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SUBSPACES OF BANACH SPACES WITH BIG SLICES

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Abstract. We study when diameter 2 properties can be inherited by subspaces. We obtain that the slice diameter 2 property (resp., the diameter 2 property, strong diameter 2 property) passes from a Banach space X to a subspace Y whenever X/Y is finite-dimensional and Y is complemented by a norm 1 projection (resp., the quotient X/Y is finite-dimensional and strongly regular). Also, we study the same problem for the dual properties of diameter 2 properties, such as having octahedral, weakly octahedral, or 2-rough norm.

1. Introduction

We recall that a Banach space X satisfies the strong diameter 2 property (SD2P) (resp., the diameter 2 property, D2P; the slice diameter 2 property, slice-D2P) if every convex combination of slices (resp., every nonempty relatively weakly open subset, every slice) in the unit ball of X has diameter two. The weak-star slice diameter 2 property (w^*-slice-D2P), weak-star diameter 2 property (w^*-D2P), and weak-star strong diameter 2 property (w^*-SD2P) for a dual Banach space are defined as usual, changing slices by w^*-slices and weakly open subsets by w^*-open subsets in the unit ball. It is known that the Daugavet property implies the SD2P, and therefore the D2P and slice-D2P, too. The above connection between the Daugavet property and the diameter 2 properties was discovered in [14]. In fact, the dual of a Banach space with the Daugavet property also satisfies the w^*-SD2P (see [4]). It is also known that the above six properties are extremely different, as proved in [5].

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As can easily be seen, the above properties are not inherited by subspaces, and so, given a Banach space X with some of the above properties, a natural question is to wonder which subspaces Y of X also have that property. This problem has been considered in [2] in terms of *almost isometric ideals* (see definition below) in the case of diameter 2 properties. Also, it is known that the Daugavet property is inherited by rich subspaces. However, only Banach spaces with the Daugavet property can contain rich subspaces (see [11]). The aim of this note is to study when the diameter 2 properties pass to subspaces regarding the behavior of the quotient space. We show that the above properties are not 3-space properties. We also obtain that a subspace Y of a Banach space X with the SD2P has the SD2P whenever the quotient X/Y is strongly regular, which is a weaker property than the Radon–Nikodym property (RNP). In particular, we show that the above holds if X/Y does not contain ℓ_1^n uniformly. Moreover, we get that a subspace Y of a Banach space X with D2P has D2P whenever X/Y is finite-dimensional, and we also obtain that a subspace Y of a Banach space X with the slice-D2P has slice-D2P if Y is finite-codimensional and complemented by a norm 1 projection.

Finally, we study the same kind of problems for the dual properties of diameter 2 properties, such as having octahedral, weakly octahedral, or 2-rough norm. Recall that a Banach space X satisfies the SD2P if and only if the norm on X^* is octahedral, and the norm on X is octahedral if and only if X^* has the w^*-SD2P (see [4]).

The norm on a Banach space X is octahedral if, for every $\varepsilon > 0$ and for every finite-dimensional subspace M of X, there is some y in the unit sphere of X such that

$$\|x + \lambda y\| \geq (1 - \varepsilon)\big(\|x\| + |\lambda|\big)$$

holds for every $x \in M$ and for every scalar λ (see [7]).

Similarly, following [10] and [7, Proposition I.1.11], a Banach space X satisfies the D2P (resp., the slice-D2P) if the norm on the dual space X^* is weakly octahedral (resp., 2-rough). Also, the norm on X is weakly octahedral (resp., 2-rough) if and only if X^* has the w^*-D2P (resp., the w^*-slice-D2P).

The norm on a Banach space X is weakly octahedral (see [10]) if for every finite-dimensional subspace Y of X, every $x^* \in B_{X^*}$, and every $\varepsilon \in \mathbb{R}^+$ there exists $y \in S_X$ satisfying

$$\|x + y\| \geq (1 - \varepsilon)\big(\big|x^*(x)\big| + \|y\|\big) \quad \forall x \in Y.$$

