



Banach J. Math. Anal. 10 (2016), no. 3, 621–637

<http://dx.doi.org/10.1215/17358787-3639646>

ISSN: 1735-8787 (electronic)

<http://projecteuclid.org/bjma>

NORM-ATTAINING LIPSCHITZ FUNCTIONALS

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Communicated by D. Werner

ABSTRACT. We prove that for a given Banach space X , the subset of norm-attaining Lipschitz functionals in $\text{Lip}_0(X)$ is weakly dense but not strongly dense. Then we introduce a weaker concept of directional norm attainment and demonstrate that for a uniformly convex X the set of directionally norm-attaining Lipschitz functionals is strongly dense in $\text{Lip}_0(X)$ and, moreover, that an analogue of the Bishop–Phelps–Bollobás theorem is valid.

1. INTRODUCTION AND MOTIVATION

In this text, the letter X stands for a real Banach space. We denote, as usual, by S_X and B_X the unit sphere and the closed unit ball of X , respectively. A functional $x^* \in X^*$ *attains its norm* if there is $x \in S_X$ with $x^*(x) = \|x^*\|$. If X is reflexive, then all $x^* \in X^*$ attain their norms and, according to the famous James theorem (see [8, Chapter 1, Theorem 3]), in every nonreflexive space there are functionals that do not attain their norm. Nevertheless, in every Banach space there are “many” norm-attaining functionals. Namely, the classical Bishop–Phelps theorem (see [5], [8, Chapter 1]) states that the set of norm-attaining functionals on a Banach space is norm-dense in the dual space. Moreover, for every closed bounded convex set $C \subset X$, the collection of functionals that attain their maximum on C is norm-dense in X^* .

The fact that every functional can be approximated by those that are norm-attaining is quite useful, but sometimes one needs more. Namely, sometimes (in

Copyright 2016 by the Tusi Mathematical Research Group.

Received Nov. 19, 2015; Accepted Nov. 28, 2015.

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2010 *Mathematics Subject Classification.* Primary 46B04; Secondary 46B20, 46B22, 47A30.

Keywords. Bishop–Phelps–Bollobás theorem, norm-attaining functional, Lipschitz-free space, Lipschitz functional, uniformly convex Banach space.

particular, when one works with the numerical radius of operators) one needs to approximate a pair $(x, x^*) \in S_X \times S_{X^*}$ such that $x^*(x) \sim 1$ by a pair $(y, y^*) \in S_X \times S_{X^*}$ such that $y^*(y) = 1$. Such a modification of the Bishop–Phelps theorem was given by Bollobás in [6]. Below we cite it in a slightly modified form with sharp estimates from [7].

Theorem 1.1 ([7, Corollary 2.4]). *Let X be a Banach space. Suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy $x^*(x) \geq 1 - \delta$ for $\delta \in (0, 2)$. Then there exists $(y, y^*) \in X \times X^*$ with $\|y\| = \|y^*\| = y^*(y) = 1$ such that*

$$\max\{\|x - y\|, \|x^* - y^*\|\} \leq \sqrt{2\delta}. \quad (1.1)$$

In this project we are searching for possible extensions of the Bishop–Phelps theorem and the Bishop–Phelps–Bollobás theorem for nonlinear Lipschitz functionals $f : X \rightarrow \mathbb{R}$.

Throughout this paper, we use the letter E to denote a metric space, equipped with a distinguished point 0 and such that $E \setminus \{0\} \neq \emptyset$, and such that ρ denotes the given distance of E . Recall that to such a *pointed metric space*, one associates the Banach space $\text{Lip}_0(E)$ that consists of functions $f : E \rightarrow \mathbb{R}$ with $f(0) = 0$ which satisfy (globally) the Lipschitz condition. This space is equipped with the norm

$$\|f\| = \sup\left\{\frac{|f(x) - f(y)|}{\rho(x, y)} : x, y \in E, x \neq y\right\}. \quad (1.2)$$

In other words, $\|f\|$ is the smallest Lipschitz constant of f . (We refer the reader to the book [15] for background on Lipschitz spaces.)

The most interesting results of the present paper are related to Definition 1.3 below, which does not make sense for functions on general pointed metric spaces, but only in the Banach space setting. That is why we mainly concentrate here on the case of $\text{Lip}_0(X)$, where X is a Banach space. Note that in this case, evidently, X^* is a closed subspace of $\text{Lip}_0(X)$ with equality of norms.

Since we deal here with possible extensions of the Bishop–Phelps and Bishop–Phelps–Bollobás theorems, we first need to state what we understand by the term *norm-attaining Lipschitz functional*. We have a couple of possible definitions for this, the first being the following.

Definition 1.2. A functional $f \in \text{Lip}_0(E)$ *attains its norm in the strong sense* if there are $x, y \in E, x \neq y$, such that $\|f\| = \frac{|f(x) - f(y)|}{\rho(x, y)}$. The subset of all functionals $f \in \text{Lip}_0(E)$ that attain their norm in the strong sense is denoted $\text{SA}(E)$.

Unfortunately, in the sense of the Bishop–Phelps theorem, this definition is too restrictive. Even in the 1-dimensional case ($X = \mathbb{R}$), the subset $\text{SA}(X)$ is not dense in $\text{Lip}_0(X)$ (see Example 2.1 and Theorem 2.3). For every Banach space X , nevertheless, $\text{SA}(X)$ is weakly sequentially dense in $\text{Lip}_0(X)$ (see Theorem 2.6). This is the content of our Section 2, where the results are actually proved for metrically convex metric spaces.

It is then clear that a less restrictive way for a Lipschitz functional to attain its norm is needed to get norm density. We will use the following definition.

Definition 1.3. A functional $g \in \text{Lip}_0(X)$ attains its norm at the direction $u \in S_X$ if there is a sequence of pairs $\{(x_n, y_n)\}$ in $X \times X$, with $x_n \neq y_n$, such that

$$\lim_{n \rightarrow \infty} \frac{x_n - y_n}{\|x_n - y_n\|} = u \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} = \|g\|.$$

In this case, we say that g attains its norm directionally. The set of all those $f \in \text{Lip}_0(X)$ that attain their norm directionally is denoted by $\text{DA}(X)$.

We start our consideration with two reasons why the directional approach is natural in our framework.

- (a) If X is finite-dimensional, then $\text{DA}(X) = \text{Lip}_0(X)$ by a compactness argument, so at least in this easiest case the directional Bishop–Phelps theorem does not fail.
- (b) A linear functional attains its norm at direction u if and only if $f(u) = \|f\|$ (i.e., if it attains its norm in the usual sense).

