Banach J. Math. Anal. 10 (2016), no. 4, 686–702
http://dx.doi.org/10.1215/17358787-3649854
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

IDEAL STRUCTURES IN VECTOR-VALUED POLYNOMIAL SPACES

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Communicated by D. Werner

Abstract. This paper is concerned with the study of geometric structures in spaces of polynomials. More precisely, we discuss for E and F Banach spaces, whether the class of n-homogeneous polynomials, $\mathcal{P}_w(^nE,F)$, which are weakly continuous on bounded sets, is an HB-subspace or an $M(1,C)$-ideal in the space of continuous n-homogeneous polynomials, $\mathcal{P}(^nE,F)$. We establish sufficient conditions under which the problem can be positively solved. Some examples are given. We also study when some ideal structures pass from $\mathcal{P}_w(^nE,F)$ as an ideal in $\mathcal{P}(^nE,F)$ to the range space F as an ideal in its bidual F^{**}.

1. Introduction

Let X be a (real or complex) Banach space, and let J be a closed subspace of X. According to the Hahn–Banach theorem, every continuous linear functional $g \in J^*$ has an extension $f \in X^*$ with the same norm. A long-standing problem is to determine when every functional on J has a *unique* norm-preserving extension to X. This question is closely related to geometric properties of both spaces which, in many cases, imply the existence of a norm 1 projection on X^* whose kernel is $J^\perp := \{x^* \in X^* \colon x^*(y) = 0,\ \text{for all } y \in J\}$, the annihilator of J. When there exists such a projection, J is said to be an *ideal* in X. A canonical example of this fact is that X is always an ideal in its bidual X^{**}.

The notion of M-*ideal*, introduced by Alfsen and Effros and widely studied by Harmand, Werner, and Werner in [18], is one of these geometric properties

Received Dec. 3, 2015; Accepted Dec. 22, 2015.
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2010 *Mathematics Subject Classification.* Primary 46G25; Secondary 47H60, 46B04, 47L22.
Keywords. HB-subspaces, homogeneous polynomials, weakly continuous on bounded sets polynomials.

ensuring unique Hahn–Banach extensions. Recall that J *is an M-ideal in* X if it is an ideal in X with associated projection q such that for each $f \in X^*$ one has

$$\|f\| = \|qf\| + \|f - qf\|.$$

The fact that J is an M-ideal in X has a strong impact on both J and X, and sometimes seems to be too restrictive. So, we will be interested in studying some weaker properties among those implying unique norm-preserving extensions.

Recall that a closed subspace J is *HB-smooth* in X if every element in J^* has a unique norm-preserving extension to an element in X^*. A closed subspace J is *strongly HB-smooth* in X if there exists a linear projection q on X^* whose kernel is $J^\perp$ such that for each $f \in X^*$ with $f \neq qf$ one has

$$\|qf\| < \|f\|.$$

The interplay between uniqueness of the extension and strong HB-smoothness was clarified by Oja [22]. Namely, the uniqueness of the extensions and being an ideal are independent notions for a subspace J, strong HB-smoothness implies both, and if J is an HB-smooth ideal in X, then J is strongly HB-smooth in X.

A particular case of HB-smoothness is the notion of *HB-subspace*, introduced by Hennefeld [19]. A closed subspace J *is an HB-subspace of* X if there exists a projection q on X^* whose kernel is $J^\perp$ such that for each $f \in X^*$ with $f \neq qf$ one has

$$\|qf\| < \|f\| \qquad \text{and} \qquad \|f - qf\| \leq \|f\|.$$

Finally, given $C \in (0, 1]$, a closed subspace J is an $M(1, C)$*-ideal* in X if J is an ideal of X with associated projection q on X^* such that for each $f \in X^*$ one has

$$\|qf\| + C\|f - qf\| \leq \|f\|.$$

The last inequality is called the $M(1, C)$*-inequality*. Note that when $C = 1$, the notion of M-ideal is covered and to be $M(1, C)$-ideal immediately implies strong HB-smoothness. However, the notions of $M(1, C)$-ideal and HB-subspace are independent. On the one hand, Cabello and Nieto [6, Example 3.7] showed that if X is a nonreflexive separable M-ideal in its bidual, then $\ell_p(X)$ as a subspace of its bidual, $1 < p < \infty$, is an HB-subspace that cannot be renormed to be an $M(1, C)$-ideal for any $0 < C < 1$. On the other hand, Cabello, Nieto, and Oja [7, Example 4.3] showed that for any $0 < C < 1$, there is a renorming of c_0, $\hat{c}_0$ due to Johnson and Wolfe such that the space of compact operators on $\hat{c}_0$, $\mathcal{K}(\hat{c}_0)$ is an $M(1, C)$-ideal in the space of all continuous operators $\mathcal{L}(\hat{c}_0)$ without being an HB-subspace.

Several authors have been interested in this kind of property for arbitrary subspaces of Banach spaces and also for distinguished particular cases. The space of compact operators $\mathcal{K}(E, F)$ between Banach spaces E and F as a subspace of the space of all continuous linear operators $\mathcal{L}(E, F)$ received special interest (see, e.g., [6], [7], [19], [21]–[24], [26]). The strongest of the abovementioned properties is the one of being an M-ideal. All other properties which are more flexible still allow us to deal with uniqueness of Hahn–Banach extensions.

Here, we will be concerned with $\mathcal{P}({}^nE,F)$, the space of continuous n-homogeneous polynomials between Banach spaces E and F. In the polynomial context, the space of *compact mappings* is usually replaced by $\mathcal{P}_w({}^nE,F)$, the subspace of *homogeneous polynomials which are weakly continuous on bounded sets*. Recall that a polynomial $P \in \mathcal{P}({}^nE,F)$ is in $\mathcal{P}_w({}^nE,F)$ if it maps bounded weakly convergent nets into convergent nets. Note that we could have considered polynomials in $\mathcal{P}({}^nE,F)$ mapping bounded sets into relatively compact sets, which are called *compact polynomials*. For linear operators to be compact and to be weakly continuous on bounded sets are equivalent notions. For n-homogeneous polynomials with $n > 1$, every polynomial in $\mathcal{P}_w({}^nE,F)$ is compact (as can be derived from results in [3] and [4]), but the converse might not be true. Every scalar-valued continuous polynomial is compact but it is not necessarily weakly continuous on bounded sets. The prototypical example of this situation is given by $P(x) = \sum_k x_k^2$, for all $x = (x_k)_k \in \ell_2$. Therefore, we will focus our attention on determining the presence of ideal structures for $\mathcal{P}_w({}^nE,F)$ as a subspace of $\mathcal{P}({}^nE,F)$. To be more precise, our main concern is to study the notion of HB-subspace in the polynomial setting.

Some previous results in this direction can be found in [12], where the problem of determining when $\mathcal{P}_w({}^nE)$ is an M-ideal in $\mathcal{P}({}^nE)$ was considered. A vector-valued approach of the same question was treated in [14]. Note that the searching of ideal structures for $\mathcal{P}_w({}^nE,F)$ as a subspace of $\mathcal{P}({}^nE,F)$ makes sense when the spaces $\mathcal{P}_w({}^nE,F)$ and $\mathcal{P}({}^nE,F)$ do not coincide. The equality $\mathcal{P}_w({}^nE,F) = \mathcal{P}({}^nE,F)$ is a long-standing nontrivial problem considered, for instance, in [1], [5], [16], and [17].

The plan of the paper is as follows. First, we review the notation and the basic facts that will be used in Sections 3 and 4. Then, in Section 3, we investigate sufficient conditions under which the subspace $\mathcal{P}_w({}^nE,F)$ enjoys an additional geometric structure inside $\mathcal{P}({}^nE,F)$, and we present some particular examples. In the last section, we study some ideal structures for the range space F as a subspace of F^{**} when they are fulfilled by $\mathcal{P}_w({}^nE,F)$ as a subspace of $\mathcal{P}({}^nE,F)$.

