$. Hence we have

$$\begin{aligned}\|S_P^L f\|_{L^2}^2 &\le C \int_{\mathbb{R}^n} \Big[\int_0^\infty \int_{|x-y|<t} \big|D_t^L f(y)\big|^2 \frac{d\mu(y)\,dt}{t\mu(B(y,t))}\Big]\,d\mu(x)\\ &\le C\big\|g_P^L f\big\|_{L^2}^2 \le C\|f\|_{L^2}. \end{aligned}$$

□

Theorem 3.11. *Suppose that $V \in B_{q,\mu}$, $q>1$. Let L be the degenerate Schrödinger operator defined by (1.1). If $f \in H_L^1(d\mu)$, then $S_P^L(f) \in L^1(d\mu)$.*

Proof. We only need to prove that, for any H_L^1-atom, $\|S_P^L a\|_{L^1} \le C$. We denote by $\chi_{\Gamma(x)}(y,t)$ the character function of the cone $\Gamma(x) = \{(y,t) \in \mathbb{R}^n : |y-x|<t\}$. Because $|y-x|<t$ and $z \in B(y,t)$, by Theorem 3.10 we can get

$$\begin{aligned}\big\|S_P^L a\big\|_{L^2(\mathbb{R}^n,d\mu)}^2 &\le C_\gamma \int_{\mathbb{R}^n}\int_0^\infty\int_{\mathbb{R}^n} \big|D_t^L a(y)\big|^2 \chi_{\Gamma(x)}(y,t) \frac{d\mu(x)}{\mu(B(y,t))}\frac{d\mu(y)\,dt}{t}\\ &\le C_\gamma \int_{\mathbb{R}^n}\int_0^\infty \big|D_t^L a(y)\big|^2 \frac{d\mu(y)\,dt}{t}\\ &\le C_\gamma \Big(\int_{\mathbb{R}^n} \big|a(y)\big|^2\,d\mu(y)\Big)^{1/2} \le C_\gamma \mu\big(B(x,r)\big)^{-1/2}.\end{aligned}$$

Notice that $\operatorname{supp} a(x) = B(x_0,r)$. We can get

$$\big\|S_P^L a\big\|_{L^1} = \int_{B(x_0,4r)} \big|S_P^L a(x)\big|\,d\mu(x) + \int_{B^c(x_0,4r)} \big|S_P^L a(x)\big|\,d\mu(x) := I + II.$$

Clearly, the L^2-boundedness of S_P^L implies that

$$I \le \mu\big(B(x_0,4r)\big)^{1/2}\big\|S_P^L a\big\|_{L^2} \le C_\gamma \mu\big(B(x_0,4r)\big)^{1/2}\mu\big(B(x,r)\big)^{-1/2} \le C_\gamma.$$

For the estimate of II, we divide the discussion into two cases.

Case I: $r<\rho(x)$. The atom a has the canceling property. Then we can get

$$\begin{aligned}S_P^L a(x) &= \Big[\int_0^\infty \int_{|x-y|<t} \Big|\int_{B(x_0,r)} \big[D_t^L(y,z) - D_t^L(y,x_0)\big] a(z)\,d\mu(z)\Big|^2 \frac{d\mu(y)\,dt}{t\mu(B(x,t))}\Big]^{1/2}\\ &\le II_1 + II_2,\end{aligned}$$

where

$$II_1 = \Big(\int_0^{\frac{|x-x_0|}{2}} \int_{|x-y|<t} \Big(\int_{B(x_0,r)} \big|D_t^L(y,z) - D_t^L(y,x_0)\big| \frac{d\mu(z)}{\mu(B(x_0,r))}\Big)^2 \frac{d\mu(y)\,dt}{t\mu(B(x,t))}\Big)^{1/2}$$

and

$$II_2 = \Big(\int_{\frac{|x-x_0|}{2}}^\infty \int_{|x-y|<t} \Big(\int_{B(x_0,r)} \big|D_t^L(y,z) - D_t^L(y,x_0)\big| \frac{d\mu(z)}{\mu(B(x_0,r))}\Big)^2 \frac{d\mu(y)\,dt}{t\mu(B(x,t))}\Big)^{1/2}.$$