The norm on X is said to be 2-*rough* if, for every u in the unit sphere of X, one has

$$\limsup_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|} = 2.$$

Recall that a closed, bounded, and convex subset C of a Banach space X is said to be *strongly regular* if every closed and convex subset of C has convex combinations of slices with arbitrarily small diameter (resp., convex combinations of relatively weakly open subsets with arbitrarily small diameter) since it is known that every nonempty relatively weakly open subset contains a convex combination of slices (see [6, Lemma 5.3]). We refer to this fact as the *Bourgain lemma*. A point

x of such a subset C of X is said to be a *point of strong regularity* if there are convex combinations of slices in C containing x with an arbitrarily small diameter. If C is strongly regular, then C contains a norm dense subset of points of strong regularity (see [9, Proposition 3.6]). The strong regularity is a strictly weaker property than the RNP, and it is known that, for a Banach space X, X^* is strongly regular if and only if X does not contain any isomorphic copy of ℓ_1 (see [9, Corollary 6.18]). Also, it is known that Banach spaces not containing ℓ_1^n uniformly are strongly regular (see [12, Proposition 2.14]). Banach spaces not containing ℓ_1^n uniformly are exactly the K-convex Banach spaces.

Next, we introduce some notation. We consider real Banach spaces B_X (resp., S_X) to denote the closed unit ball (resp., sphere) of the Banach space X. All subspaces of a Banach space will be considered closed subspaces. If Y is a subspace of a Banach space X, then X^* stands for the dual space of X, and the annihilator of Y is the subspace of X^* given by

$$Y^\circ = \big\{x^* \in X^* : x^*(Y) = \{0\}\big\}.$$

A slice of a bounded subset C of X is the set

$$S(C, f, \alpha) := \big\{x \in C : f(x) > M - \alpha\big\},$$

where $f \in X^*$, $f \neq 0$, $M = \sup_{x\in C} f(x)$, and $\alpha > 0$. If $X = Y^*$ is a dual space for some Banach space Y and if C is a bounded subset of X, then a w^*-slice of C is the set

$$S(C, y, \alpha) := \big\{f \in C : f(y) > M - \alpha\big\},$$

where $y \in Y$, $y \neq 0$, $M = \sup_{f\in C} f(y)$, and $\alpha > 0$. The weak (resp., weak-star) topology of a Banach space is denoted by w (resp., w^*).

According to [2], given a Banach space X and a subspace $Y \subseteq X$, it is said that Y is an *almost ideal in* X if, for every $\varepsilon > 0$ and every finite-dimensional subspace $E \subseteq X$, there exists a bounded and linear operator $T : E \longrightarrow Y$ satisfying the following conditions:

(1) $T(e) = e$ for every $e \in E \cap Y$.
(2) For each $e \in E$, one has

$$\frac{1}{1+\varepsilon}\|e\| \leq \big\|T(e)\big\| \leq (1+\varepsilon)\big\|T(e)\big\|.$$

The well-known principle of *local reflexivity* (see [3, Chapter 11]) asserts that a Banach space X is an almost isometric ideal in its bidual (see [2, Theorem 1.4]), which provides us with a wide class of examples of almost isometric ideals.

It is known that the slice-D2P, the D2P, the SD2P, and the Daugavet property are inherited by almost isometric ideals (see [2, Propositions 3.1, 3.2, 3.8 and Corollary 3.4]).

2. Main results

We shall begin with the following question: Can a closed subspace satisfying any diameter 2 property force the space to have any diameter 2 property? Although the answer to the above question is negative, we will go further, proving that diameter 2 properties are not 3-space properties.

Indeed, it is possible to construct an example $Z = X \times Y$ for $X = Y = C([0,1])$ and a norm $\|\cdot\|$ on Z such that Z fails every diameter 2 property in spite of the fact that X and Y have the SD2P under the norm of Z restricted to X and Y, respectively. In order to provide such a norm, we give a proof, for the reader's convenience, of a special case of the well-known Bourgain–Namioka super lemma (see [8, p. 157]).