2. Notation and basic facts

Before proceeding, we fix some notation. Every time we write E or F we will be considering Banach spaces over the real or complex field $\mathbb{K}$. The closed unit ball of E will be denoted by B_E and the unit sphere by S_E. As usual, E^* and E^{**} stand for the dual and bidual of E, respectively. The space of linear bounded operators from E to F will be denoted by $\mathcal{L}(E,F)$ (and $\mathcal{L}(E)$ when $E = F$); its subspace of compact mappings will be denoted by $\mathcal{K}(E,F)$ ($\mathcal{K}(E)$ in the case $E = F$).

A function $P\colon E \to F$ is an n-homogeneous polynomial if there exists a (unique) symmetric n-linear form $A\colon \underbrace{E \times \cdots \times E}_{n} \to F$ such that

$$P(x) = A(x, \ldots, x),$$

for all $x \in E$. The space of all continuous n-homogeneous polynomials from E to F, $\mathcal{P}(^nE, F)$, endowed with the supremum norm

$$\|P\| = \sup\{\|P(x)\|: x \in B_E\},$$

is a Banach space.

Every polynomial P in $\mathcal{P}(^nE; F)$ can be associated with a linear operator in $\mathcal{L}(\widehat{\otimes}^{n,s}_{\pi_s} E; F)$, where π_s is the symmetric projective tensor norm. We will identify P with its linearization without further mention. Even though this identification preserves the norm, there is no Hahn–Banach theorem for homogeneous polynomials of degree 2 or greater. However, Aron and Berner [2] and Davie and Gamelin [9] showed that for every $P \in \mathcal{P}(^nE, F)$ there is a norm-preserving extension of P to $\overline{P} \in \mathcal{P}(^nE^{**}, F^{**})$ such that $\overline{P}(x) = P(x)$ for all $x \in E$. The construction of this *canonical extension* is based on the Arens extension of the symmetric mapping A associated to the polynomial P. To obtain the Arens extension, we simply extend by weak-star continuity, one variable at a time, the n variables of A. This process depends on the order that the variables are extended and the final result might not be a symmetric mapping. However, the $n!$ possible extensions coincide on the diagonal and $\overline{P}$ is well defined. For the particular case in which P belongs to $\mathcal{P}_w(^nE, F)$, the range of $\overline{P}$ is also in F (as can be derived from [3] and [8, Proposition 2.5]). This fact will be used repeatedly in Section 4.

In this paper, we will present several results in which at least one of the spaces involved enjoys the metric compact approximation property. Recall that a Banach space E has the *metric compact approximation property* if there is a net of compact operators (K_α) on E such that $K_\alpha \to \mathrm{Id}_E$ pointwise and $\sup_\alpha \|K_\alpha\| \leq 1$. Usually, the net (K_α) is called a *metric compact approximation of the identity.* If in addition $K_\alpha^* \to \mathrm{Id}_{E^*}$ pointwise, the net (K_α) is called a *shrinking* metric compact approximation of the identity. As usual, K^α denotes the operator $\mathrm{Id}_E - K_\alpha$. For dual spaces, we have the following intermediate property. We say that E^* has a *metric compact approximation of the identity with adjoint operators* if there exists a net $(K_\alpha) \subset \mathcal{K}(E)$ such that K_α^* converges to Id_{E^*} pointwise and $\sup_\alpha \|K_\alpha\| \leq 1$. These notions are closely related with ideal structures on Banach spaces. For instance, [21, Theorem 1.1] asserts that the following conditions are equivalent:

(i) F has the metric compact approximation property,
(ii) $\mathcal{K}(E, F)$ is an ideal in $\mathcal{L}(E, F)$ for every Banach space E.

So, it is natural to expect that the metric compact approximation property shows up when describing $\mathcal{P}_w(^nE, F)$ as an ideal in $\mathcal{P}(^nE, F)$.

One further ingredient will appear in our discussion. In [10], Delpech obtained an appropriate connection between the moduli of asymptotic uniform smoothness and convexity and weak sequential continuity of polynomials. In [13], Dimant, Gonzalo, and Jaramillo followed his approach to obtain results on compactness or weak-sequential continuity of multilinear mappings. Here, we will impose restrictions on the growth of the moduli of the underlying spaces E or F to ensure that $\mathcal{P}_w(^nE, F)$ enjoys an appropriate property in $\mathcal{P}(^nE, F)$ (see Theorem 3.4 and Proposition 3.13). Some definitions are in order.

For an infinite-dimensional Banach space E, the *modulus of asymptotic pointwise smoothness* is defined for $\|x\| = 1$ and $t > 0$ by

$$\overline{\rho}_E(t;x) = \inf_{\dim(E/H)<\infty} \sup_{h\in H, \|h\|\leq t} \|x+h\| - 1,$$

and the *modulus of asymptotic uniform smoothness* is defined for $t > 0$ by

$$\overline{\rho}_E(t) = \sup_{\|x\|=1} \overline{\rho}_E(t;x).$$

The space E is *asymptotically uniformly smooth* if $\lim_{t\to 0} \frac{\overline{\rho}_E(t)}{t} = 0$.

For an infinite-dimensional Banach space, the *modulus of asymptotic pointwise convexity* is defined for $\|x\| = 1$ and $t > 0$ by

$$\overline{\delta}_E(t;x) = \sup_{\dim(E/H)<\infty} \inf_{h\in H, \|h\|\geq t} \|x+h\| - 1,$$

and the *modulus of asymptotic uniform convexity* is defined for $t > 0$ by

$$\overline{\delta}_E(t) = \inf_{\|x\|=1} \overline{\delta}_E(t;x).$$

The space E is *asymptotically uniformly convex* if $\overline{\delta}_E(t) > 0$, for every $0 < t \leq 1$. Finally, E has modulus of *asymptotic uniform convexity of power p* if there exists $C > 0$ such that $\overline{\delta}_E(t) \geq Ct^p$, for all $0 < t \leq 1$.

We refer to [15] for the necessary background on polynomials on Banach spaces.

3. Sufficient conditions

When working with polynomials, the lack of linearity provides, in many cases, difficulties that can be overcome not without certain detours. The value of n for which $\mathcal{P}_w(^nE, F)$ has the chance to be a nontrivial M-ideal in $\mathcal{P}(^nE, F)$ cannot be chosen arbitrarily. In fact, in the scalar-valued case (see [12]) it was proved that, whenever $\mathcal{P}(^mE) \setminus \mathcal{P}_w(^mE) \neq \emptyset$ for some m, there exists a unique value n, called the *critical degree*, for which $\mathcal{P}_w(^nE)$ can be a nontrivial M-ideal in $\mathcal{P}(^nE)$. The critical degree of E is defined as

$$\operatorname{cd}(E) := \min\{k \in \mathbb{N} \colon \mathcal{P}_w(^kE) \neq \mathcal{P}(^kE)\}.$$

In the vector-valued case, the critical degree is defined by analogy (see [14]) as

$$\operatorname{cd}(E,F) := \min\{k \in \mathbb{N} \colon \mathcal{P}_w(^kE,F) \neq \mathcal{P}(^kE,F)\},$$

and the problem of whether $\mathcal{P}_w(^nE, F)$ is an M-ideal in $\mathcal{P}(^nE, F)$ is worth being studied only for polynomials of degree n with $\operatorname{cd}(E, F) \leq n \leq \operatorname{cd}(E)$. Although we are interested in studying ideal structures which are more flexible than to be an M-ideal, in order to show positive results we could not get rid of some restrictions on the degree of homogeneity. We start with a lemma which, under certain conditions on n, gives a version of Johnson's projection (see [20, Lemma 1.1]) for the polynomial case.