We devote Section 3 to norm-attaining seminorms. Continuous seminorms on a Banach space X form a closed cone in $\text{Lip}_0(X)$, and the Lipschitz norm coincides with the uniform norm (i.e., the supremum on the unit sphere of the space). Moreover, the respective sets of continuous seminorms that attain the norm strongly, that attain the norm directionally, and that attain the norm uniformly coincide (see Lemma 3.2). We provide a general Bishop–Phelps–Bollobás theorem for seminorms, but getting uniform density instead of density in the Lipschitz norm (see Proposition 3.4). Besides, we prove the Lipschitz-norm density of norm-attaining seminorms for Banach spaces with the Radon–Nikodým property (see Proposition 3.6) by using Stegall’s version of the classical Bourgain–Steagall non-linear optimization principle. Finally, we show that in every infinite-dimensional Banach space, there is a continuous seminorm which does not attain its norm (Example 3.3), so item (a) above actually characterizes finite dimension.

The main result of this article is a Bishop–Phelps–Bollobás-type theorem for Lipschitz functionals on uniformly convex spaces. Let us introduce the following definition.

Definition 1.4. A Banach space X has the *directional Bishop–Phelps–Bollobás property for Lipschitz functionals* ($X \in \text{LipBPB}$ for short) if for every $\varepsilon > 0$ there is such a $\delta > 0$ that, for every $f \in \text{Lip}_0(X)$ with $\|f\| = 1$ and for every $x, y \in X$ with $x \neq y$ satisfying $\frac{f(x) - f(y)}{\|x - y\|} > 1 - \delta$, there is $g \in \text{Lip}_0(X)$ with $\|g\| = 1$ and there is $u \in S_X$ such that g attains its norm at the direction u , $\|g - f\| < \varepsilon$, and $\left\| \frac{x - y}{\|x - y\|} - u \right\| < \varepsilon$.

With this notation, the main result of the paper is to prove that uniformly convex spaces have the LipBPB (Theorem 5.3), and even a stronger property called *local directional Bishop–Phelps–Bollobás property for Lipschitz functionals* introduced in Definition 4.3. This is the content of Section 5. In proving such a result, we need to provide a weak version of the property (see Lemma 4.1), valid for all Banach spaces, which is proved using the Lipschitz-free space (see the definition in Section 4). We also prove (see Lemma 4.4) that the requirements for a general Banach space to have the (local) Bishop–Phelps–Bollobás property

for Lipschitz functionals can be relaxed in the sense that only “approximate” directional norm attainment of g is required. These two preliminary results are the content of Section 4.

We should make it clear that we are not able to answer some natural easy-looking questions related to our results. For example, we are not able to construct a Banach space which has no (local) directional Bishop–Phelps–Bollobás property for Lipschitz functionals. On the other hand, we refer to [9] for some negative results on norm-attaining Lipschitz maps between Banach spaces which do not overlap with the results of this manuscript.

We finish our preliminaries by recalling an important tool for constructing Lipschitz functionals: the classical *McShane’s extension theorem*. It states that, if M is a subspace of a metric space E and if $f : M \rightarrow \mathbb{R}$ is a Lipschitz functional, then there is an extension to a Lipschitz functional $F : E \rightarrow \mathbb{R}$ with the same Lipschitz constant (see [15, Theorem 1.5.6] or [4, p. 12, 13]).

2. STRONGLY ATTAINING LIPSCHITZ FUNCTIONALS

As we stated above, there is no Bishop–Phelps-type theorem for Lipschitz functionals in the strong sense of the attainment, even in the 1-dimensional case.

Example 2.1. $\text{SA}([0, 1])$ is not dense in $\text{Lip}_0([0, 1])$.

In order to demonstrate this, we need the following easy lemma.

Lemma 2.2. *If $f \in \text{Lip}_0(E)$ attains its norm on a pair $(x, y) \in E \times E$, $x \neq y$, and if $z \in E \setminus \{x, y\}$ is such an element that $\rho(x, y) = \rho(x, z) + \rho(z, y)$, then f strongly attains its norm on the pairs (x, z) and (y, z) , and*

$$f(z) = \frac{\rho(z, y)f(x) + \rho(x, z)f(y)}{\rho(x, y)}. \quad (2.1)$$

In particular, if E is a convex subset of a Banach space, then f is affine on the closed segment $\text{conv}\{x, y\}$; that is, $f(\theta x + (1 - \theta)y) = \theta f(x) + (1 - \theta)f(y)$ for every $\theta \in [0, 1]$.

Proof. We may (and do) assume without loss of generality that $f(x) - f(y) \geq 0$ (otherwise we multiply f by -1). Since f attains its norm on the pair (x, y) , we have

$$\begin{aligned} \|f\|\rho(x, y) &= f(x) - f(y) = f(x) - f(z) + f(z) - f(y) \\ &\leq \|f\|\rho(x, z) + \|f\|\rho(z, y) = \|f\|\rho(x, y). \end{aligned}$$

This means that the inequalities

$$f(x) - f(z) \leq \|f\|\rho(x, z) \quad \text{and} \quad f(z) - f(y) \leq \|f\|\rho(z, y)$$

which we used above are, in fact, equalities. So, $f(x) = f(z) + \|f\|\rho(x, z)$ and $f(y) = f(z) - \|f\|\rho(z, y)$, and so f attains its norm on the pairs (x, z) and (y, z) . Substituting the last two formulas into the right-hand side of (2.1), we get the desired result. \square

Proof of Example 2.1. It is well known (see, e.g., [15, Example 1.6.5] or [12, Propositions 6 and 2]) that $\text{Lip}_0([0, 1])$ is isometric to $L_\infty([0, 1])$, and the corresponding bijective isometry $U : \text{Lip}_0([0, 1]) \rightarrow L_\infty([0, 1])$ is just the differentiation operator (the derivative of a Lipschitz function $f : [0, 1] \rightarrow \mathbb{R}$ exists almost everywhere). Under this isometry, every $f \in \text{SA}([0, 1])$ maps to a function, which is equal either to $\|f\|$ or to $-\|f\|$ on some nonvoid interval (here we use Lemma 2.2). Denote A a nowhere dense closed subset of $[0, 1]$ of positive Lebesgue measure, and let $g \in \text{Lip}_0([0, 1])$ be the function, whose derivative equals $\mathbf{1}_A$ (the characteristic function of A) almost everywhere. Then g cannot be approximated by functions from $\text{SA}([0, 1])$. Actually, we claim that

$$\|g - f\| = \|\mathbf{1}_A - f'\|_\infty \geq \frac{1}{2} \tag{2.2}$$

for every $f \in \text{SA}(\mathbb{R})$. In fact, $\|g\| = \|\mathbf{1}_A\|_\infty = 1$; so if $\|f'\|_\infty \leq \frac{1}{2}$, then (2.2) follows from the triangle inequality. If $\|f'\|_\infty > \frac{1}{2}$, then, as we remarked before, $|f'(t)| > \frac{1}{2}$ on some open interval (a, b) . But, since A is nowhere dense, there is a smaller interval $(c, d) \subset (a, b)$ such that $\mathbf{1}_A(t) = 0$ for $t \in (c, d)$. So $|\mathbf{1}_A - U(f)](t)| > \frac{1}{2}$ on (c, d) , which implies (2.2). Hence, $\text{SA}([0, 1])$ is not dense in $\text{Lip}_0([0, 1])$. \square

Recall, that a metric space E is said to be *metrically convex* if for every pair of distinct points $x, y \in E$ there is a curve $\ell \subset E$ which connects x and y and is isometric to the segment $[0, \rho(x, y)] \subset \mathbb{R}$. The next theorem demonstrates that Example 2.1 extends to all nontrivial metrically convex pointed metric spaces, and, in particular, to all Banach spaces.