Lemma 2.1. *Let X be a Banach space, and assume that $C = \operatorname{co}(B \cup \{x_0\})$ for some closed, bounded, and convex subset B of X and $x_0 \in X \setminus B$. Then x_0 is a denting point of C. Moreover, if B is the unit ball of X, then $\operatorname{co}(C \cup \{-x_0\})$ is the unit ball of some equivalent norm on X and x_0 is a denting point of the unit ball for this new norm.*

Proof. As $x_0 \notin B$, we can find by a separation argument $x^* \in S_{X^*}$ such that $x^*(x_0) > \sup x^*(B) = M$. Hence $\sup x^*(C) = x^*(x_0)$. Fix $\varepsilon > 0$. Let $\beta := \sup_{x \in B} \|x\|$, and let $0 < \alpha < \frac{x^*(x_0) - M}{2(\beta + \|x_0\|)}\varepsilon$. Consider $S = \{x \in C : x^*(x) > x^*(x_0) - \alpha\}$. Now S is a slice of C. Pick $y, z \in \operatorname{co}(B \cup \{x_0\}) \cap S$ with $y = \lambda b_1 + (1-\lambda)x_0$ and $z = \mu b_2 + (1-\mu)x_0$ for some $0 \leq \lambda, \mu \leq 1$, and $b_1, b_2 \in B$. As $y, z \in S$, we deduce that $\lambda, \mu < \frac{\varepsilon}{2(\beta + \|x_0\|)}$. Hence, $\|y - z\| < \frac{\varepsilon}{2(\beta + \|x_0\|)}(\|b_1\| + \|b_2\| + \|x_0\|) < \varepsilon$. This proves that $\operatorname{co}(B \cup \{x_0\}) \cap S$ has diameter less than ε, and so S is a slice of C containing x_0 with diameter less than ε and x_0 is a denting point of C.

In the case that B is in particular the unit ball of X, it is easy to see that the above set S is a slice of $\operatorname{co}(C \cup \{-x_0\})$ containing x_0 with diameter less than ε for α small enough. □

In order to exhibit the announced example, let B be the closed unit ball of $C([0,1]) \oplus_1 C([0,1])$ and let $C = \operatorname{co}(B \cup \{(x_0, x_0)\} \cup \{(-x_0, -x_0)\})$, where x_0 is a point in $C([0,1])$ whose usual norm in $C([0,1])$ is 1. From Lemma 2.1, C is the unit ball of some norm on $C([0,1]) \times C([0,1])$, failing every diameter 2 property whose restriction to the factor spaces has the SD2P.

Recall that a property $(\mathcal{P})$ is said to be a 3-*space property* if a Banach space X satisfies $(\mathcal{P})$ whenever there exists a closed subspace $Y \subseteq X$ such that Y and X/Y enjoy the property $(\mathcal{P})$.

As a consequence of the previous lemma, neither the Daugavet property nor the diameter 2 properties are 3-space properties.

We now study the following question: Given a Banach space X satisfying some diameter 2 property, which closed subspaces of X enjoy this diameter 2 property? The following result enables us to draw such conclusions, assuming quite simple properties on the quotient X/Y.

Theorem 2.2. *Let X be a Banach space, and let Y be a subspace of X.*

(i) *If X has the slice-D2P and if X/Y is finite-dimensional, and there exists a linear and norm 1 projection π from X onto Y, then Y has the slice-D2P.*
(ii) *If X has the D2P and X/Y is finite-dimensional, then Y has the D2P.*
(iii) *If X has the SD2P and X/Y is strongly regular, then Y has the SD2P.*

Proof. (i) Note that, since $\|\pi\| = 1$,

$$\pi(B_X) = B_Y.$$

Under the hypotheses of (i), it is proved in [13, Theorem 5.3] that

$$2 = \inf\{\operatorname{diam}(S) : S \text{ slice of } B_X\} \leq \inf\{\operatorname{diam}(T) : T \text{ slice of } \pi(B_X) = B_Y\}.$$

Thus Y has the slice-D2P, as desired.

(ii) Consider

$$W := \{y \in Y : |y_i^*(y - y_0)| < \varepsilon_i \quad \forall i \in \{1, \dots, n\}\}$$

for $n \in \mathbb{N}, \varepsilon_i \in \mathbb{R}^+, y_i^* \in Y^*$ for each $i \in \{1, \dots, n\}$ and $y_0 \in Y$ such that

$$W \cap B_Y \neq \emptyset.$$

Let us prove that $W \cap B_Y$ has diameter 2. To this aim, pick an arbitrary $0 < \delta < 1$.

There is no loss of generality by the Hahn–Banach theorem if we assume that $y_i^* \in X^*$ for each $i \in \{1, \dots, n\}$.