Lemma 3.1. *Let E, F be Banach spaces, and let $n < \operatorname{cd}(E)$. Suppose that F has the metric compact approximation property. Then $\mathcal{P}_w(^nE, F)$ is an ideal in $\mathcal{P}(^nE, F)$.*

Proof. Let $(L_\beta) \subset \mathcal{K}(F)$ be a metric compact approximation of the identity. As (L_β) is bounded, there exists a subnet, still denoted by (L_β), that converges w^* to some $L_0 \in \mathcal{K}(F)^{**}$. Define $\Lambda\colon \mathcal{P}(^nE, F)^* \to \mathcal{P}(^nE, F)^*$ by

$$\Lambda(f)(P) = \lim_\beta f(L_\beta \circ P). \tag{3.1}$$

Note that Λ is well defined. In fact, if $P \in \mathcal{P}(^nE, F)$ and $\tau_P\colon \mathcal{K}(F) \to \mathcal{P}(^nE, F)$ is the composition operator, $\tau_P(K) = K \circ P$, then its transpose τ_P^* satisfies that $\tau_P^*(f) \in \mathcal{K}(F)^*$ for any $f \in \mathcal{P}(^nE, F)^*$. So

$$\lim_\beta f(L_\beta \circ P) = \lim_\beta \tau_P^*(f)(L_\beta) = L_0\big(\tau_P^*(f)\big).$$

It is clear that Λ is linear and $\|\Lambda\| \leq 1$. It is also a projection: since any $P \in \mathcal{P}_w(^nE, F)$ is compact and (L_β) converges to the identity on compact sets, we see that $\lim_\beta L_\beta \circ P = P$. Thus,

$$\Lambda(f)(P) = \lim_\beta f(L_\beta \circ P) = f(P)$$

for $P \in \mathcal{P}_w(^nE, F)$. Now, by [14, Lemma 1.8], as $n < \mathrm{cd}(E)$, the net of polynomials $(L_\beta \circ Q)$ belongs to $\mathcal{P}_w(^nE, F)$ for every $Q \in \mathcal{P}(^nE, F)$. Hence,

$$\Lambda\big(\Lambda(f)\big)(Q) = \lim_\beta \Lambda(f)(L_\beta \circ Q) = \lim_\beta f(L_\beta \circ Q) = \Lambda(f)(Q),$$

and $\Lambda^2 = \Lambda$. Finally, it is easy to check that $\ker \Lambda = \mathcal{P}_w(^nE, F)^\perp$. Then, Λ is a norm 1 projection on $\mathcal{P}(^nE, F)^*$ with $\ker \Lambda = \mathcal{P}_w(^nE, F)^\perp$. $\square$

Remark 3.2. Every time $\mathcal{P}_w(^nE, F)$ is an ideal in $\mathcal{P}(^nE, F)$ with associated projection Λ, we have the decomposition

$$\mathcal{P}(^nE, F)^* = \mathcal{P}_w(^nE, F)^* \oplus \mathcal{P}_w(^nE, F)^\perp,$$

and any $f \in \mathcal{P}(^nE, F)^*$ has a unique representation such that

$$f = g + h, \quad \text{with } g = \Lambda(f) \in \mathcal{P}_w(^nE, F)^*, h = f - g \in \mathcal{P}_w(^nE, F)^\perp. \tag{3.2}$$

Now, if F has a metric compact approximation of the identity $(L_\beta) \subset \mathcal{K}(F)$, we may (and will) suppose that (L_β) is w^*-convergent in $\mathcal{K}(F)^{**}$ and that the projection Λ is defined as in (3.1), $\Lambda(f)(P) = \lim_\beta f(L_\beta \circ P)$. Then, with $L^\beta = \mathrm{Id}_F - L_\beta$, we have the following facts that were already used:

- $\lim_\beta \Lambda(f)(L^\beta \circ P) = 0$ for all $f \in \mathcal{P}(^nE, F)^*$ and all $P \in \mathcal{P}(^nE, F)$;
- $\lim_\beta L_\beta \circ Q = Q$ for all $Q \in \mathcal{P}_w(^nE, F)$.

Indeed, the first assertion follows by [14, Lemma 1.8]. For the second one, note that L_β converges uniformly to the identity on compact sets.

Finally, note that since $\|\Lambda\| \leq 1$, we automatically have $\|g\| \leq \|f\|$. If in addition $\|L^\beta\| \leq 1$, we also obtain $\|h\| \leq \|f\|$ since $h(P) = (f - \Lambda f)(P) = \lim_\beta f(L^\beta \circ P)$ for all $P \in \mathcal{P}(^nE, F)$.

As mentioned before, Delpech used the moduli of asymptotic uniform convexity and smoothness of a Banach space to obtain properties of weak sequential continuity of polynomials. Dimant, Gonzalo, and Jaramillo [13] showed the connection between these moduli and compactness or weak sequential continuity of multilinear mappings. The moduli play their role when dealing with $\mathcal{P}_w(^nE, F)$

as an HB-subspace of $\mathcal{P}(^nE,F)$. We present a refinement of [10, Lemme 10.3] that will be used later in this article.

Lemma 3.3. *Let E be an infinite-dimensional Banach space, and let $(w_\alpha)\subset E$ be a weakly null bounded net.*

(a) *If $x\in B_E$, then $\overline{\lim}_\alpha\|x+w_\alpha\|\le 1+\overline{\rho}_E(\overline{\lim}_\alpha\|w_\alpha\|)$.*
(b) *If $(x_\alpha)\subset B_E$ is a net contained in a compact set, then*

$$\overline{\lim}_\alpha\|x_\alpha+w_\alpha\|\le 1+\overline{\rho}_E\big(\overline{\lim}_\alpha\|w_\alpha\|\big).$$

Proof. To prove (a) first note that [10, Lemme 10.3] remains valid if we consider weakly null nets instead of weakly null sequences. That is, $\overline{\lim}_\alpha\|x+w_\alpha\|\le 1+\overline{\rho}_E(\overline{\lim}_\alpha\|w_\alpha\|)$, for any $x\in S_E$. Now, fix a nonzero $x\in B_E$, and consider each $x+w_\alpha$ as a convex combination of $\frac{x}{\|x\|}+w_\alpha$ and $\frac{-x}{\|x\|}+w_\alpha$. Applying the above inequality to $\frac{\pm x}{\|x\|}$, we get

$$\overline{\lim}_\alpha\Big\|\frac{\pm x}{\|x\|}+w_\alpha\Big\|\le 1+\overline{\rho}_E\big(\overline{\lim}_\alpha\|w_\alpha\|\big),$$

and the statement follows.

Now, suppose that (b) does not hold. Then, we may find subnets $(x_\beta),(w_\beta)$, and $x_0\in B_E$ so that $\lim_\beta x_\beta=x_0$ and $\overline{\lim}_\beta\|x_\beta+w_\beta\|>1+\overline{\rho}_E(\overline{\lim}_\alpha\|w_\alpha\|)$. As $\overline{\rho}_E$ is increasing, $\overline{\lim}_\beta\|x_\beta+w_\beta\|>1+\overline{\rho}_E(\overline{\lim}_\beta\|w_\beta\|)$. Note that for any subnet (β_i) such that $\lim_i\|x_{\beta_i}+w_{\beta_i}\|$ exists, so too does the limit $\lim_i\|x_0+w_{\beta_i}\|$, and both coincide. This implies that $\overline{\lim}_\beta\|x_0+w_\beta\|=\overline{\lim}_\beta\|x_\beta+w_\beta\|$. It follows that $\overline{\lim}_\beta\|x_0+w_\beta\|>1+\overline{\rho}_E(\overline{\lim}_\beta\|w_\beta\|)$, which contradicts (a). □

We are ready to describe $\mathcal{P}_w(^nE,F)$ as an HB-subspace of $\mathcal{P}(^nE,F)$, under certain conditions on F and n.

Theorem 3.4. *Let E be a Banach space, and let $n<\mathrm{cd}(E)$. Let F be an infinite-dimensional Banach space with a shrinking metric compact approximation of the identity $(L_\beta)\subset\mathcal{K}(F)$ such that $\sup_\beta\|L^\beta\|\le 1$, and suppose that F is asymptotically uniformly smooth. Then, $\mathcal{P}_w(^nE,F)$ is an HB-subspace of $\mathcal{P}(^nE,F)$.*

Proof. Consider the projection Λ, given in (3.1), under which $\mathcal{P}_w(^nE,F)$ is an ideal in $\mathcal{P}(^nE,F)$. For any $f\in\mathcal{P}(^nE,F)^*$, write $f=g+h$ as in (3.2). Then, as we commented in Remark 3.2, $\|g\|\le\|f\|$ and $\|h\|\le\|f\|$. In order to finish, we have to prove that $\|g\|<\|f\|$ for $h\ne 0$.