Theorem 2.3. *Let E be a metrically convex pointed metric space. Then $\text{SA}(E)$ is not dense in $\text{Lip}_0(E)$.*

Proof. Fix $x_0 \in E \setminus \{0\}$; without loss of generality, assume that $\rho(0, x_0) = 1$. Denote $\ell \subset E$ the isometric copy of $[0, 1]$ which connects 0 and x_0 . Again without loss of generality, we may assume that $\ell = [0, 1] \subset E$. Let $u : E \rightarrow [0, 1]$, $\|u\| = 1$, be a Lipschitz function whose restriction to $[0, 1]$ is the identity map (here we apply McShane’s extension theorem), and let $g \in \text{Lip}_0([0, 1])$ be the function from Example 2.1. We are going to demonstrate that $h := g \circ u \in \text{Lip}_0(E) \setminus \overline{\text{SA}(E)}$.

Consider an arbitrary $f \in \text{SA}(E)$. As in Example 2.1, we will show that $\|h - f\| \geq \frac{1}{2}$. Assume that

$$\|h - f\| < \frac{1}{2}. \tag{2.3}$$

Then $\|f\| > 1/2$. Denote $(x, y) \in E \times E$, $x \neq y$ a pair at which f attains its norm. Due to metric convexity of E , there is an isometric copy γ of $[0, \rho(x, y)]$ which connects x and y . According to Lemma 2.2,

$$|f(z_1) - f(z_2)| = \|f\|\rho(z_1, z_2)$$

for every $z_1, z_2 \in \gamma$, $z_1 \neq z_2$. Consequently,

$$|h(z_1) - h(z_2)| \geq |f(z_1) - f(z_2)| - \|h - f\|\rho(z_1, z_2) > 0$$

for all such z_1, z_2 . This, in turn, implies that $g(v_1) \neq g(v_2)$ for every pair of distinct points v_1, v_2 in the nonvoid interval $u(\gamma) \subset [0, 1]$. But, according to the definition of g , this is impossible. \square

Even though in the sense of Definition 1.2 the Bishop–Phelps theorem does not transfer to Lipschitz functionals, $\text{SA}(X)$ cannot be too small. Namely, we are going to demonstrate that $\text{SA}(X)$ is weakly sequentially dense in $\text{Lip}_0(X)$ for every Banach space X . Even more, we will prove an analogous fact for a wider class of “local” metric spaces.

According to [11, Definition 2.2], a pointed metric space E is said to be *local* if for every $\varepsilon > 0$ and for every function $f \in \text{Lip}_0(E)$ there are two distinct points $t_1, t_2 \in E$ such that $\rho(t_1, t_2) < \varepsilon$ and

$$\frac{f(t_2) - f(t_1)}{\rho(t_1, t_2)} > \|f\| - \varepsilon. \quad (2.4)$$

Every local space evidently is infinite, and it is also easy to see that locality implies the absence of isolated points (the function $f = \mathbf{1}_{\{\tau\}}$ where τ is an isolated point does not fit to the definition). Every metrically convex E is local (see [11, Proposition 2.3]) and a partial converse statement is known (see [11, Proposition 2.9]): let E be a metric subspace of a smooth locally uniformly rotund Banach space. If E is compact and local, then E is convex. We begin with a helpful lemma.

Lemma 2.4. *Let E be a local metric space, let $\{f_n\}$ be a sequence in $S_{\text{Lip}_0(E)}$, and for each $n \in \mathbb{N}$ let $U_n := \{x \in E : f_n(x) \neq 0\}$ be the corresponding supports. Suppose that the sets U_n are pairwise separated; that is,*

$$d_{n,m} = \inf\{\rho(x, y) : x \in U_n, y \in U_m\} > 0$$

for every $n \neq m$. Then, the sequence $\{f_n\}$ is isometrically equivalent to the canonical basis of c_0 ; that is, for any finite collection $\{a_j\}_{j=1}^n$ of reals, we have

$$\left\| \sum_{j=1}^n a_j f_j \right\| = \max_k |a_k|. \quad (2.5)$$

Proof. Denote $f = \sum_{j=1}^n a_j f_j$, $d_n = \min\{d_{k,j} : k, j \in \{1, 2, \dots, n\}\}$. Fix an $\varepsilon > 0$ satisfying $\varepsilon < d_n$. According to the definition of locality, there are points $t_1, t_2 \in E$ such that $0 < \rho(t_1, t_2) < \varepsilon$ which fulfill (2.4). If one of t_i belongs to some U_m , then the other one either belongs to the same U_m , or lies outside of f 's support. Consequently, for this m ,

$$\|f\| < \frac{f(t_2) - f(t_1)}{\rho(t_1, t_2)} + \varepsilon = a_m \frac{f_m(t_2) - f_m(t_1)}{\rho(t_1, t_2)} + \varepsilon \leq |a_m| + \varepsilon.$$

By the arbitrariness of ε , this implies that $\|f\| \leq \max_k |a_k|$.

In order to get the reverse inequality, we fix such an m that $\max_k |a_k| = |a_m|$ and we apply the locality condition to f_m . We get points $t_1, t_2 \in E$ such that $0 < \rho(t_1, t_2) < \varepsilon$ and

$$\frac{f_m(t_2) - f_m(t_1)}{\rho(t_1, t_2)} > 1 - \varepsilon.$$

If ε is small enough, this condition again means that one of t_i belongs to U_m and the other one either belongs to U_m , or lies outside of f 's support. Consequently,

$$\begin{aligned} \max_k |a_k| &= |a_m| \leq \frac{1}{1-\varepsilon} |a_m| \frac{|f_m(t_2) - f_m(t_1)|}{\rho(t_1, t_2)} \\ &= \frac{1}{1-\varepsilon} \frac{|f(t_2) - f(t_1)|}{\rho(t_1, t_2)} \leq \frac{1}{1-\varepsilon} \|f\|. \end{aligned} \quad \square$$

As an obvious consequence, we obtain the following.

Corollary 2.5. *If E is a local metric space, then every bounded separately supported sequence in $\text{Lip}_0(E)$ converges weakly to zero.*

We are now ready to prove the weak sequential density of strongly attaining Lipschitz functionals.