Define

$$U := \{x \in X : |y_i^*(x - y_0)| < \varepsilon_i \quad \forall i \in \{1, \dots, n\}\},$$

which is a weakly open set in X such that $U \cap B_X \neq \emptyset$.

Let $p : X \longrightarrow X/Y$ be the quotient map, which is a $w - w$ open map. Then $p(U)$ is a weakly open set in X/Y. In addition,

$$\emptyset \neq p(U \cap B_X) \subseteq p(U) \cap p(B_X) \subseteq p(U) \cap B_{X/Y}.$$

Defining $A := p(U) \cap B_{X/Y}$, we see that A is a nonempty, relatively weakly open and convex subset of $B_{X/Y}$ which contains 0. Hence, as X/Y is finite-dimensional, we can find a weakly open set V of X/Y, in fact a ball centered at 0, such that $V \subset A$ and such that

$$\operatorname{diam}(V \cap p(U) \cap B_{X/Y}) = \operatorname{diam}(V) < \frac{\delta}{16}. \tag{2.1}$$

As $V \subset A$, we have $B := p^{-1}(V) \cap U \cap B_X \neq \emptyset$. Hence B is a nonempty relatively weakly open subset of B_X. Using the fact that X satisfies the D2P, we can assure the existence of $x, y \in B$ such that

$$\|x - y\| > 2 - \frac{\delta}{16}. \tag{2.2}$$

Note that $x \in B$ implies that $p(x) \in V = V \cap P(U) \cap B_{X/Y}$. In view of (2.1), it follows that

$$\|p(x)\| \leq \operatorname{diam}(V \cap p(U) \cap B_{X/Y}) < \frac{\delta}{16}.$$

Hence there exists $v \in Y$ such that $\|x - v\| < \frac{\delta}{16}$, and so $\|v\| < 1 + \frac{\delta}{16}$. Letting $u = \frac{v}{\|v\|}$, we have

$$\begin{aligned}\|x - u\| &\leq \|x - v\| + \left\|v - \frac{v}{\|v\|}\right\| \\ &< \frac{\delta}{16} + \|v\|(\|v\| - 1)\end{aligned}$$

$$
\begin{aligned}
&< \frac{\delta}{16} + \Big(1+\frac{\delta}{16}\Big)\frac{\delta}{16}\\
&= \frac{\delta}{16}\Big(2+\frac{\delta}{16}\Big),
\end{aligned}
$$

and so

$$\|x-u\| < \frac{\delta}{4}. \tag{2.3}$$

Again using (2.1), by a similar argument we can find $v \in B_Y$ satisfying

$$\|v-y\| < \frac{\delta}{4}. \tag{2.4}$$

Note that, given $i \in \{1,\ldots,n\}$ and keeping in mind (2.3), one has

$$\big|y_i^*(u-y_0)\big| \leq \big|y_i^*(y-x)\big| + \big|y_i^*(x-y_0)\big| \leq \|y_i^*\|\frac{\delta}{4} + \varepsilon_i$$

using the fact that $x \in U$. Thus, if we define

$$W_\delta := \Big\{y \in Y / \big|y_i^*(y-y_0)\big| < \varepsilon_i + \|y_i^*\|\frac{\delta}{4} \quad \forall i \in \{1,\ldots,n\}\Big\},$$

then it follows that $u, v \in W_\delta \cap B_Y$. On the other hand, in view of (2.2), (2.3), and (2.4), we can estimate

$$\operatorname{diam}(W_\delta \cap B_Y) \geq \|u-v\| \geq \|x-y\| - \|x-u\| - \|y-v\| \geq 2 - \frac{\delta}{16} - \frac{\delta}{4} - \frac{\delta}{4} > 2-\delta.$$

As $0<\delta<1$ was arbitrary, we deduce that $\operatorname{diam}(W \cap B_Y) = 2$, as requested.

(iii) Assume that X has the strong-D2P and that X/Y is strongly regular.

Let $C := \sum_{i=1}^n \lambda_i S(B_Y, y_i^*, \varepsilon) = \sum_{i=1}^n \lambda_i S_i$ be a convex combination of slices of B_Y. Let us prove that $\operatorname{diam}(C) = 2$. To this aim, pick an arbitrary $0<\delta<1$.