Fix $\varepsilon>0$, and take $P\in\mathcal{P}(^nE,F)$ and $Q\in\mathcal{P}_w(^nE,F)$ such that

$$\|P\|=\|Q\|=1,\qquad h(P)>\|h\|-\varepsilon,\qquad\text{and}\qquad g(Q)>\|g\|-\varepsilon.$$

Since $\lim_\beta L_\beta\circ Q=Q$, we can choose β_0 satisfying $|g(L_{\beta_0}\circ Q)|>\|g\|-2\varepsilon$. Change, if necessary, Q to λQ (with $|\lambda|=1$) to obtain $g(L_{\beta_0}\circ Q)>\|g\|-2\varepsilon$. For $t>0$, consider $L_{\beta_0}\circ Q+tL^\beta\circ P$, and take a net $(x_\beta)\subset B_E$ with

$$\overline{\lim}_\beta\|L_{\beta_0}\circ Q+tL^\beta\circ P\|_{\mathcal{P}(^nE,F)}=\overline{\lim}_\beta\big\|(L_{\beta_0}\circ Q)(x_\beta)+t(L^\beta\circ P)(x_\beta)\big\|_F.$$

Now, note that $((L^\beta \circ P)(x_\beta))$ is weakly null. Indeed, $(P(x_\beta))$ is bounded and for any $y^* \in F^*$, $\lim_\beta (L^\beta)^* y^* = 0$. Then,

$$\lim_\beta \big\langle (L^\beta \circ P)(x_\beta), y^* \big\rangle = \lim_\beta \big\langle P(x_\beta), (L^\beta)^* y^* \big\rangle = 0$$

for any $y^* \in F^*$. On the other hand, the compact set $L_{\beta_0}(B_F) \subset B_F$ contains the net $((L_{\beta_0} \circ Q)(x_\beta))$. Therefore, Lemma 3.3 can be applied to get

$$\overline{\lim}_\beta \| L_{\beta_0} \circ Q + tL^\beta \circ P \|_{\mathcal{P}(^n E,F)} \leq 1 + \overline{\rho}_F \big(\overline{\lim}_\beta \big\| t (L^\beta \circ P)(x_\beta) \big\|_F \big) \leq 1 + \overline{\rho}_F(t).$$

As observed in Remark 3.2, $\lim_\beta g(L^\beta \circ P) = 0$. Then, $|g(L^\beta \circ P)| < \varepsilon$ for $\beta \geq \beta_1$. Also, $h(L^\beta \circ P) > \|h\| - \varepsilon$, since $L_\beta \circ P$ belongs to $\mathcal{P}_w(^n E, F)$ and $h(P) = h(L^\beta \circ P)$.

Combining the previous estimates, we conclude for $t > 0$ and $\beta \geq \beta_1$ that

$$\begin{aligned} \big(\|g\| - 2\varepsilon\big) - t\varepsilon + t\big(\|h\| - \varepsilon\big) &< g(L_{\beta_0} \circ Q) - t\big|g(L^\beta \circ P)\big| + th(L^\beta \circ P) \\ &\leq \big| f(L_{\beta_0} \circ Q + tL^\beta \circ P) \big|. \end{aligned}$$

Then, for $t > 0$ and $\varepsilon > 0$,

$$\begin{aligned} \|g\| + t\|h\| - 2\varepsilon(1+t) &\leq \overline{\lim}_\beta \big| f(L_{\beta_0} \circ Q + tL^\beta \circ P) \big| \\ &\leq \|f\| \overline{\lim}_\beta \| L_{\beta_0} \circ Q + tL^\beta \circ P \| \\ &\leq \|f\| \big(1 + \overline{\rho}_F(t)\big). \end{aligned}$$

Thus, $\|g\| + t\|h\| \leq \|f\|(1 + \overline{\rho}_F(t))$. Now, suppose that $\|g\| = \|f\|$; then $t\|h\| \leq \|f\| \overline{\rho}_F(t)$ for $t > 0$. Since F is asymptotically smooth, $\lim_{t \to 0} \frac{\overline{\rho}_F(t)}{t} = 0$ and $h = 0$, which completes the proof. □

Our next result gives another set of sufficient conditions, also related with the notion of smoothness, under which $\mathcal{P}_w(^n E, F)$ is an HB-subspace in $\mathcal{P}(^n E, F)$. It is reminiscent of [23, Theorem 1]. The proof is similar to that of the preceding theorem and we omit it.

Theorem 3.5. *Let E, F be Banach spaces, and let $n < \mathrm{cd}(E)$. Suppose that there exists a metric compact approximation of the identity $(L_\beta) \subset \mathcal{K}(F)$ such that $\sup_\beta \|L^\beta\| \leq 1$ and for any $\varepsilon > 0$ there exist $\mu > 0$ and β_0 so that*

$$\sup_{\|y\|, \|z\| \leq 1} \| L_\beta y + \mu L^\beta z \| \leq 1 + \varepsilon \mu \quad \textit{for all } \beta \geq \beta_0. \tag{3.3}$$

Then, $\mathcal{P}_w(^n E, F)$ is an HB-subspace of $\mathcal{P}(^n E, F)$.

Following [19, Definition 3.6] we say that a Schauder basis of a Banach space is *uniformly smooth* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x+y\| + \|x-y\| < 2 + \varepsilon\|y\|$, whenever x and y have disjoint supports with respect to the basis, $\|x\| = 1$ and $\|y\| < \delta$. Note that by using convex combinations, the definition can be restated for x with $\|x\| \leq 1$. The next result should be compared with [19, Theorem 3.7].

Corollary 3.6. *Let E, F be Banach spaces, and let $n < \mathrm{cd}(E)$. Suppose that F has a uniformly smooth 1-unconditional basis. Then $\mathcal{P}_w(^n E, F)$ is an HB-subspace of $\mathcal{P}(^n E, F)$.*

Proof. Let Π_N be the natural projection on F onto the subspace generated by the first N elements of the 1-unconditional basis. Then, (Π_N) is a metric compact approximation of the identity and satisfies $\sup_N \|\Pi^N\| \leq 1$. Also, (3.3) in Theorem 3.5 holds. Indeed, take $\varepsilon > 0$, and consider $\mu = \delta$ as in the definition of uniform smoothness of the basis. For any $y, z \in B_F$ and $N \in \mathbb{N}$, being the basis 1-unconditional, we have $\|\Pi_N y + \mu \Pi^N z\| \leq 1 + \mu\varepsilon/2$. Thus, an immediate application of Theorem 3.5 gives the result. □

The above corollary can be applied to show some examples of HB-subspaces of polynomials where the spaces ℓ_p and the Lorentz sequence spaces $d(w,p)$ appear. Recall that, for $1 < p < \infty$, both ℓ_p and $d(w,p)$ have uniformly smooth 1-unconditional bases. Also, the critical degree of ℓ_p is the integer number satisfying $p \leq \mathrm{cd}(\ell_p) < p+1$, and $\mathrm{cd}(\ell_p, \ell_q)$ is the integer satisfying $\frac{p}{q} \leq \mathrm{cd}(\ell_p, \ell_q) < \frac{p}{q}+1$. For the case of $\mathrm{cd}(\ell_p, d(w,q))$, a restatement of (I) and (II) in [14, p. 705] reads as

$$\mathrm{cd}\big(\ell_p, d(w,q)\big) = \max\Big\{k \in \mathbb{N} : k < \frac{p}{q} + 1 \text{ and } w \notin \ell_{(\frac{p}{(k-1)q})^*}\Big\}.$$

Example 3.7. Let $1 < p, q < +\infty$.

(a) Let E be a Banach space, and let $n < \mathrm{cd}(E)$. Then,
  - $\mathcal{P}_w(^nE, \ell_q)$ is an HB-subspace of $\mathcal{P}(^nE, \ell_q)$,
  - $\mathcal{P}_w(^nE, d(w,q))$ is an HB-subspace of $\mathcal{P}(^nE, d(w,q))$.