For II_1, because $0<t<\frac{|x-x_0|}{2}$ and $|x-y|<t$ imply that $|x-y|<\frac{|x-x_0|}{2}$, we can get $|y-x_0| \sim |x-x_0|$. Then, for $z \in B(x_0,r)$ and $x \in B^c(x_0,4r)$, we have $|y-x_0| \ge 4r$ and $|x_0-z|<r \le \frac{|y-x_0|}{4}$. Hence, by part (2) of Proposition 3.9 and

the symmetry of the kernel $D_t^L(x,y)$, we have

$$\left|D_t^L(y,z) - D_t^L(y,x_0)\right| \le \frac{Ct(|z-x_0|/t)^{\delta'}}{\sqrt{t^2+|y-x_0|^2}} \frac{(1+t/\rho(y))^{-M}(1+t/\rho(x_0))^{-M}}{\mu(B(x_0,\sqrt{t^2+|y-x_0|^2}))}.$$

The above estimate of D_t^L gives

$$\begin{aligned} II_1 &\le \frac{1}{\mu(B(x_0,\sqrt{t^2+|x-x_0|^2}))} \\ &\quad \times \Big(\int_0^{\frac{|x-x_0|}{2}}\int_{|x-y|<t}\Big(\frac{t}{|x-x_0|}\Big)^2\Big(\frac{r}{t}\Big)^{2\delta'}\frac{d\mu(y)\,dt}{t\mu(B(x,t))}\Big)^{1/2} \\ &\le \frac{1}{\mu(B(x_0,|x-x_0|))}\frac{r^{\delta'}}{|x-x_0|^{\delta'}}. \end{aligned}$$

Now we deal with II_2. Because $|z-x_0| \le r < |x-x_0|/2 \le t$, we apply condition (2) of Proposition 3.9 to deduce that

$$\begin{aligned} II_2 &\le \Big(\int_{|x-x_0|/2}^{\infty}\int_{|x-y|<t}\frac{(r/t)^{2\delta'}}{\mu(B(x_0,\sqrt{t^2+|y-x_0|^2}))^2}\frac{t^2}{t^2+|y-x_0|^2}\frac{d\mu(y)\,dt}{t\mu(B(x,t))}\Big)^{1/2} \\ &\le \frac{C}{\mu(B(x_0,|x-x_0|))}\frac{r^{\delta'}}{|x-x_0|^{\delta'}}. \end{aligned}$$

The estimates of II_1 and II_2 show that

$$\int_{B^c(x_0,4r)}\left|S_P^L a(x)\right| d\mu(x) \le \sum_{k=1}^{\infty}\int_{4^k r\le|x-x_0|<4^{k+1}r}\frac{r^{\delta'}}{(4^k r)^{\delta'}}\frac{d\mu(x)}{\mu(B(x_0,4^k r))} \le C_\gamma.$$

Case II: $\rho(x_0) \le r < 4\rho(x_0)$. In this case, the atom a has no canceling property. We have

$$\begin{aligned} \big(S_P^L a(x)\big)^2 &= \int_0^\infty\int_{|x-y|<t}\left|Q_t^L a(y)\right|^2\frac{d\mu(y)\,dt}{t\mu(B(x,t))} \\ &\le \Big(\int_0^{r/2}\int_{|x-y|<t} + \int_{r/2}^{\frac{|x-x_0|}{4}}\int_{|x-y|<t} + \int_{\frac{|x-x_0|}{4}}^{\infty}\int_{|x-y|<t}\Big) \\ &\quad \times \left|Q_t^L a(y)\right|^2\frac{d\mu(y)\,dt}{t\mu(B(x,t))} \\ &:= II_3 + II_4 + II_5. \end{aligned}$$

We begin to estimate II_i, $i=1,2,3$, separately. For $|x-x_0| > 4r$ and $|x-y| < t < r/2$, we have $|y-x_0| > 7r/2$. By condition (1) of Proposition 3.9, we get

$$\begin{aligned} \left|D_t^L(x,y)\right| &\le \frac{C_N t}{(t^2+|x-y|^2)^{1/2}}\frac{1}{\mu(B(x,\sqrt{t^2+|x-y|^2}))} \\ &\quad \times \Big(1+\frac{t}{\rho(x)}\Big)^{-N}\Big(1+\frac{t}{\rho(y)}\Big)^{-N}. \end{aligned}$$