Let $\pi : X \longrightarrow X/Y$ be the quotient map. Again, there is no loss of generality by the Hahn–Banach theorem if we assume that $y_i^* \in X^*$ for each $i \in \{1,\ldots,n\}$.

For each $i \in \{1,\ldots,n\}$ consider $A_i := \pi(S(B_X, y_i^*, \varepsilon))$, which is a convex subset of $B_{X/Y}$ containing 0. By [9, Proposition III.6], $\overline{A_i}$ is equal to the closure of the set of its strongly regular points. As a consequence, for each $i \in \{1,\ldots,n\}$, there exists a strongly regular point a_i of $\overline{A_i}$ such that

$$\|a_i\| < \frac{\delta}{32}. \tag{2.5}$$

For every $i \in \{1,\ldots,n\}$ we can find $n_i \in \mathbb{N}$, $\mu_1^i,\ldots,\mu_{n_i}^i \in (0,1]$ such that $\sum_{j=1}^{n_i} \mu_j^i = 1$ and $(a_1^i)^*,\ldots,(a_{n_i}^i)^* \in S_{(X/Y)^*}$, $\eta_j^i \in \mathbb{R}^+$ satisfying

$$a_i \in \sum_{j=1}^{n_i} \mu_j^i \big(S\big(B_{X/Y}, (a_j^i)^*, \eta_j^i\big) \cap \overline{A_i}\big)$$

and also satisfying

$$\operatorname{diam}\Big(\sum_{j=1}^{n_i} \mu_j^i \big(S\big(B_{X/Y}, (a_j^i)^*, \eta_j^i\big) \cap \overline{A_i}\big)\Big) < \frac{\delta}{32}. \tag{2.6}$$

It is clear that, for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n_i\}$, one has

$$S\big(B_{X/Y}, (a_j^i)^*, \eta_j^i\big) \cap A_i \neq \emptyset \Rightarrow S\big(B_X, \pi^*\big((a_j^i)^*\big), \eta_j^i\big) \cap S(B_X, y_i^*, \varepsilon) \neq \emptyset.$$

Now $\sum_{i=1}^n \lambda_i \sum_{j=1}^{n_i} \mu_j^i (S(B_X, \pi^*((a_j^i)^*), \eta_i) \cap S(B_X, y_i^*, \varepsilon))$ is a convex combination of nonempty relatively weakly open subsets of B_X. The last set contains a convex combination of slices of B_X (see [9, Lemma 5.3]), and, as a consequence, the last set has diameter two. Hence we can find, for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n_i\}$, elements $x_j^i, z_j^i \in S(B_X, \pi^*((a_j^i)^*), \eta_i) \cap S(B_X, y_i^*, \varepsilon)$ satisfying

$$\Big\| \sum_{i=1}^n \lambda_i \sum_{j=1}^{n_i} \mu_j^i x_j^i - \sum_{i=1}^n \lambda_i \sum_{j=1}^{n_i} \mu_j^i z_j^i \Big\| > 2 - \frac{\delta}{16}. \tag{2.7}$$

On the one hand, given $i \in \{1, \dots, n\}$, one has

$$\sum_{j=1}^{n_i} \mu_j^i x_j^i \in \sum_{j=1}^{n_i} \mu_j^i S\big(B_X, \pi^*\big((a_j^i)^*\big), \eta_j^i\big) \cap S(B_X, y_i^*, \varepsilon)$$
$$\Rightarrow \pi\Big(\sum_{j=1}^{n_i} \mu_j^i x_j^i\Big) \in \sum_{j=1}^{n_i} \mu_j^i S\big(S_{X/Y}, (a_j^i)^*, \eta_j^i\big) \cap A_i;$$

thus, since (2.5) and (2.6), we have

$$\Big\| \pi\Big(\sum_{j=1}^{n_i} \mu_j^i x_j^i\Big) \Big\| \leq \Big\| \sum_{j=1}^{n_i} \mu_j^i a_i \Big\| + \operatorname{diam}\Big(\sum_{j=1}^{n_i} \mu_j^i \big(S\big(S_{X/Y}, (a_j^i)^*, \eta_j^i\big) \cap A_i\big)\Big) < \frac{\delta}{16}.$$