(b) Let $\mathrm{cd}(\ell_p, \ell_q) < n < \mathrm{cd}(\ell_p)$. Then, $\mathcal{P}_w(^n\ell_p, \ell_q)$ is an HB-subspace but not an M-ideal in $\mathcal{P}(^n\ell_p, \ell_q)$.

(c) Let $\mathrm{cd}(\ell_p, d(w,q)) < n < \mathrm{cd}(\ell_p)$. Then, $\mathcal{P}_w(^n\ell_p, d(w,q))$ is an HB-subspace but not an M-ideal in $\mathcal{P}(^n\ell_p, d(w,q))$. The same result holds for $n = \mathrm{cd}(\ell_p, d(w,q))$ for the case $\mathrm{cd}(\ell_p, d(w,q)) < \frac{p}{q}$.

The statements about not being M-ideals in the previous examples are proved in [14, Theorems 3.2 and 3.9].

Now we consider conditions satisfied by the domain space E so that we also have geometric structures in $\mathcal{P}(^nE; F)$. Similarly to what happens in Lemma 3.1, we will describe $\mathcal{P}_w(^nE, F)$ as an ideal of $\mathcal{P}(^nE, F)$ whenever E^* has a metric compact approximation of the identity with adjoint operators. Here, no restrictions on the degree of the polynomials are imposed.

Lemma 3.8. *Let E, F be Banach spaces such that E^* has a metric compact approximation of the identity with adjoint operators. Then $\mathcal{P}_w(^nE, F)$ is an ideal of $\mathcal{P}(^nE, F)$ for all $n \in \mathbb{N}$.*

Proof. Let $(K_\alpha) \subset \mathcal{K}(E)$ be a net satisfying $\lim_\alpha K_\alpha^* x^* = x^*$ for all $x^* \in E^*$ and $\sup_\alpha \|K_\alpha\| \leq 1$. Without loss of generality, we may assume that (K_α) is weak*-convergent in $\mathcal{K}(E)^{**}$. Therefore, as in Lemma 3.1, the mapping

$$\Lambda\colon \mathcal{P}(^nE, F)^* \to \mathcal{P}(^nE, F)^*$$

given by

$$\Lambda(f)(P) = \lim_\alpha f(P \circ K_\alpha) \tag{3.4}$$

is well defined. It is clear that Λ is linear and $\|\Lambda\| \leq 1$. By [14, Lemma 2.1], $\lim_\alpha \|P - P \circ K_\alpha\| = 0$ for every $P \in \mathcal{P}_w(^nE, F)$. Then, $\Lambda(f)(P) = f(P)$ for all $P \in \mathcal{P}_w(^nE, F)$. Furthermore, Λ is a projection: for all $Q \in \mathcal{P}(^nE, F)$, $(Q \circ K_\alpha)$ belongs to $\mathcal{P}_w(^nE, F)$. Thus, for all $Q \in \mathcal{P}(^nE, F)$,

$$\Lambda\big(\Lambda(f)\big)(Q) = \lim_\alpha \Lambda(f)(Q \circ K_\alpha) = \lim_\alpha f(Q \circ K_\alpha) = \Lambda(f)(Q),$$

and $\Lambda^2 = \Lambda$. It is easy to check that $\ker \Lambda = \mathcal{P}_w(^nE, F)^\perp$, and the result follows. □

The next result gives a sufficient condition to obtain the dual space $\mathcal{P}_w(^nE, F)^*$ as a quotient. We denote by π the projective tensor norm. Recall that $\overline{P}$ denotes the canonical extension of P in $\mathcal{P}(^nE, F)$ to $\mathcal{P}(^nE^{**}, F^{**})$.

Proposition 3.9. *Let E, F be Banach spaces such that $\mathcal{P}_w(^nE, F)$ does not contain ℓ_1. Then, the application $j\colon \widehat{\otimes}^{n,s}_{\pi_s} E^{**} \widehat{\otimes}_\pi F^* \to \mathcal{P}_w(^nE, F)^*$, given on any elementary tensor $u \otimes y^*$ by $j(u \otimes y^*)(P) = y^*(\overline{P}(u))$, is a quotient mapping.*

Proof. Take $v \in \widehat{\otimes}^{n,s}_{\pi_s} E^{**} \widehat{\otimes}_\pi F^*$. For each representation of v of the form $\sum_i u_i \otimes y_i^*$, with $u_i \in \widehat{\otimes}^{n,s}_{\pi_s} E^{**}$ and $y_i^* \in F^*$ for all i, we have

$$\Big| j\Big(\sum_i u_i \otimes y_i^*\Big)(P)\Big| = \Big|\sum_i y_i^*\big(\overline{P}(u_i)\big)\Big| \leq \|P\| \sum_i \|u_i\| \|y_i^*\|.$$

So, j is continuous and $\|j\| = 1$. Using Haydon's characterization of spaces not containing ℓ_1, we may write the unit ball of $\mathcal{P}_w(^nE, F)^*$ as the closed convex hull of its extreme points. Now, by [14, Proposition 1.2], with $e_z(P) = \overline{P}(z)$ for $z \in E^{**}$, we obtain

$$\begin{aligned} B_{\mathcal{P}_w(^nE,F)^*} &= \overline{\Gamma(Ext_{\mathcal{P}_w(^nE,F)^*})} \subset \overline{\Gamma(e_z \otimes y^*\colon z \in S_{E^{**}}, y^* \in S_{F^*})} \\ &\subset \overline{j(B_{\widehat{\otimes}^{n,s}_{\pi_s} E^{**} \widehat{\otimes}_\pi F^*})} \subset B_{\mathcal{P}_w(^nE,F)^*}. \end{aligned}$$

Then, all the inclusions are (actually) equalities and j is a quotient mapping. □

In the next result, we show that the natural hypotheses on E and F guarantee that $\mathcal{P}_w(^nE, F)$ does not contain ℓ_1, and the preceding proposition can be applied. We will appeal to the result by Stegall which asserts that if a Banach space E has a separable subspace whose dual is nonseparable, then E^* lacks the Radon–Nikodym property (see, e.g., [11, Theorem VII.2.6]).

Proposition 3.10. *Let E, F be Banach spaces such that E^{**} and F^* have the Radon–Nikodym property. Then $\mathcal{P}_w(^nE, F)^*$ has the Radon–Nikodym property, and hence $\mathcal{P}_w(^nE, F)$ does not contain ℓ_1 for all $n \in \mathbb{N}$.*

Proof. For any $P \in \mathcal{P}_w(^nE, F)$ we consider its associated symmetric multilinear map A and define $T_P \in \mathcal{L}(E, \mathcal{P}_w(^{n-1}E, F))$ as the operator given by $T_P(x)(\tilde{x}) = A(x, \tilde{x}, \ldots, \tilde{x})$. By [3, Theorem 2.9], T_P is a well-defined compact operator and $\|P\| \leq \|T_P\| \leq e\|P\|$. Then, the mapping $\Phi\colon \mathcal{P}_w(^nE, F) \to \mathcal{K}(E, \mathcal{P}_w(^{n-1}E, F))$ given by $\Phi(P) = T_P$ is an isomorphism with its image. As the Radon–Nikodym property is preserved by isomorphisms, induction on n and [25, Theorem 1.9] yield the result. □

Lemma 3.11. *Let E, F be Banach spaces such that $\mathcal{P}_w({}^nE, F)$ does not contain ℓ_1 and such that $n < \mathrm{cd}(E)$. Suppose that E^* has a metric compact approximation of the identity with adjoint operators given by $(K_\alpha) \subset \mathcal{K}(E)$. Then, $\lim_\alpha P \circ K^\alpha = 0$ in the topology $\sigma(\mathcal{P}({}^nE, F), \mathcal{P}_w({}^nE, F)^*)$ for any $P \in \mathcal{P}({}^nE, F)$.*