Theorem 2.6. *If E is a local metric space, then $\text{SA}(E)$ is weakly sequentially dense in $\text{Lip}_0(E)$; that is, for every $g \in \text{Lip}_0(E)$ there is a sequence $\{g_n\}$ in $\text{SA}(E)$ which converges weakly to g .*

Proof. Without loss of generality, we may assume that $\|g\| = 1$. Let us choose a sequence of pairwise disjoint balls $\{U_n\}$ with corresponding radii $r_n > 0$, centers x_n and such that $0 \notin U_n$. For every $n \in \mathbb{N}$, select $\varepsilon_n \in (0, \frac{1}{2})$ and $y_n \in U_n$ with $0 < \rho(x_n, y_n) = \varepsilon_n r_n$ (here we use the absence of isolated points). For a fixed n , consider $E_n = (E \setminus U_n) \cup \{x_n, y_n\} \subset E$. Define $h_n : E_n \rightarrow \mathbb{R}$ as follows: $h_n(t) = g(t)$ for $t \in E_n \setminus \{x_n\}$ and $h_n(x_n) = g(y_n) - s_n(1 + 2\varepsilon_n)\rho(x_n, y_n)$, where $s_n = \text{sign}(g(y_n) - g(x_n))$. We claim that the Lipschitz constant of h_n is attained at the pair (x_n, y_n) and equals $1 + 2\varepsilon_n$. First,

$$\frac{|h_n(x_n) - h_n(y_n)|}{\rho(x_n, y_n)} = 1 + 2\varepsilon_n.$$

Second, if $x, y \in E_n \setminus \{x_n\}$, then

$$\frac{|h_n(x) - h_n(y)|}{\rho(x_n, y_n)} = \frac{|g(x) - g(y)|}{\rho(x, y)} \leq 1.$$

So, it remains to check that for every $y \in E \setminus U_n$, we have

$$\frac{|h_n(x_n) - h_n(y)|}{\rho(x_n, y)} \leq 1 + 2\varepsilon_n.$$

In fact,

$$\begin{aligned} \frac{|h_n(x_n) - h_n(y)|}{\rho(x_n, y)} &= \frac{|g(y_n) - s_n(1 + 2\varepsilon_n)\rho(x_n, y_n) - g(y)|}{\rho(x_n, y)} \\ &= \frac{|g(y_n) - s_n(1 + 2\varepsilon_n)\varepsilon_n r_n - g(y)|}{\rho(x_n, y)} \\ &\leq \frac{|g(x_n) - g(y)|}{\rho(x_n, y)} + \frac{|g(y_n) - g(x_n) - s_n(1 + 2\varepsilon_n)\varepsilon_n r_n|}{\rho(x_n, y)} \\ &\leq 1 + \frac{(1 + 2\varepsilon_n)\varepsilon_n r_n}{r_n} \leq 1 + 2\varepsilon_n. \end{aligned}$$

The claim is proved. Now, applying McShane's extension theorem, we extend h_n to a functional g_n on the whole of E preserving its Lipschitz constant. Then, $g_n \in \text{Lip}_0(X)$, $\|g_n\| = 1 + 2\varepsilon_n$, and

$$\frac{|g_n(x_n) - g_n(y_n)|}{\rho(x_n, y_n)} = \frac{|h_n(x_n) - h_n(y_n)|}{\rho(x_n, y_n)} = 1 + 2\varepsilon_n,$$

so $g_n \in \text{SA}(E)$. On the other hand, $\text{supp}(g_n - g) \subset U_n$, so the functionals $g_n - g$ are disjointly supported. According to Corollary 2.5, this implies that $\{g_n - g\}$ converges weakly to zero. \square

Remark 2.7. Choosing in the above proof sequences $\{\varepsilon_n\}$ and $\{r_n\}$ converging to zero, one gets additional properties of the approximating sequence g_n . Namely, one can get that $\{\|g_n\|\} \rightarrow \|g\|$ and that $\{g_n\}$ converges uniformly on the whole of E to g .

As Banach spaces are local metric spaces, we may particularize Theorem 2.6 and Remark 2.7 to this case.

Corollary 2.8. *Let X be a Banach space. Then, for every $g \in \text{Lip}_0(X)$ there is a sequence $\{g_n\}$ in $\text{SA}(X)$ which converges weakly to g . Moreover, the sequence $\{g_n\}$ can be chosen in such a way that one also has that $\{\|g_n\|\} \rightarrow \|g\|$ and that $\{g_n\}$ converges uniformly on the whole of X to g .*

3. BISHOP–PHELPS THEOREMS FOR SEMINORMS

Let X be a Banach space. We write $\text{Sem}(X)$ for the set of all continuous seminorms on X . As usual, we consider the set $\text{Sem}(X)$ as a closed cone of $\ell_\infty(S_X, \mathbb{R})$, where for a set Γ , $\ell_\infty(\Gamma, \mathbb{R})$ denotes the Banach space of all bounded functions from Γ into \mathbb{R} endowed with the uniform norm $\|\cdot\|_\infty$. We need a couple of easy remarks on continuous seminorms: they can be viewed as Lipschitz functionals with equality of norms, and they can be expressed in terms of the norm of a bounded linear operator. We use the notation $L(X, Y)$ for the Banach space of all bounded linear operators between the Banach spaces X and Y . Only in this section, we will write $\|\cdot\|_{\text{Lip}}$ to denote the Lipschitz norm when there is a possible confusion.

Remark 3.1. Let X be a Banach space. Then we have the following.

- (a) $\text{Sem}(X) \subseteq \text{Lip}_0(X)$ and $\|p\|_{\text{Lip}} = \|p\|_\infty$ for every $p \in \text{Sem}(X)$.
- (b) For every $p \in \text{Sem}(X)$ there exist a Banach space Y and $T \in L(X, Y)$ such that $p(x) = \|Tx\|$ for all $x \in X$, which obviously satisfies $\|T\| = \|p\|_\infty$. Actually, one can consider $Y = \ell_\infty(\Gamma, \mathbb{R})$, where Γ is a set whose cardinality equals the density character of X .

Proof. (a). This is a consequence of the triangle inequality. Indeed, for $p \in \text{Sem}(X)$ and $x, y \in X$ with $x \neq y$, we have

$$\frac{|p(x) - p(y)|}{\|x - y\|} \leq \frac{p(x - y)}{\|x - y\|} = p\left(\frac{x - y}{\|x - y\|}\right) \leq \|p\|_\infty,$$

so $\|p\|_{\text{Lip}} \leq \|p\|_{\infty}$. Conversely, for every $z \in S_X$ we have

$$p(z) = \frac{p(z) - p(0)}{\|z - 0\|} \leq \|p\|_{\text{Lip}}.$$

(b). Let Y be the completion of the quotient $(X, p)/\ker p$, and let $T \in L(X, Y)$ be the composition of the natural quotient map from X onto $(X, p)/\ker p$ and the inclusion into Y . It is then obvious that $p(x) = \|Tx\|$ for every $x \in X$. For the moreover part, just observe that the density character of Y is smaller or equal than that of X , so Y embeds isometrically into $\ell_{\infty}(\Gamma, \mathbb{R})$ and one can view T as an element of $L(X, \ell_{\infty}(\Gamma, \mathbb{R}))$. \square

Our goal in this section is to study norm-attaining seminorms. As we have two norms defined on $\text{Sem}(X)$ and also two ways for a Lipschitz functional to attain the norm, we have three possibilities. As a matter of fact, all of them are the same for continuous seminorms.