Hence, from similar computations to (ii), we conclude that, for each $i \in \{1, \dots, n\}$, there exists $a_i \in B_Y$ such that

$$\Big\| a_i - \sum_{j=1}^{n_i} \mu_j^i x_j^i \Big\| < \frac{\delta}{4}. \tag{2.8}$$

By a similar argument we can find, for every $i \in \{1, \dots, n\}$, an element $b_i \in B_Y$ satisfying

$$\Big\| b_i - \sum_{j=1}^{n_i} \mu_j^i z_j^i \Big\| < \frac{\delta}{4}. \tag{2.9}$$

Thus, given $i \in \{1, \dots, n\}$, we deduce, in view of (2.8),

$$y_i^*(a_i) = y_i^*\Big(\sum_{j=1}^{n_i} \mu_j^i x_j^i\Big) + y_i^*\Big(a_i - \sum_{j=1}^{n_i} \mu_j^i x_j^i\Big) > 1 - \varepsilon - \frac{\delta}{4}.$$

In a similar way, using (2.9), we get

$$y_i^*(b_i) > 1 - \varepsilon - \frac{\delta}{4}.$$

Summarizing,

$$a_i, b_i \in S\Big(B_Y, y_i^*, \varepsilon + \frac{\delta}{4}\Big).$$

On the other hand, in view of (2.7), we deduce that

$$\begin{aligned}\operatorname{diam}\Big(\sum_{i=1}^{n}\lambda_i S\Big(B_Y, y_i^*, \varepsilon+\frac{\delta}{4}\Big)\Big) &\geq \Big\|\sum_{i=1}^{n}\lambda_i a_i-\sum_{i=1}^{n}\lambda_i b_i\Big\| \\ &\geq \Big\|\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{n_i}\mu_j^i x_j^i-\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{n_i}\mu_j^i z_j^i\Big\| \\ &\quad -\Big\|\sum_{i=1}^{n}\lambda_i a_i-\sum_{j=1}^{n_i}\mu_j^i x_j^i\Big\|-\Big\|b_i-\sum_{j=1}^{n_i}\mu_j^i z_j^i\Big\| \\ &> 2-\delta.\end{aligned}$$

As $0<\delta<1$ was arbitrary, we conclude that $\operatorname{diam}(C)=2$, as desired. □

Remark 2.3. Consider $X := c_0 \oplus_\infty \ell_2$ and consider $Y := 0 \oplus \ell_2 \subseteq X$. Although X has the strong-D2P, Y fails to have the slice-D2P. This shows that the above theorem is false if we delete the assumptions on X/Y.

Note that the proof of Theorem 2.2 can be adapted in order to get a similar result related to dual versions of diameter 2 properties.

Proposition 2.4. *Let X be a Banach space, and let $Y \subseteq X$ be a subspace.*

- (i) *If X^* has the w^*-slice-D2P, Y is finite-dimensional, and there exists a norm 1 linear and continuous projection $\pi : X \longrightarrow Y$, then Y° has the w^*-slice-D2P.*
- (ii) *If X^* has the w^*-D2P and Y is finite-dimensional, then Y° has the w^*-D2P.*
- (iii) *If X^* has the w^*-SD2P and Y is reflexive, then Y° has the w^*-SD2P.*

Proof. (i) If we define $Z = \ker(\pi)$, then we get that Z° is finite-dimensional since Y is also finite-dimensional. From the w^* version of part (i) in Theorem 2.2, it is enough to see that there is a norm 1 projection from X^* onto Y° with $\ker(p) = Z^\circ$. Consider the inclusion i of Y into X. Letting $p = \pi^* \circ i^*$, we obtain the desired projection.

(ii) Consider a weak-star open subset W of $Y^\circ = (X/Y)^*$ such that

$$W \cap B_{Y^\circ} \neq \emptyset.$$

Now we can extend W to a weak-star open subset of X^*, say U, as it is done in Theorem 2.2(ii) satisfying $U \cap B_{X^*} \neq \emptyset$.

Let $p : X^* \longrightarrow X^*/Y^\circ$ be the quotient map, which is a $w^* - w^*$ open map. Then $p(U)$ is a weak-star open set of X^*/Y° that meets with B_{X^*/Y°.