On the one hand, because $|x_0 - z| < r < \frac{|x-x_0|}{4}$, we have $|y-z| \sim |x-x_0|$. Then the term II_3 can be estimated as follows:

$$\begin{aligned} II_3 &\leq \int_0^{r/2}\int_{|x-y|<t} \frac{t^2}{t^2+|x-x_0|^2}\frac{1}{\mu(B(x,\sqrt{t^2+|x-x_0|^2}))^2}\frac{d\mu(y)\,dt}{t\mu(B(x,t))} \\ &\leq \int_0^{r/2}\int_{|x-y|<t}\Big(\frac{t}{|x-x_0|}\Big)^2\frac{1}{\mu(B(y,|x-x_0|))^2}\frac{d\mu(y)\,dt}{t\mu(B(x,t))}. \end{aligned}$$

For every $z \in B(x_0,|x-x_0|)$, we have that $|z-y| \leq |x-x_0| + |x_0-y|$. Because $|y-x| < t < r/2$, we have $|y-x_0| < r/2 < \frac{1}{8}|x-x_0|$ for $x \in B^c(x_0,4r)$. Then we conclude that $|y-x_0| \sim |x-x_0|$ and that $\mu(B(x_0,|x-x_0|)) \leq \mu(B(y,2|x-x_0|))$. Hence

$$\begin{aligned} II_3 &\leq \int_0^{r/2}\int_{|x-y|<t}\Big(\frac{t}{|x-x_0|}\Big)^2\frac{1}{\mu(B(x_0,|x-x_0|))^2} \\ &\qquad\times\Big(\frac{\mu(B(y,2|x-x_0|))}{\mu(B(y,|x-x_0|))}\Big)^2\frac{d\mu(y)\,dt}{t\mu(B(x,t))} \\ &\leq \frac{1}{\mu(B(x_0,|x-x_0|))^2}\Big(\frac{r}{|x-x_0|}\Big)^2. \end{aligned}$$

For II_4, because $z \in B(x_0,r)$ implies that $|z-x_0| < r < 4\rho(x_0)$, we have $\rho(x_0) \sim \rho(z)$. Also, for $r/2 < t < \frac{|x-x_0|}{4}$ and $|z-x_0| < r < \frac{|x-x_0|}{4}$, it holds that $|y-z| \sim |x-x_0|$. Then, by condition (1) of Proposition 3.9, we obtain

$$\begin{aligned} II_4 &\leq \int_{r/2}^{|x-x_0|/4}\int_{|x-y|<t}\Big|\int_{B(x_0,r)} D_t^L(y,z)a(z)\,d\mu(z)\Big|^2\frac{d\mu(y)\,dt}{t\mu(B(x,t))} \\ &\leq \int_{r/2}^{|x-x_0|/4}\int_{|x-y|<t}\frac{(1+t/\rho(x_0))^{-M}}{\mu(B(y,|x-x_0|))^2}\frac{t^2}{t^2+|x-x_0|^2}\frac{d\mu(y)\,dt}{t\mu(B(x,t))}. \end{aligned}$$

For every $z \in B(x,|x-x_0|)$, we have $|y-z| \leq |x-y| + |x-x_0|$. Then, by use of the fact that $|x-y| < t < \frac{|x-x_0|}{4}$, we have $|y-z| \leq \frac{5}{4}|x-x_0|$. By the double property of the measure μ, we get $\frac{1}{\mu(B(y,|x-x_0|))} \leq \frac{C_\gamma}{\mu(B(x_0,|x-x_0|))}$. Taking M large enough, we have

$$\begin{aligned} II_4 &\leq \int_{r/2}^{|x-x_0|/4}\frac{1}{\mu(B(x,|x-x_0|))^2}\frac{t^2}{|x-x_0|^2}\Big(1+\frac{t}{\rho(x_0)}\Big)^{-2M}\frac{dt}{t} \\ &\leq \frac{1}{\mu(B(x,|x-x_0|))^2}\Big(\frac{r}{|x-x_0|}\Big)^2. \end{aligned}$$

At last, we estimate II_5. Because $|z-x_0| < r < 4\rho(x_0)$ implies that $\rho(x_0) \sim \rho(z)$, then similarly we have

$$\begin{aligned} II_5 &\leq \int_{\frac{|x-x_0|}{4}}^{\infty}\int_{|x-y|<t}\Big(\int_{B(x_0,r)}\frac{(1+t/\rho(x_0))^{-M}}{\mu(B(y,\sqrt{t^2+|y-z|^2}))}\frac{d\mu(z)}{\mu(B(x_0,r))}\Big)^2\frac{d\mu(y)\,dt}{t\mu(B(x,t))} \\ &\leq \frac{1}{\mu(B(x_0,|x-x_0|))^2}\int_{|x-x_0|/4}^{\infty}\Big(\frac{\rho(x_0)}{t}\Big)^{2M}\frac{\mu(B(x_0,|x-x_0|))^2}{\mu(B(x,t))^2}\frac{dt}{t}. \end{aligned}$$