Lemma 3.2. *Let X be a Banach space and let $p \in \text{Sem}(X)$. Then, the following conditions are equivalent:*

- (i) p attains its norm as an element of $\ell_{\infty}(S_X, \mathbb{R})$ (i.e. there exists $z \in S_X$ such that $p(z) = \|p\|$),
- (ii) $p \in \text{SA}(X)$ (i.e., there exists a pair $(x, y) \in X \times X$ with $x \neq y$ such that $\frac{|p(x)-p(y)|}{\|x-y\|} = \|p\|$),
- (iii) $p \in \text{DA}(X)$ (i.e., there exists a sequence of pairs $\{(x_n, y_n)\}$ in $X \times X$ with $x_n \neq y_n$ for all n such that $\{\frac{x_n - y_n}{\|x_n - y_n\|}\}$ is convergent and $\{\frac{p(x_n) - p(y_n)}{\|x_n - y_n\|}\} \rightarrow \|p\|$),
- (iv) for every Banach space Y and every such operator $T \in L(X, Y)$ such that $p(x) = \|Tx\|$ ($x \in X$), T attains its norm (i.e., there is $z \in S_X$ such that $\|Tz\| = \|T\| = \|p\|$),
- (v) there exist a Banach space Y and a norm-attaining operator $T \in L(X, Y)$ such that $p(x) = \|Tx\|$ ($x \in X$).

In the case that p satisfies any (all) of the above conditions, we will say that p is a norm-attaining seminorm.

Proof. For (i) \Rightarrow (ii), consider the pair $(z, 0)$ (otherwise, (ii) \Rightarrow (iii), (iv) \Rightarrow (v), and (v) \Rightarrow (i) are evident).

(iii) \Rightarrow (iv). Write $u = \lim \frac{x_n - y_n}{\|x_n - y_n\|}$, so $T(u) = \lim \frac{T(x_n) - T(y_n)}{\|x_n - y_n\|}$. Then

$$\begin{aligned} \|T(u)\| &= \lim \frac{\|T(x_n) - T(y_n)\|}{\|x_n - y_n\|} \geq \lim \frac{|\|T(x_n)\| - \|T(y_n)\||}{\|x_n - y_n\|} \\ &= \lim \frac{|p(x_n) - p(y_n)|}{\|x_n - y_n\|} = \|p\|. \end{aligned} \quad \square$$

As a consequence of this result, we may provide an interesting example which shows that there is a Lipschitz version of the James theorem, but in this case it characterizes finite-dimensionality instead of reflexivity.

Example 3.3. For every infinite-dimensional Banach space X , there is a continuous seminorm $p \in \text{Sem}(X)$ which does not attain its norm. As a consequence, if $\text{DA}(X) = \text{Lip}_0(X)$ for a Banach space X , then X is finite-dimensional.

The example is based on [13, Lemma 2.2], but we include the details for the sake of completeness.

Proof. Since X is infinite-dimensional, there is a sequence $\{x_n^*\}$ in S_{X^*} which is weak-star convergent to zero (this is the Josefson–Nissenzweig theorem). Now, consider the seminorm $p : X \rightarrow \mathbb{R}$ given by

$$p(x) = \max \left\{ \frac{n|x_n^*(x)|}{n+1} : n \in \mathbb{N} \right\} \quad (x \in X)$$

and observe that $\|p\| = 1$ but $p(x) < 1$ for every $x \in S_X$. \square

We are ready to prove the uniform density of the set of norm-attaining seminorms. Actually, we may prove more.

Proposition 3.4 (Bishop–Phelps–Bollobás theorem for seminorms in the uniform norm). *Let X be a Banach space. Then for every $\varepsilon > 0$ there is such a $\delta > 0$ that, for every $p_0 \in \text{Sem}(X)$ with $\|p_0\| = 1$ and every $x_0 \in S_X$ with $p_0(x_0) > 1 - \delta$, there exist $p \in \text{Sem}(X)$ with $\|p\| = 1$ and $x \in S_X$ such that*

$$\begin{aligned} p(x) &= 1 = \|p\|, & \|x - x_0\| &< \varepsilon & \text{ and} \\ \|p - p_0\|_\infty &= \sup_{x \in S_X} |p(x) - p_0(x)| &< \varepsilon. \end{aligned}$$

Proof. Let Γ be a set whose cardinality equals the density character of X . It is shown in [1, Theorem 2.2] that for every $\varepsilon > 0$ there is $\delta > 0$ such that, whenever $T_0 \in L(X, \ell_\infty(\Gamma, \mathbb{R}))$ with $\|T_0\| = 1$ and $x_0 \in S_X$ satisfy that $\|T_0(x_0)\| > 1 - \delta$, there exist $T \in L(X, \ell_\infty(\Gamma, \mathbb{R}))$ with $\|T\| = 1$ and $x \in S_X$ such that

$$\|Tx\| = 1 = \|T\|, \quad \|x - x_0\| < \varepsilon, \quad \|T - T_0\| < \varepsilon. \quad (3.1)$$

Now, for a given $\varepsilon > 0$, consider such a $\delta > 0$. By Remark 3.1, there is $T_0 \in L(X, \ell_\infty(\Gamma, \mathbb{R}))$ such that $p_0(x) = \|T_0(x)\|$ for all $x \in X$. As $p_0(x_0) > 1 - \delta$, we have $\|T_0(x_0)\| > 1 - \delta$ and we may find $T \in L(X, \ell_\infty(\Gamma, \mathbb{R}))$ with $\|T\| = 1$ and $x \in S_X$ satisfying (3.1). Consider $p \in \text{Sem}(X)$ be given by $p(z) = \|T(z)\|$ for every $z \in X$. Then, $\|p\| = 1 = p(x)$, $\|x - x_0\| < \varepsilon$, and

$$|p(z) - p_0(z)| = \left| \|T(z)\| - \|T_0(z)\| \right| \leq \|T(z) - T_0(z)\| \leq \|T - T_0\| < \varepsilon$$

for every $z \in S_X$, so $\|p - p_0\|_\infty < \varepsilon$, finishing the proof. \square

One may wonder whether the above result is also true replacing $\|p - p_0\|_\infty$ by $\|p - p_0\|_{\text{Lip}}$. In general, $\|p - q\|_\infty \leq \|p - q\|_{\text{Lip}}$ for all $p, q \in \text{Sem}(X)$, so the uniform convergence of seminorms is weaker than the Lipschitz convergence, and the next example shows that it is indeed strictly weaker.

Remark 3.5. The uniform convergence of seminorms does not force the Lipschitz convergence, even in the finite-dimensional case. Indeed, let X be the 2-dimensional space \mathbb{R}^2 endowed with the maximum norm, let $p_0 \in \text{Sem}(X)$ be defined by $p_0(x_1, x_2) = |x_1|$ for every $(x_1, x_2) \in X$, and for every $n \in \mathbb{N}$ let

$p_n \in \text{Sem}(X)$ be defined by $p_n(x_1, x_2) = \max\{|x_1|, \frac{1}{n}|x_2|\}$ for every $(x_1, x_2) \in X$. On the one hand, $\|p_n - p_0\|_\infty \leq 1/n$, so $\{p_n\}$ is uniformly convergent to p_0 . On the other hand,

$$\|p_n - p_0\|_{\text{Lip}} \geq \frac{|(p_n(0, n) - p_0(0, n)) - (p_n(1, n) - p_0(1, n))|}{\|(0, n) - (1, n)\|} = 1.$$

Our last result in this section shows that the Radon–Nikodým property is sufficient to assure that norm-attaining seminorms are dense in the Lipschitz sense.