Proof. By Proposition 3.9, the application $j\colon \widehat{\otimes}^{n,s}_{\pi_s} E^{**} \widehat{\otimes}_\pi F^* \to \mathcal{P}_w({}^nE, F)^*$, defined by $j(u \otimes y^*)(P) = y^*(\overline{P}(u))$, is a quotient mapping. We will show for $\tilde{u} = \sum_{i=1}^M w_i \otimes w_i \otimes \cdots \otimes w_i \otimes y_i^*$ that $\lim_\alpha \langle \tilde{u}, P \circ K^\alpha \rangle = 0$. So, as for any $h \in \mathcal{P}_w({}^nE, F)^*$ there exists $u \in \widehat{\otimes}^{n,s}_{\pi_s} E^{**} \widehat{\otimes}_\pi F^*$ so that $j(u) = h$ and u can be approximated by such $\tilde{u}$'s, the result follows. Take $\tilde{u}$ as above. Then $\langle \tilde{u}, P \circ K^\alpha \rangle = \sum_{i=1}^M \overline{P \circ K^\alpha}(w_i)(y_i^*) = \sum_{i=1}^M \overline{P}((K^\alpha)^{**}(w_i))(y_i^*)$. As $n < \mathrm{cd}(E)$, $\overline{P}$ is w^*-w^* continuous, and $\lim_\alpha (K^\alpha)^{**}(w_i) = 0$ in the w^*-topology, we obtain $\lim_\alpha \langle \tilde{u}, P \circ K^\alpha \rangle = 0$, which completes the proof. $\square$

Proposition 1.4 in [13] provides an appropriate equivalence of asymptotic uniform convexity of power p. With a slight modification of its proof we drop the hypothesis of separability and obtain a refinement, analogous to condition (c) in Lemma 3.3, as follows.

Lemma 3.12. *Let E be an infinite-dimensional Banach space, and let $1 < p < \infty$. The following statements are equivalent.*

(a) *E has modulus of asymptotic uniform convexity of power p.*

(b) *There exists a constant $C > 0$ such that for every $x \in S_E$ and every bounded weakly null net (w_α) in E, we have*

$$\overline{\lim}_\alpha \|x + w_\alpha\|^p \geq 1 + C \overline{\lim}_\alpha \|w_\alpha\|^p.$$

(c) *There exists a constant $C > 0$ such that for every net (x_α) in a compact set of B_E and every bounded weakly null net (w_α) in E, we have*

$$\overline{\lim}_\alpha \|x_\alpha + w_\alpha\|^p \geq \overline{\lim}_\alpha \|x_\alpha\|^p + C\|w_\alpha\|^p.$$

Proposition 3.13. *Let E, F be Banach spaces such that E^{**} and F^* enjoy the Radon–Nikodym property. Suppose that E has modulus of asymptotic uniform convexity of power $n = \mathrm{cd}(E, F)$, with $n < \mathrm{cd}(E)$, and suppose that E has a shrinking metric compact approximation of the identity $(K_\alpha) \subset \mathcal{K}(E)$ satisfying $\sup_\alpha \|K^\alpha\| \leq 1$ and $\overline{\lim}_\alpha \|\mathrm{Id}_E - 2K_\alpha\| \leq 1$.*

Then there exists $C > 0$ such that $\mathcal{P}_w({}^nE, F)$ is an $M(1, C)$-ideal of $\mathcal{P}({}^nE, F)$. Furthermore, $\mathcal{P}_w({}^nE, F)$ is an HB-subspace of $\mathcal{P}({}^nE, F)$.

Proof. We proceed as in Theorem 3.4. Consider Λ as in (3.4), under which $\mathcal{P}_w({}^nE, F)$ is an ideal in $\mathcal{P}({}^nE, F)$, and write $f \in \mathcal{P}({}^nE, F)^*$, $f = g + h$ as in (3.2), where $\|g\| = \|\Lambda(f)\| \leq \|f\|$.

Consider $\varepsilon > 0$, and take $P \in \mathcal{P}({}^nE, F)$, $Q \in \mathcal{P}_w({}^nE, F)$ with

$$\|P\| = \|Q\| = 1, \qquad h(P) \geq \|h\| - \varepsilon, \qquad \text{and} \qquad g(Q) \geq \|g\| - \varepsilon.$$

For any α, the polynomial $P - P \circ K^\alpha$ is weakly continuous on bounded sets at 0 (see, e.g., the proof of [12, Proposition 2.2]). Since $n = \mathrm{cd}(E, F)$, the net $(P - P \circ K^\alpha)_\alpha$ is in $\mathcal{P}_w({}^nE, F)$. Then, with $h \in \mathcal{P}_w({}^nE, F)^\perp$, $h(P) = h(P \circ K^\alpha) \geq \|h\| - \varepsilon$

for all α. Also, as $\lim_\alpha Q \circ K_\alpha = Q$, there exists α_0 so that $|g(Q \circ K_\alpha)| > \|g\| - 2\varepsilon$ and $\|\mathrm{Id}_E - 2K_{\alpha_0}\| < 1 + \varepsilon$ for all $\alpha \geq \alpha_0$. Changing $Q \circ K_{\alpha_0}$ to $\lambda Q \circ K_{\alpha_0}$ with $|\lambda| = 1$, if necessary, we may assume that $g(Q \circ K_{\alpha_0}) > \|g\| - 2\varepsilon$. Now, with $C > 0$ (to be fixed later) we have

$$\begin{aligned}\|f\| \|Q \circ K_{\alpha_0} + CP \circ K^\alpha\| &\geq \big|f(Q \circ K_{\alpha_0} + CP \circ K^\alpha)\big| \\ &\geq g(Q \circ K_{\alpha_0}) + Ch(P \circ K^\alpha) - C\big|g(P \circ K^\alpha)\big| \\ &\geq \|g\| + C\|h\| - (2 + C)\varepsilon - C\big|g(P \circ K^\alpha)\big|.\end{aligned}$$

By Lemma 3.11, we may find $\alpha_1 \geq \alpha_0$ so that $|g(P \circ K^\alpha)| < \varepsilon$ for $\alpha \geq \alpha_1$. Thus, we obtain

$$\|f\| \|Q \circ K_{\alpha_0} + CP \circ K^\alpha\| \geq \|g\| + C\|h\| - 2\varepsilon(1 + C) \quad \text{for all } \alpha \geq \alpha_1.$$

Now, take $(x_\alpha) \subset B_E$ such that $\overline{\lim}_\alpha \|Q \circ K_{\alpha_0} + CP \circ K^\alpha\| = \overline{\lim}_\alpha \|Q(K_{\alpha_0} x_\alpha) + CP(K^\alpha x_\alpha)\|$, and note that $(K_{\alpha_0} x_\alpha)$ is contained in a compact subset of B_E and that $(K^\alpha x_\alpha)$ is weakly null. Since E has modulus of asymptotic convexity of power n, we apply Lemma 3.12, with $C > 0$ as in item (c), and get

$$\begin{aligned}\overline{\lim}_\alpha \|Q \circ K_{\alpha_0} + CP \circ K^\alpha\| &\leq \overline{\lim}_\alpha \|Q\| \|K_{\alpha_0} x_\alpha\|^n + C\|P\| \|K^\alpha x_\alpha\|^n \\ &= \overline{\lim}_\alpha \|K_{\alpha_0} x_\alpha\|^n + C\|K^\alpha x_\alpha\|^n \\ &\leq \overline{\lim}_\alpha \|K_{\alpha_0} x_\alpha + K^\alpha x_\alpha\|^n \\ &\leq \overline{\lim}_\alpha \|K_{\alpha_0} + K^\alpha\|^n \\ &\leq \|\mathrm{Id}_E - 2K_{\alpha_0}\|^n,\end{aligned}$$

where the last inequality, being standard, can be found, for instance, in [18, p. 300]. Then, we may find $\alpha_2 > \alpha_1$ so that

$$\|QK_{\alpha_0} + CPK^{\alpha_2}\| < 1 + \varepsilon,$$

and therefore,

$$\|g\| + C\|h\| - 2\varepsilon(1 + C) \leq \|f\|(1 + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, $\|g\| + C\|h\| \leq \|f\|$ and $\mathcal{P}_w(^nE, F)$ is an $M(1, C)$-ideal in $\mathcal{P}(^nE, F)$.