For every $z \in B(x_0, |x-x_0|)$, we have $\mu(B(x_0,|x-x_0|)) \lesssim \mu(B(x, 2|x-x_0|))$. If $t > 2|x-x_0|$, by use of $\rho(x_0) \le r$, then we have

$$II_5 \le \frac{r^{2M}}{|x-x_0|^{2M}} \frac{1}{\mu(B(x_0,|x-x_0|))^2}.$$

If $t < 2|x-x_0|$, then $(\mu(B(x_0,|x-x_0|))/\mu(B(x,t)))^2 \le (2|x-x_0|/t)^{2\gamma}$, and hence

$$II_5 \le \frac{1}{\mu(B(x_0,|x-x_0|))^2}\Big(\frac{r}{|x-x_0|}\Big)^{2M}.$$

Finally, we obtain

$$\begin{aligned}\int_{B^c(x_0,4r)} S_P^L a(x)\,d\mu(x) &\le \sum_{k=2}^{\infty}\int_{2^k r\le|x-x_0|<2^{k+1}r}\Big[\frac{r}{|x-x_0|}+\Big(\frac{r}{2^k r}\Big)^M\Big]\frac{d\mu(x)}{\mu(B(x_0,2^k r))}\\ &\le \sum_{k=2}^{\infty}\frac{1}{2^k}+\sum_{k=2}^{\infty}\frac{1}{2^{kM}} \le C.\end{aligned}$$

This completes the proof of Theorem 3.11. □

Now we give the converse of Theorem 3.11. First, we need a reproducing formula associated with D_t^L. We introduce the following weak-type convergence related to the dual of $\mathrm{BMO}_L(d\mu)$.

Definition 3.12. We say that $f \in (\mathrm{BMO}_L(d\mu))^*$ is equal to zero weakly at ∞ associated with L if

$$\lim_{A\to\infty}\int_A^\infty (D_t^L)^2 f(x)\frac{dt}{t} = 0,$$

where the above limit holds in the sense of $(\mathrm{BMO}_L(d\mu))^*$.

We can prove the following reproducing formula.

Theorem 3.13. *Suppose that f is equal to zero weakly at ∞ associated with L. We have*

$$f(x) = 8\int_0^\infty (D_t^L)^2 f(x)\frac{dt}{t}, \tag{3.4}$$

where the integral means, in $(\mathrm{BMO}_L(d\mu))^*$,

$$\lim_{\varepsilon\longrightarrow 0}\lim_{A\longrightarrow\infty} 8\int_\varepsilon^A (D_t^L)^2 f(x)\frac{dt}{t} = f(x).$$

Proof. It is easy to see that

$$8\int_\epsilon^A (D_t^L)^2 f(x)\frac{dt}{t} = 8\int_\epsilon^\infty (D_t^L)^2 f(x)\frac{dt}{t} - 8\int_A^\infty (D_t^L)^2 f(x)\frac{dt}{t} = I_1 - I_2.$$

Because f is equal to zero weakly at ∞ associated with L, we have $\lim_{A\to\infty} I_2 = 0$. For any $\phi \in \mathrm{BMO}_L(d\mu)$,

$$\Big\langle 8\int_\epsilon^\infty (D_t^L)^2 f(x)\frac{dt}{t}, \phi(x)\Big\rangle = \Big\langle f(x), 8\int_\epsilon^\infty (D_t^L)^2\phi(x)\frac{dt}{t}\Big\rangle = \big\langle f(x), \phi(x)\big\rangle.$$

The last equality holds since $\lim_{\epsilon\to 0} 8\int_\epsilon^\infty (D_t^L)^2\frac{dt}{t} = I$ in the sense of $(\mathrm{BMO}_L(d\mu))^*$ and I is the identity operator in $(\mathrm{BMO}_L(d\mu))^*$. □

For the converse of Theorem 3.11, we assume that $f \in (\mathrm{BMO}_L(d\mu))^* \cap L^1(d\mu)$. On the one hand, if $f \in H^1(d\mu)$, then it is obvious that $f \in (\mathrm{BMO}_L(d\mu))^* \cap L^1(d\mu)$. Theorem 3.11 guarantees that $S_P^L(f) \in L^1(d\mu)$. Conversely, if $f \in (\mathrm{BMO}_L(d\mu))^* \cap L^1(d\mu)$ and $S_P^L(f) \in L^1(d\mu)$, we will use the reproducing formula (3.4) to derive that f can be represented as the linear combination of H^1-atoms and the scalars. By Proposition 1.3, this means that $f \in H^1_L(d\mu)$. Precisely, we have the following theorem.