Proposition 3.6 (Bishop–Phelps theorem for seminorms in the Lipschitz norm for RNP spaces). *Let X be a Banach space with the Radon–Nikodým property. Then, the set of norm-attaining seminorms is dense in $(\text{Sem}(X), \|\cdot\|_{\text{Lip}})$.*

Proof. This is an easy consequence of the following (not quite easy) Stegall’s result from [14, Theorem, p. 174], from which we are citing only the part which we need: let D be an RNP set and let $f : D \rightarrow \mathbb{R}$ be upper semicontinuous and bounded above. Then, for every $\varepsilon > 0$, there exists $x^* \in X^*$ with $\|x^*\| < \varepsilon$ such that $f + |x^*|$ attains its supremum on D . Now, let us apply this theorem to $D = B_X$ and $f = p \in \text{Sem}(X)$. Then, the corresponding $q := p + |x^*|$ is again a continuous seminorm, and q is norm-attaining. Finally,

$$\|p - q\|_{\text{Lip}} = \||x^*\||_{\text{Lip}} = \||x^*\||_\infty = \|x^*\| < \varepsilon. \quad \square$$

4. TWO PRELIMINARY RESULTS

In this section, we demonstrate two preliminary results on the way to Theorem 5.3. The first one is a weak version of the Bishop–Phelps–Bollobás theorem for Lipschitz functionals, valid for all Banach spaces, which can be of independent interest.

Lemma 4.1 (Preliminary LipBPB Theorem). *Let X be a Banach space, let $f \in \text{Lip}_0(X)$, let $\|f\| = 1$, $\delta \in (0, 2)$, and let $x, y \in X, x \neq y$ be such elements that*

$$\frac{f(x) - f(y)}{\|x - y\|} > 1 - \delta. \tag{4.1}$$

Then for every $h \in \text{Lip}_0(X)$ with $\|h\| = 1$, such that $\frac{h(x)-h(y)}{\|x-y\|} = 1$ there exists $g \in \text{Lip}_0(X)$ with $\|g\| = 1$, $\|f - g\| < \sqrt{2\delta}$, and there exists a sequence of pairs $\{(v_n, w_n)\}$ in $X \times X$ with $v_n \neq w_n$ for every n such that

$$\frac{h(v_n) - h(w_n)}{\|v_n - w_n\|} > 1 - \sqrt{2\delta} \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{g(v_n) - g(w_n)}{\|v_n - w_n\|} = 1.$$

The proof of this result is based on the *Lipschitz-free space technique*, so let us first recall the relevant definitions and basic facts. For every $x \in X$, we denote \hat{x} the corresponding evaluation functional on $\text{Lip}_0(X)$, that is, $\hat{x}(f) = f(x)$. Then \hat{x} is an element of $\text{Lip}_0(X)^*$. The subspace $\overline{\text{Lin}}\{\hat{x} : x \in X\}$ of $\text{Lip}_0(X)^*$ is denoted by $\mathcal{F}(X)$. The most common name for $\mathcal{F}(X)$ is the *Lipschitz-free space* of X . This object was studied under various names by several authors (see [3], [12], [10]), and is known to be useful for Lipschitz maps study.

The elements of $\text{Lip}_0(X)$ are continuous linear functionals on $\mathcal{F}(X)$; moreover, $\mathcal{F}(X)^* = \text{Lip}_0(X)$ as Banach spaces. The map $x \mapsto \hat{x}$ is a nonlinear isometric embedding of X into $\mathcal{F}(X)$ since $\|\hat{x} - \hat{y}\|_{\mathcal{F}(X)} = \|x - y\|_X$ for all $x, y \in X$.

The action of $f \in \text{Lip}_0(X)$ on $w \in \mathcal{F}(X)$ is denoted by $\langle f, w \rangle$. With this notation, $\langle f, \hat{x} \rangle = f(x)$ and the formula (1.2) can be re-written as follows:

$$\|f\| = \sup \left\{ \left| \left\langle f, \frac{\hat{x} - \hat{y}}{\|x - y\|} \right\rangle \right| : x, y \in X, x \neq y \right\}. \quad (4.2)$$

Denote $W = \left\{ \frac{\hat{x} - \hat{y}}{\|x - y\|} : x, y \in X, x \neq y \right\}$ and observe that W is a symmetric subset of $S_{\mathcal{F}(X)}$. The Hahn–Banach theorem, together with formula (4.2), gives us

$$B_{\mathcal{F}(X)} = \overline{\text{conv}} W. \quad (4.3)$$

We are now ready to prove our result.

Proof of Lemma 4.1. Consider $w = \frac{\hat{x} - \hat{y}}{\|x - y\|} \in S_{\mathcal{F}(X)}$. The condition (4.1) gives us $\langle f, w \rangle > 1 - \delta$. Since $f \in \text{Lip}_0(X) = \mathcal{F}(X)^*$, the Bishop–Phelps–Bollobás theorem (Theorem 1.1) is applicable. So, there are $g \in \text{Lip}_0(X)$ with $\|g\| = 1$ and $z \in \mathcal{F}(X)$ with $\|z\| = 1$ such that $\|w - z\| < \sqrt{2\delta}$, $\|f - g\| < \sqrt{2\delta}$, and $\langle g, z \rangle = 1$. Let $\nu > 0$ be such a number that $\|w - z\| < \nu < \sqrt{2\delta}$. Fix a sequence $\{\delta_n\}$ of positive numbers converging to zero. The formula (4.3) implies that we can select a sequence $\{z_n\}$ in $\text{conv } W$ converging to z in such a way that

$$\|w - z_n\| < \nu \quad \text{and} \quad \langle g, z_n \rangle > 1 - \delta_n. \quad (4.4)$$

The condition on h means $\langle h, w \rangle = 1$, and consequently

$$\langle h, z_n \rangle \geq \langle h, w \rangle - \|w - z_n\| > 1 - \nu. \quad (4.5)$$

Choose a sequence $\{\alpha_n\}$ in $(0, 1)$ satisfying the conditions

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} = 0. \quad (4.6)$$

Fix $n \in \mathbb{N}$. Combining inequalities (4.4) and (4.5), we obtain

$$\alpha_n \langle h, z_n \rangle + (1 - \alpha_n) \langle g, z_n \rangle > \alpha_n(1 - \nu) + (1 - \alpha_n)(1 - \delta_n) = 1 - \alpha_n \nu - (1 - \alpha_n) \delta_n.$$