To prove that $\mathcal{P}_w(^nE, F)$ is also an HB-subspace of $\mathcal{P}(^nE, F)$, note that for $h \neq 0$, $\|g\| < \|g\| + C\|h\| \leq \|f\|$. On the other hand, for $\alpha > \alpha_1$,

$$\|f\| \geq \big|f(P \circ K^\alpha)\big| \geq h(P \circ K^\alpha) - \big|g(P \circ K^\alpha)\big| \geq \|h\| - 2\varepsilon,$$

implying that $\|f\| \geq \|h\|$, which completes the proof. □

4. Ideal structures inherited by the range space

Our purpose in this section is to give sufficient conditions on the spaces E and F under which those geometric properties enjoyed by $\mathcal{P}_w(^nE, F)$ as a subspace of $\mathcal{P}(^nE, F)$ are inherited by the range space F as a subspace of F^{**}. We start with HB-smoothness presenting an extension to the polynomial setting of [24, Theorem 7]. Our proof also follows their ideas.

Proposition 4.1. *Let E, F be Banach spaces such that there exists a surjection $\rho\colon E \to F$. If $\mathcal{P}_w(^nE, F)$ is HB-smooth in $\mathcal{P}(^nE, F)$ for some $n \in \mathbb{N}$, then F is HB-smooth in F^{**}.*

Proof. Denote by $NA(E^{**})$ the subset of norm-attaining elements in E^{**}, and consider

$$A := \big\{\rho^{**}(x^{**})\colon x^{**} \neq 0, x^{**} \in NA(E^{**})\big\}.$$

As ρ is surjective, ρ^{**} is also surjective and, by the Bishop–Phelps theorem, $\overline{A} = F^{**}$. By [24, Theorem 1(c)] it is enough to show that, for every $\rho^{**}(x^{**}) \in A$ and any sequence $(y_k) \subset F$ with $\|y_1\| < 1$, $\|y_{k+1} - y_k\| < 1$, there are $y \in F$ and $k_0 \in \mathbb{N}$ so that $\|\rho^{**}x^{**} - y \pm y_{k_0}\| < k_0$.

Fix $x_0^{**} \neq 0$ in $NA(E^{**})$, take $x_0^* \in S_{E^*}$ such that $x_0^{**}(x_0^*) = \|x_0^{**}\|$, and define, for $k \in \mathbb{N}$, the n-homogeneous polynomial $P_k(x) = x_0^*(x)^n y_k \in \mathcal{P}_w(^nE, F)$. It is clear that

$$\|P_1\| < 1 \qquad \text{and} \qquad \|P_{k+1} - P_k\| \leq \|y_{k+1} - y_k\| < 1 \quad \text{for all } k \in \mathbb{N}.$$

Now, define $Q(x) = \rho(x)x_0^*(x)^{n-1}\|x_0^{**}\|$ and consider its Aron–Berner extension $\overline{Q}$ given by $\overline{Q}(x^{**}) = \rho^{**}(x^{**})x^{**}(x_0^*)^{n-1}\|x_0^{**}\|$.

By [24, Theorem 1(a)], there exist $R \in \mathcal{P}_w(^nE, F)$ and $k_0 \in \mathbb{N}$ with $\|Q - R \pm P_{k_0}\| < k_0$. As the Aron–Berner extension preserves the norm, we obtain $\|\overline{Q}(x_0^{**}) - \overline{R}(x_0^{**}) \pm \overline{P}_{k_0}(x_0^{**})\| < k_0\|x_0^{**}\|^n$. The result follows by taking $y = \frac{\overline{R}(x_0^{**})}{\|x_0^{**}\|^n}$. ∎

As an immediate consequence, we have the following result.

Corollary 4.2. *Let E be a Banach space. If $\mathcal{P}_w(^nE, E)$ is HB-smooth in $\mathcal{P}(^nE, E)$ for some $n \in \mathbb{N}$, then E is HB-smooth in E^{**}.*

Note that the above corollary says that $\mathcal{P}_w(^n\ell_1, \ell_1)$ is not HB-smooth in $\mathcal{P}(^n\ell_1, \ell_1)$ for any $n \in \mathbb{N}$.

Now we address the notion of HB-subspace, when the range space F is a quotient of the space E. The following technical result, inspired by [26, Proposition 2.3], will be useful.

Lemma 4.3. *Let J be an ideal in the Banach space E under the projection q. Suppose that J is HB-smooth and that $\lambda \in (0, 2]$. The following statements are equivalent.*

(i) $\|\mathrm{Id}_{E^*} - \lambda q\| \leq 1$.

(ii) *For each $x \in B_E$ there exists a net $(y_\alpha) \subset J$ such that $\lim_\alpha y_\alpha = x$ in the $\sigma(E, J^*)$-topology and $\overline{\lim}_\alpha \|x - \lambda y_\alpha\| \leq 1$.*

(iii) *For each $x \in B_E$ and $\varepsilon > 0$ there exists a net $(y_\alpha) \subset J$ such that $\lim_\alpha y_\alpha = x$ in the $\sigma(E, J^*)$-topology and $\overline{\lim}_\alpha \|x - \lambda y_\alpha\| \leq 1 + \varepsilon$.*

Proof. To prove that (i) implies (ii), consider the set of indices $A := \{\alpha = (N, M, \varepsilon)\colon N \in \mathrm{FIN}(E^{**}), M \in \mathrm{FIN}(E^*), \varepsilon > 0\}$, where FIN denotes the set of all finite-dimensional subspaces, with the usual order. By the principle of local reflexivity, for any $\alpha \in A$ there exists $T_\alpha\colon N \to E$ so that

- $\langle T_\alpha x^{**}, x^* \rangle = \langle x^{**}, x^* \rangle$ for $x^{**} \in N, x^* \in M$,

- $\|T_\alpha\| \le 1+\varepsilon$,
- $T_\alpha|_{N\cap E} = \mathrm{Id}_E$.

Fix $x \in B_E$, and consider $y_\alpha = T_\alpha(q^* x) \in E$, defined for α large enough. Fix $y^* \in J^*$ and $\varepsilon > 0$; then if $\alpha \ge (\{q^*x\}, \{y^*\}, \varepsilon)$, as J^* is the range of q, we have

$$\langle y^*, y_\alpha\rangle = \big\langle y^*, T_\alpha(q^*x)\big\rangle = \langle y^*, q^*x\rangle = \langle qy^*, x\rangle = \langle y^*, x\rangle,$$

and $\lim_\alpha y_\alpha = x$ in the $\sigma(E, J^*)$-topology. On the other hand, also for α large enough,

$$\begin{aligned}\|x - \lambda y_\alpha\| &= \big\|T_\alpha(x - \lambda q^* x)\big\| \le (1+\varepsilon)\|x - \lambda q^* x\| \\ &\le (1+\varepsilon)\|\mathrm{Id}_{E^{**}} - \lambda q^*\| = (1+\varepsilon)\|\mathrm{Id}_{E^*} - \lambda q\| \le 1+\varepsilon.\end{aligned}$$

Then, $\overline{\lim}_\alpha\|x - \lambda y_\alpha\| \le 1$ and (ii) follows.

Clearly, (ii) implies (iii). To prove that (iii) implies (i), fix $\varepsilon > 0$, $x \in B_E$, and choose $(y_\alpha) \subset J$ satisfying (iii). For each $x^* \in B_{E^*}$, as $\lim_\alpha x^*(y_\alpha) = \lim_\alpha qx^*(y_\alpha) = qx^*(x)$, we have

$$\big|x^*(x) - \lambda qx^*(x)\big| = \lim_\alpha\big|x^*(x) - \lambda x^*(y_\alpha)\big| \le \|x^*\|\overline{\lim}_\alpha\|x - \lambda y_\alpha\| \le 1+\varepsilon.$$

Since ε is arbitrary, the implication follows. □

Proposition 4.4. *Let E, F be Banach spaces such that there exists a quotient mapping $\rho\colon E \to F$. If $\mathcal{P}_w({}^nE, F)$ is an HB-subspace of $\mathcal{P}({}^nE, F)$ for some $n \in \mathbb{N}$, then F is an HB-subspace of F^{**}.*

Proof. First, note that F is an ideal in its bidual, and call q the *associated projection*. By Proposition 4.1, F is HB-smooth in F^{**}. Then, by [22, Theorem], F is strongly HB-smooth in F^{**} and we only have to show that $\|f - qf\| \le \|f\|$ for all $f \in F^{***}$. In order to do so, we will see that condition (iii) in Lemma 4.3 is satisfied for $\lambda = 1$.