Theorem 3.14. *Suppose that $V \in B_{q,\mu}, q > 1$. Let L be the degenerate Schrödinger operator defined by (1.1) for every $f \in (\mathrm{BMO}_L(d\mu))^* \cap L^1(d\mu)$ and equal to zero weakly at ∞ associated with L. If $S_P^L f \in L^1(d\mu)$, then we have $f \in H^1_L(d\mu)$.*

Proof. We can see that

$$\int_{\mathbb{R}^n} \left|S_P^L f(x)\right| d\mu(x) = \int_{\mathbb{R}^n} \Big(\int_0^\infty \int_{B(x,t)} \left|D_t^L f(y)\right|^2 \frac{d\mu(y)\, dt}{t\mu(B(x,t))}\Big)^{1/2} d\mu(x),$$

and so $D_t^L f(x)$ belongs to the tent space T_2^1. By the atomic decomposition of T_2^1, we have $D_t^L f(x) = \sum_i \lambda_i a_i(x,t)$, where $a_i(x,t)$ are T_2^1-atoms and $\sum_i |\lambda_i| < \infty$. We assume that the atom $a(x,t)$ is supported on $\widehat{B}(x_0, r)$. By the reproducing formula (3.4),

$$f(x) = 4\int_0^\infty D_t^L\Big(\sum_{i=1}^\infty \lambda_i a_i(x,t)\Big)\frac{dt}{t} := \sum_{i=1}^\infty \lambda_i \alpha_i(x),$$

where $\alpha_i(x) = \int_0^\infty D_t^L a_i(x,t)\frac{dt}{t}$. We have

$$\begin{aligned}\big\|\sup_{t>0}\left|e^{-tL}\alpha(x)\right|\big\|_{L^1} &\le \big\|\big(\sup_{t>0}\left|e^{-tL}\alpha(x)\right|\big)\chi_{B^*}\big\|_{L^1} + \big\|\big(\sup_{t>0}\left|e^{-tL}\alpha(x)\right|\big)\chi_{(B^*)^c}\big\|_{L^1}\\ &:= I_1 + I_2.\end{aligned}$$

For I_1, we use Hölder's inequality to deduce that

$$\begin{aligned}\|\alpha\|_2 &= \sup_{\|\beta\|_2\le 1}\int_{\mathbb{R}^n}\Big(\int_0^\infty D_t^L a(x,t)\frac{dt}{t}\Big)\bar{\beta}(x)\, d\mu(x)\\ &\le \sup_{\|\beta\|_2\le 1}\Big(\int_0^\infty\int_{\mathbb{R}^n}\left|a(x,t)\right|^2\frac{dt\, d\mu(x)}{t}\Big)^{1/2}\Big(\int_0^\infty\int_{\mathbb{R}^n}\left|D_t^L\bar{\beta}(x)\right|^2\frac{dt\, d\mu(x)}{t}\Big)^{1/2}\\ &\le \sup_{\|\beta\|_2\le 1}\mu(B)^{-1/2}\|\beta\|_2 \le \mu(B)^{-1/2},\end{aligned}$$

which gives $I_1 \le \mu(B^*)^{1/2}\mu(B)^{-1/2} \le C_\gamma$.

Now we deal with I_2. For $s > 0$, by functional calculus, we have

$$\begin{aligned}&\Big|e^{-sL}\Big(\int_0^\infty D_t^L a(x,t)\frac{dt}{t}\Big)\Big|\\ &\quad= \Big|c\int_0^\infty\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{t^2 L}{4u}} t^2 L e^{-sL} a(x,t)\frac{dt}{t}\, du\Big|\\ &\quad\le \int_0^\infty\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\frac{t^2}{s+\frac{t^2}{4u}}\int_{\mathbb{R}^n}\big|Q_{\sqrt{s+\frac{t^2}{4u}}}(x,y)\big|\left|a(y,t)\right|\frac{d\mu(y)\, dt\, du}{t}.\end{aligned}$$