If we define $A := p(U) \cap B_{X^*/Y^\circ}$, then we have that A is a relatively weak-star open and convex subset of B_{X^*/Y° that contains 0.

As $X^*/Y^\circ = Y^*$ is finite-dimensional, we can find a weak-star open subset V of X^*/Y°, in fact a ball centered at zero, such that $V \subset A$ and whose diameter is as close to zero as desired.

From here, it is straightforward to check that computations of Theorem 2.2(ii) work, and this allows us to conclude that $\operatorname{diam}(W \cap B_{Y^\circ}) = 2$.

(iii) Consider $C := \sum_{i=1}^{n} \lambda_i S(B_{Y^\circ}, x_i, \alpha)$, a convex combination of weak-star slices in B_{Y°; let us prove that $\operatorname{diam}(C) = 2$.

Define $S_i := S(B_{X^*}, x_i, \alpha)$ for each $i \in \{1, \ldots, n\}$. Let $\pi : X^* \longrightarrow X^*/Y^\circ$ be the quotient map, and define $A_i := \pi(S_i)$.

As $X^*/Y^\circ = Y^*$ is reflexive, X^*/Y° is strongly regular, and so we can find for each $i \in \{1, \ldots, n\}$ a strongly regular point a_i of $\overline{A_i}$ whose norm is as close to zero as desired. Given $i \in \{1, \ldots, n\}$, as a_i is a point of strong regularity, we can find convex combinations of slices containing a_i and whose diameters are as small as desired. In addition, from the reflexivity of X^*/Y°, convex combinations of slices are indeed convex combinations of weak-star slices, and so we can actually find convex combinations of weak-star slices containing a_i and whose diameters are as close to zero as desired for each $i \in \{1, \ldots, n\}$.

Using the previous ideas, the result can be concluded following word by word the proof of Theorem 2.2(iii). $\square$

As we said in the Introduction, the three weak-star diameter 2 properties are dual properties of different kinds of octahedrality of the norm (see [10], [7, Proposition I.1.11], and [4] for details). From these facts we conclude the following.

Theorem 2.5. *Let X be a Banach space, and let Y be a subspace of X.*

(i) *If X has a 2-rough norm, Y is a finite-dimensional subspace of X, and $\pi : X \to Y$ is a norm 1 projection, then X/Y has a 2-rough norm.*
(ii) *If X has a weakly octahedral norm and Y is finite-dimensional, then X/Y has a weakly octahedral norm.*
(iii) *If X has an octahedral norm and Y is reflexive, then X/Y has an octahedral norm.*

Now, using the identification $Y^\circ = (X/Y)^*$, we get the following.

Corollary 2.6. *Let X be a Banach space such that X^* has the SD2P, and let Y be a subspace of X. If Y does not contain any copy of ℓ_1, then Y° has the SD2P.*

Proof. Assume that Y does not contain any isomorphic copy of ℓ_1. Then $Y^* = X^*/Y^\circ$ is strongly regular (see [9]). By Theorem 2.2(iii), we deduce that $Y^\circ = (X/Y)^*$ has the SD2P. $\square$

Note that, taking into account the duality between SD2P and octahedrality, we deduce from Corollary 2.6 that the norm on $(X/Y)^{**} = X^{**}/Y^{\circ\circ}$ is octahedral whenever the norm on X^{**} is octahedral and Y does not contain isomorphic copies of ℓ_1.

Corollary 2.7. *Let X be a Banach space satisfying the SD2P, and let Y be a subspace of X. If Y° does not contain isomorphic copies of ℓ_1, then Y satisfies the SD2P.*

Proof. As Y° does not contain isomorphic copies of ℓ_1, we deduce that $(Y^\circ)^*$ is strongly regular, and so X/Y is also strongly regular as a subspace of $(Y^\circ)^*$. Now Theorem 2.2 applies. $\square$

Note that, again taking into account the duality between the SD2P and octahedrality, we deduce from the above corollary that the norm on Y^* is octahedral whenever the norm on X^* is octahedral and Y° does not contain isomorphic copies of ℓ_1.