Since $z_n \in \text{conv } W$, the last inequality implies that there exists $u_n \in W$ such that

$$\alpha_n \langle h, u_n \rangle + (1 - \alpha_n) \langle g, u_n \rangle > 1 - \alpha_n \nu - (1 - \alpha_n) \delta_n.$$

Combining this fact with the evident estimates $\langle h, u_n \rangle \leq 1$ and $\langle g, u_n \rangle \leq 1$, we deduce that

$$\langle g, u_n \rangle > 1 - \delta_n - \frac{\alpha_n}{1 - \alpha_n} \sqrt{2\delta}$$

and

$$\langle h, u_n \rangle > 1 - \nu - \delta_n \frac{1 - \alpha_n}{\alpha_n}.$$

These inequalities, together with (4.6), imply that

$$\langle g, u_n \rangle \longrightarrow 1$$

and that

$$\langle h, u_n \rangle > 1 - \sqrt{2\delta}$$

for $n \in \mathbb{N}$ large enough. In order to complete the proof, it remains to recall that every $u_n \in W$ is of the form $\frac{\hat{v}_n - \hat{w}_n}{\|v_n - w_n\|}$ for some $v_n, w_n \in X$ with $v_n \neq w_n$. \square

Before stating the second result, we need a couple of definitions.

Definition 4.2. A functional $g \in \text{Lip}_0(X)$ attains its norm at a point $v \in X$ in the direction $u \in S_X$ if there is a sequence of pairs $\{(x_n, y_n)\}$ in $X \times X$, with $x_n \neq y_n$, such that

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} y_n = v, \\ \lim_{n \rightarrow \infty} \frac{x_n - y_n}{\|x_n - y_n\|} &= u \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} = \|g\|. \end{aligned}$$

In this case, we say that g attains its norm locally/directionally. The set of all those $f \in \text{Lip}_0(X)$ that attain their norm locally/directionally is denoted by $\text{LDA}(X)$.

Definition 4.3. A Banach space X has the local directional Bishop–Phelps–Bollobás property for Lipschitz functionals (“ $X \in \text{LLipBPB}$ ” for short), if for every $\varepsilon > 0$ there is such a $\delta > 0$ that, for every $f \in \text{Lip}_0(X)$ with $\|f\| = 1$ and for every $x, y \in X$ with $x \neq y$ satisfying $\frac{f(x) - f(y)}{\|x - y\|} > 1 - \delta$, there is $g \in \text{Lip}_0(X)$ with $\|g\| = 1$ and there are $v \in X$, $u \in S_X$ such that g attains its norm in the point v at the direction u , $\|g - f\| < \varepsilon$, $\|\frac{x - y}{\|x - y\|} - u\| < \varepsilon$, and $\text{dist}(v, \text{conv}\{x, y\}) < \varepsilon$.

The second preliminary result of this section, which also can be of independent interest, is a relaxation of the requirements for a Banach space to have the LLipBPB .

Lemma 4.4. *Let X be a Banach space. Suppose that for every $\varepsilon > 0$ there is such a $\delta > 0$ that, for every $f \in \text{Lip}_0(X)$ with $\|f\| = 1$ and for every pair $(x, y) \in X \times X$, $x \neq y$, with $\frac{f(x) - f(y)}{\|x - y\|} > 1 - \delta$, there is $g \in \text{Lip}_0(X)$ with $\|g\| = 1$ and a sequence of pairs $\{(v_n, w_n)\}$ in $X \times X$ with $v_n \neq w_n$ for every n such that*

$$\lim_{n \rightarrow \infty} \frac{g(v_n) - g(w_n)}{\|v_n - w_n\|} = 1, \tag{4.7}$$

$\|g - f\| < \varepsilon$, $\|\frac{x - y}{\|x - y\|} - \frac{v_n - w_n}{\|v_n - w_n\|}\| < \varepsilon$, $\|v_n - w_n\| < \varepsilon$, and $\text{dist}(v_n, \text{conv}\{x, y\}) < \varepsilon$. Then $X \in \text{LLipBPB}$.

Observe that the difference between the requirements of the lemma above and the local directional Bishop–Phelps–Bollobás property is that here the convergence of the sequences $\{v_n\}$, $\{w_n\}$, and $\{\frac{v_n - w_n}{\|v_n - w_n\|}\}$ is not required since these sequences depend upon ε (but they are still well controlled).

Proof. For a fixed $\varepsilon > 0$, let us select a decreasing sequence $\{\varepsilon_n\}$ of positive number such that $\sum_{n=1}^\infty \varepsilon_n < \varepsilon/4$. For every $n \in \mathbb{N}$, let $\delta_n = \delta(\varepsilon_n)$ be from the assumptions of the lemma for ε_n . We will demonstrate that $\delta = \delta_1$ satisfies conditions of Definition 4.3 for ε .

To this end, let us fix $f \in \text{Lip}_0(X)$ with $\|f\| = 1$ and a pair $(x, y) \in X \times X$, $x \neq y$, such that $\frac{f(x)-f(y)}{\|x-y\|} > 1 - \delta$. Applying the hypotheses to $\varepsilon_1 = \varepsilon$, $\delta_1 = \delta(\varepsilon_1)$, the norm 1 Lipschitz functional $f_1 = f$, and the pair $(x_1, y_1) = (x, y)$ in $X \times X$ which satisfy $\frac{f_1(x_1)-f_1(y_1)}{\|x_1-y_1\|} > 1 - \delta_1$, we get the corresponding g_1 and the sequence of pairs $\{(v_n, w_n)\}$. Thanks to (4.7), we can find such an $n_1 \in \mathbb{N}$ that

$$\frac{g_1(v_{n_1}) - g_1(w_{n_1})}{\|v_{n_1} - w_{n_1}\|} > 1 - \delta_2.$$

Let us denote $f_2 = g_1$, $x_2 = v_{n_1}$, and $y_2 = w_{n_1}$. Then, $\|f_1 - f_2\| < \varepsilon_1$, $\left\| \frac{x_1 - y_1}{\|x_1 - y_1\|} - \frac{x_2 - y_2}{\|x_2 - y_2\|} \right\| < \varepsilon_1$, $\text{dist}(x_2, \text{conv}\{x_1, y_1\}) < \varepsilon_1$, $\|x_2 - y_2\| < \varepsilon_1$, and

$$\frac{f_2(x_2) - f_2(y_2)}{\|x_2 - y_2\|} > 1 - \delta_2.$$

The last condition enables us to apply again the hypotheses of the lemma to ε_2 , δ_2 , f_2 , and (x_2, y_2) in order to get the corresponding f_3 and (x_3, y_3) . Repeating this process, we obtain sequences $\{f_n\}$ in $\text{Lip}_0(X)$ with $\|f_n\| = 1$ and $\{(x_n, y_n)\}$ in $X \times X$ with $x_n \neq y_n$ having the following properties:

- (a) $\|f_n - f_{n+1}\| < \varepsilon_n$,
- (b) $\left\| \frac{x_n - y_n}{\|x_n - y_n\|} - \frac{x_{n+1} - y_{n+1}}{\|x_{n+1} - y_{n+1}\|} \right\| < \varepsilon_n$,
- (c) $\frac{f_n(x_n) - f_n(y_n)}{\|x_n - y_n\|} > 1 - \delta_n$,
- (d) $\text{dist}(x_{n+1}, \text{conv}\{x_n, y_n\}) < \varepsilon_n$,
- (e) $\|x_n - y_n\| < \varepsilon_{n-1}$.