By condition (1) of Proposition 3.9, we have

$$\begin{aligned}
&\Big|e^{-sL}\Big(\int_0^\infty D_t^L a(x,t)\frac{dt}{t}\Big)\Big| \\
&\quad\le \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\Big(\int_0^r\int_B\Big(\frac{t^2}{s+t^2/4u}\Big)^2\frac{1}{\mu(B(x,\sqrt{s+t^2/4u}))^2} \\
&\qquad\times e^{-\frac{2|x-y|^2}{s+t^2/4u}}\frac{d\mu(y)\,dt}{t}\Big)^{1/2}\Big(\int_0^r\int_B|a(y,t)|^2\frac{d\mu(y)\,dt}{t}\Big)^{1/2}du \\
&\quad:= \int_0^\infty \frac{e^{-u}}{\sqrt{u}} I_{2,1}\times I_{2,2}\,du.
\end{aligned}$$

Clearly,

$$I_{2,2} = \Big(\int_0^r\int_B\big|a(y,t)\big|^2\frac{d\mu(y)\,dt}{t}\Big)^{1/2} \le \mu\big(B(x_0,r)\big)^{-1/2}.$$

For $x\in B^c(x_0,2r)$ and $y\in B(x_0,r)$, we have $|x-y|\sim|x-x_0|$. Therefore, by the doubling property of the measure μ, we get

$$\begin{aligned}
I_{2,1} &\le \Big(\int_0^r\int_B\Big(\frac{t^2}{s+t^2/4u}\Big)^2\frac{1}{\mu(B(x,\sqrt{s+t^2/4u}))^2}e^{-\frac{2|x-x_0|^2}{s+t^2/4u}}\frac{d\mu(y)\,dt}{t}\Big)^{1/2} \\
&\le \frac{u}{\mu(B(x_0,|x-x_0|))}\Big(\int_0^r\frac{t^2/4u}{s+t^2/4u}\Big(1+\frac{2|x-x_0|}{\sqrt{s+t^2/4u}}\Big)^\gamma e^{-\frac{2|x-x_0|^2}{s+t^2/4u}}\frac{dt}{t}\Big)^{1/2}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Big|e^{-sL}\Big(\int_0^\infty D_t^L a(x,t)\frac{dt}{t}\Big)\Big| \\
&\quad\le \int_0^\infty \frac{\sqrt{u}e^{-u}}{\mu(B(x_0,|x-x_0|))}\Big(\int_0^r\frac{t^2/4u}{s+t^2/4u}\Big(1+\frac{2|x-x_0|}{\sqrt{s+t^2/4u}}\Big)^\gamma e^{-\frac{2|x-x_0|^2}{s+t^2/4u}}\frac{dt}{t}\Big)^{\frac12}du \\
&\quad\le \int_0^\infty \frac{\sqrt{u}e^{-u}}{\mu(B(x_0,|x-x_0|))}I(u)\,du,
\end{aligned}$$

where

$$I(u) = \Big(\int_0^r\frac{t^2/4u}{s+t^2/4u}\Big(1+\frac{2|x-x_0|}{\sqrt{s+t^2/4u}}\Big)^\gamma\frac{1}{(1+\frac{|x-x_0|^2}{s+t^2/4u})}\frac{dt}{t}\Big)^{1/2}.$$

Choose l large enough. We have

$$I(u) \le \Big(\int_0^r\frac{t^2}{4u}\frac{1}{(s+t^2/4u+|x-x_0|^2)}\frac{dt}{t}\Big)^{1/2} \le \frac{r}{\sqrt{u}|x-x_0|},$$

and hence

$$\begin{aligned}
\Big|e^{-sL}\Big(\int_0^\infty D_t^L a(x,t)\frac{dt}{t}\Big)\Big| &\le \frac{1}{\mu(B(x_0,|x-x_0|))}\int_0^\infty\sqrt{u}e^{-u}\frac{r}{|x-x_0|}\frac{1}{\sqrt{u}}\,du \\
&\le \frac{r}{|x-x_0|}\frac{1}{\mu(B(x_0,|x-x_0|))}.
\end{aligned}$$

Finally, we get

$$I_2 \leq \int_{B^c(x_0,r)} \frac{r}{|x-x_0|} \frac{1}{\mu(B(x_0,|x-x_0|))}\, d\mu(x)$$
$$\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\mu(B(x_0,2^{k+1}r))}{\mu(B(x_0,2^k r))} \leq C.$$

This completes the proof of Theorem 3.14. □

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