Take $y^{**} \in B_{F^{**}}$ and $\varepsilon > 0$. Choose $w^* \in S_{F^*}$ so that $y^{**}(w^*) > 0$ and $(\frac{\|y^{**}\|}{y^{**}(w^*)})^n < 1+\varepsilon$. Now, with $\mu = y^{**}(w^*)$ define $P \in \mathcal{P}({}^nE, F)$ by $P(x) = \mu\rho(x)(\rho^* w^*)^{n-1}(x)$, for each $x \in E$. By Lemma 4.3, there exists a net $(P_\alpha) \subset \mathcal{P}_w({}^nE, F)$ converging to P in the $\sigma(\mathcal{P}({}^nE, F), \mathcal{P}_w({}^nE, F)^*)$-topology and such that $\overline{\lim}_\alpha\|P - P_\alpha\| \le 1$. As ρ is a quotient mapping, ρ^* is an isometry and we may find $x^{**} \in E^{**}$ with $\rho^{**}(x^{**}) = y^{**}$ and $\|x^{**}\| = \|y^{**}\|$. Since the Aron–Berner extension of each P_α has range in F, we define $y_\alpha = \overline{P}_\alpha(x^{**}/\mu) \in F$ for all α.

Note that each $x^{**} \otimes y^* \in E^{**} \otimes F^*$ acts in a natural way as an element of $\mathcal{P}({}^nE, F)^*$ and, therefore, as an element of $\mathcal{P}_w({}^nE, F)^*$. Then, as $\overline{P}(z) = \mu\rho^{**}(z)z^{n-1}(\rho^* w^*)$ for $z \in E^{**}$, we have $\overline{P}(x^{**}/\mu) = y^{**}$ and

$$y^{**}(y^*) = \big(\overline{P}(x^{**}/\mu)\big)(y^*) = \lim_\alpha y^*\big(\overline{P_\alpha}(x^{**}/\mu)\big) = \lim_\alpha y^*(y_\alpha).$$

Thus, $y_\alpha \to y^{**}$ in the w^*-topology. Also,

$$\begin{aligned}\overline{\lim}_\alpha\|y^{**} - y_\alpha\| &= \overline{\lim}_\alpha\big\|\overline{P}(x^{**}/\mu) - \overline{P}_\alpha(x^{**}/\mu)\big\| \\ &\le \overline{\lim}_\alpha\|P - P_\alpha\|\|x^{**}/\mu\|^n \\ &\le \big(\|y^{**}\|/\mu\big)^n < 1+\varepsilon.\end{aligned}$$

Another application of Lemma 4.3 gives that $\|\mathrm{Id}_{F^{***}} - q\| \le 1$ and, therefore, F is an HB-subspace of F^{**}. ☐

Finally, we focus on $M(1,C)$-ideal structures. Recall that if J is an ideal in a Banach space E satisfying the $M(1,C)$-inequality, then the following condition holds. For any $m \in \mathbb{N}$, $y_1, y_2, \dots, y_m \in B_J$, $x \in B_E$, and $\varepsilon > 0$, there is $z \in J$ such that $\|y_i + Cx - z\| \le 1 + \varepsilon$ for $1 \le i \le m$ (see [7, Lemma 2.2]). In fact, when dealing with $E = J^{**}$, it is true that being an $M(1,C)$-ideal is equivalent to an appropriate 2-ball property of type $(1,C)$. Namely, we have the following equivalence, which can be proved with the arguments appearing in the proof of [7, Lemma 2.3].

Lemma 4.5. *Let E be a Banach space, and let $C \in (0,1]$. The following statements are equivalent.*

(i) *E is an $M(1,C)$-ideal of E^{**}.*

(ii) *For all $x \in S_E$, $x^{**} \in S_{E^{**}}$, and $\varepsilon > 0$, there exists $x_0 \in E$ with $\|\pm x + Cx^{**} - x_0\| < 1 + \varepsilon$.*

We use the above characterization to give an analogous statement to Proposition 4.4 in the case of $M(1,C)$-ideals.

Proposition 4.6. *Let E, F be Banach spaces such that there exists a quotient mapping $\rho\colon E \to F$. If $\mathcal{P}_w(^nE, F)$ is an $M(1,C)$-ideal of $\mathcal{P}(^nE, F)$ for some $n \in \mathbb{N}$ and $C > 0$, then F is an $M(1,C)$-ideal of F^{**}.*

Proof. Let us prove that condition (ii) in Lemma 4.5 is satisfied. Fix $y \in S_F$, $y^{**} \in S_{F^{**}}$, and $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta + \frac{1+\delta}{(1-\delta)^{n-1}} < 1 + \varepsilon$ and $y^* \in S_{F^*}$ with $y^{**}(y^*) \ge 1 - \delta$. Define, for $x \in E$, $P \in \mathcal{P}(^nE, F)$ and $Q \in \mathcal{P}_w(^nE, F)$ by

$$P(x) = (\rho^* y^*)^{n-1}(x)\rho(x) \qquad \text{and} \qquad Q(x) = (\rho^* y^*)^n(x) y,$$

with $\|P\|, \|Q\| \le \|\rho\|^n = 1$. Due to [7, Lemma 2.2], there exists $R \in \mathcal{P}_w(^nE, F)$ so that $\|\pm Q + CP - R\| \le 1 + \delta$. As ρ^* is an isometry, there is $x^{**} \in S_{X^{**}}$ with $\rho^{**}(x^{**}) = y^{**}$. Extending by Aron–Berner for $z \in E^{**}$,

$$\overline{P}(z) = z(\rho^* y^*)^{n-1} \rho^{**}(z) \qquad \text{and} \qquad \overline{Q}(z) = z(\rho^* y^*)^n y.$$

Then, with $\mu = y^{**}(y^*)$, $\overline{P}(x^{**}) = \mu^{n-1} y^{**}$ and $\overline{Q}(x^{**}) = \mu^n y$,

$$\left\|\pm \mu^n y + C\mu^{n-1} y^{**} - \overline{R}(x^{**})\right\| = \left\|\pm \overline{Q}(x^{**}) + C\overline{P}(x^{**}) - \overline{R}(x^{**})\right\| \le 1 + \delta.$$

As $R \in \mathcal{P}_w(^nE, F)$, the range of $\overline{R}$ is also in F, and we may take $y_0 = \overline{R}(x^{**})/\mu^{n-1}$. Thus,

$$\|\pm \mu y + C y^{**} - y_0\| \le \frac{1+\delta}{(1-\delta)^{n-1}}.$$

Finally, $\|\pm y + Cy^{**} - y_0\| \le \|y - \mu y\| + \|\pm \mu y + Cy^{**} - y_0\| < 1 + \varepsilon$, and the result follows. ☐

In the above proposition, the case $C = 1$ corresponds to the structure of an M-ideal. The corollary follows directly and seems to be new in this context.

Corollary 4.7. *Let E, F be Banach spaces such that there exists a quotient mapping $\rho\colon E \to F$. If $\mathcal{P}_w(^nE, F)$ is an M-ideal of $\mathcal{P}(^nE, F)$ for some $n \in \mathbb{N}$, then F is an M-ideal of F^{**}.*

As we have already noted in Corollary 4.2, it is now immediate to derive versions of Proposition 4.4, Proposition 4.6, and Corollary 4.7 for the case $E = F$.

Acknowledgments. Work on this paper was initiated while the third author was visiting the Departamento de Matemática, Universidad de San Andrés. She is greatly indebted to Professors Dimant and Lassalle for the invitation during her sabbatical and for the warm hospitality extended by them and their families. The authors' work was partially supported by PAI-UdeSA 2013, CONICET-PIP 0624, ANPCyT PICT 1456, UBACyT X038, and MINECO grant MTM 2012-3431.

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