Corollary 2.8. *Let X be a Banach space with the SD2P, and let Y be a subspace of X. If X/Y does not contain ℓ_1^n uniformly, then Y has the SD2P.*

Note that, taking into account the duality between SD2P and octahedrality, we deduce from the preceding corollary that the norm on Y^* is octahedral whenever the norm on X^* is octahedral and Y° does not contain ℓ_1^n uniformly.

3. Some remarks and open questions

From the results of Section 2, the following questions remain open.

Problem 3.1. Let X be a Banach space, and let Y be a closed subspace of X.

(i) If X has the slice-D2P and X/Y is finite-dimensional, then does Y have the slice-D2P?
(ii) If X has the D2P and X/Y is reflexive (or has the RNP), then does Y have the D2P?

Note that, from Theorem 2.2, the answer to question (i) above is positive for the case of the diameter 2 property. Hence, if there were a Banach space X answering question (i) above negatively, then X would have to satisfy the slice-D2P but fail the D2P. The existence of such Banach spaces is known, but it is highly nontrivial (see [5]). Hence, although we think that the answer to problem (i) above has to be negative, finding a concrete example seems quite difficult. Similarly, if there were a Banach space X answering question (ii) above negatively, then X would have to satisfy the D2P but fail the strong-D2P, since the answer to question (ii) above is positive for Banach spaces having the SD2P according to Theorem 2.2. Hence, if there were a Banach space that provides a negative answer to question (ii), then its construction would be quite complicated.

In a very recent paper (see [1]), new stability results about octahedral norms and the SD2P have appeared. Indeed, we have the following.

Theorem 3.2 ([1, Theorem 3.9])**.** *Let X be a Banach space. The following assertions are equivalent.*

(i) *The norm on X is octahedral.*
(ii) *Each closed subspace $Y \subseteq X$ such that X/Y does not contain any isomorphic copy of ℓ_1 and has an octahedral norm.*

Note that the previous stability result complements the information of those above. For instance, while Theorem 2.5 provides stability of octahedrality to quotients with hypotheses on subspaces, the above result proves inheritance of octahedrality to subspaces with hypotheses on quotients.

In order to summarize the known results of inheritance of the SD2P and octahedrality, we formulate the following theorem.

Theorem 3.3. *Let X be a Banach space, and let $Y \subseteq X$ be a closed subspace.*

(i) *If X has the SD2P and Y is an almost isometric ideal, then Y has the SD2P.*

(ii) *If X has the SD2P and X/Y is strongly regular, then Y has the SD2P.*
(iii) *If X^* has the SD2P and Y does not contain an isomorphic copy of ℓ_1 (in particular, if Y is strongly regular), then X° has the SD2P.*
(iv) *If X has an octahedral norm and Y is reflexive, then X/Y has an octahedral norm.*
(v) *If X has an octahedral norm and X/Y does not contain any isomorphic copy of ℓ_1 (resp., $(X/Y)^* = Y^\circ$ is strongly regular), then Y has an octahedral norm.*

Proof. The assertions (ii) and (iv) have been proved in Theorems 2.2 and 2.5, respectively, whereas (i) is stated in [2, Proposition 3.3] and (v) is stated in Theorem 3.2. Finally, (iii) follows from Corollary 2.6. □

In view of the results noted above, it seems natural to pose the following question.

Problem 3.4. Let X be a Banach space, and let Y be a closed subspace.

If X has an octahedral norm and Y is strongly regular (or even Y^* does not contain any isomorphic copy of ℓ_1), then does X/Y have an octahedral norm?

Finally, we could wonder whether we can improve Theorem 2.5(i)–(ii), in the following sense.

Problem 3.5. Let X be a Banach space, and let Y be a closed subspace of X.
(i) If X has a 2-rough norm and Y is finite-dimensional (or has the RNP), then does X/Y have a 2-rough norm?
(ii) If X has a weakly-octahedral norm and Y is reflexive (or has the RNP), then does X/Y have a weakly octahedral norm?

On the one hand, as in Problem 3.1, observe that a negative answer to (i) (resp., (ii)) would imply that X^* has the w^*-slice-D2P (resp., the w^*-D2P) but does not have the w^*-D2P (resp., the w^*-SD2P) by Theorem 2.5.

On the other hand, note that we can get from (i) that Y has a 1-rough norm (see [7, Lemma III.1.1]).

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