Conditions (d) and (e) imply that $\|x_{n+1} - x_n\| < 2\varepsilon_{n-1}$, $n = 2, 3, \dots$. Consequently, the sequence $\{x_n\}$ has some limit $v \in X$, and by (e) the sequence $\{y_n\}$ has the same limit v . For this v we have

$$\begin{aligned} \text{dist}(v, \text{conv}\{x, y\}) &\leq \text{dist}(x_2, \text{conv}\{x_1, y_1\}) + \|x_2 - v\| \\ &< \varepsilon_1 + \sum_{n=2}^{\infty} \|x_n - x_{n+1}\| < \varepsilon. \end{aligned}$$

The condition (a) implies that the sequence $\{f_n\}$ has some limit $g \in \text{Lip}_0(X)$ with $\|g\| = 1$, and the condition (b) implies that the sequence $\left\{ \frac{x_n - y_n}{\|x_n - y_n\|} \right\}$ has some limit $u \in S_X$. Moreover,

$$\|f - g\| < \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon \quad \text{and} \quad \left\| \frac{x_1 - y_1}{\|x_1 - y_1\|} - u \right\| < \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon.$$

Also,

$$\frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} \geq 1 - \delta_n - \|g - f_n\|,$$

so $\lim_{n \rightarrow \infty} \frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} = 1$, which proves that g attains its norm at v in the direction u . □

5. BISHOP–PHELPS–BOLLOBÁS THEOREM FOR UNIFORMLY CONVEX SPACES

Let us recall the well-known concept of uniform convexity.

Definition 5.1. A Banach space X is said to be *uniformly convex* if for every $\varepsilon > 0$ there is such a $\delta > 0$ that for every pair $x, y \in B_X$ the condition $\|x - y\| \geq \varepsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. (Equivalently, $\|(x + y)/2\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon$.) The best possible value of δ is denoted $\delta_X(\varepsilon)$.

The unit ball of a uniformly convex space has many small slices. Recall, that if X is a Banach space, for any given $x^* \in S_{X^*}$ and $\delta > 0$ the corresponding *slice* of the unit ball is defined as $S(B_X, x^*, \delta) := \{x \in B_X : x^*(x) > 1 - \delta\}$. The following easy result states a “uniform way” to find small slices on a uniformly convex space.

Lemma 5.2 ([2, Lemma 2.1]). *Let X be a uniformly convex space and let $\varepsilon > 0$. Then*

$$\text{diam } S(B_X, f, \delta_X(\varepsilon)) < \varepsilon$$

for every $f \in S_{X^*}$ and every $\varepsilon > 0$.

We may now state and prove the main result of the paper.

Theorem 5.3. *Every uniformly convex Banach space X has the local directional Bishop–Phelps–Bollobás property for Lipschitz functionals.*

This result implies, in particular, that for a uniformly convex Banach space X , the set of those Lipschitz functions which attain their norm (locally/directionally) is dense in the space $\text{Lip}_0(X)$ (with the Lipschitz norm). We do not know whether this density holds in every Banach space X .

Proof of Theorem 5.3. For a fixed $\varepsilon \in (0, 1/2)$, let us chose such a $\delta \in (0, \varepsilon^2/2)$ that $\sqrt{2\delta} < \frac{1}{2}\delta_X(\varepsilon)$. Let $f \in \text{Lip}_0(X)$, with $\|f\| = 1$ and $x, y \in X$, $x \neq y$, such that $\frac{f(x)-f(y)}{\|x-y\|} > 1 - \delta$. Select $\tilde{x}, \tilde{y} \in \text{conv}\{x, y\}$ in such a way that $\|\tilde{x} - \tilde{y}\| < \frac{1}{4} \min\{\varepsilon, \|\tilde{x}\|, \|\tilde{y}\|\}$, the vector $\tilde{x}\tilde{y}$ looks at the same direction that xy (i.e., $\frac{\tilde{x}-\tilde{y}}{\|\tilde{x}-\tilde{y}\|} = \frac{x-y}{\|x-y\|}$), and such that still $\frac{f(\tilde{x})-f(\tilde{y})}{\|\tilde{x}-\tilde{y}\|} > 1 - \delta$.

Define $F \in \text{Lip}_0(X)$ by the formula $F(z) = \max\{\|\tilde{x} - \tilde{y}\| - \|\tilde{x} - z\|, 0\}$. Then $\|F\| = 1$ and $\frac{F(\tilde{x})-F(\tilde{y})}{\|\tilde{x}-\tilde{y}\|} = 1$. Let us denote $x^* \in S_{X^*}$ the supporting functional at the point $\frac{\tilde{x}-\tilde{y}}{\|\tilde{x}-\tilde{y}\|}$. Then, by linearity, $\frac{x^*(\tilde{x})-x^*(\tilde{y})}{\|\tilde{x}-\tilde{y}\|} = 1$, so we can apply the preliminary LipBPB Theorem (Lemma 4.1) with f , (x, y) , and $h = \frac{1}{2}(F + x^*) \in \text{Lip}_0(X)$. According to it there exist $g \in \text{Lip}_0(X)$ with $\|g\| = 1$, $\|f - g\| < \sqrt{2\delta} < \varepsilon$, and a sequence of pairs $\{(v_n, w_n)\}$ in $X \times X$ with $v_n \neq w_n$ such that

$$h\left(\frac{v_n - w_n}{\|v_n - w_n\|}\right) = \frac{1}{2}\left(\frac{F(v_n) - F(w_n)}{\|v_n - w_n\|} + \frac{x^*(v_n) - x^*(w_n)}{\|v_n - w_n\|}\right) > 1 - \sqrt{2\delta} \quad (5.1)$$

for all $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \frac{g(v_n) - g(w_n)}{\|v_n - w_n\|} = 1.$$

The inequality (5.1) and the fact that $\|x^*\| = \|F\| = 1$ imply that

$$\frac{F(v_n) - F(w_n)}{\|v_n - w_n\|} > 1 - 2\sqrt{2\delta} > 1 - 2\varepsilon \quad (5.2)$$

and that

$$\frac{x^*(v_n) - x^*(w_n)}{\|v_n - w_n\|} > 1 - 2\sqrt{2\delta} > 1 - \delta_X(\varepsilon).$$

The last condition means geometrically that $\frac{v_n - w_n}{\|v_n - w_n\|} \in S(B_X, x^*, \delta_X(\varepsilon))$. Since also $\frac{x - y}{\|x - y\|} = \frac{\tilde{x} - \tilde{y}}{\|\tilde{x} - \tilde{y}\|} \in S(B_X, x^*, \delta_X(\varepsilon))$, we get from Lemma 5.2 that

$$\left\| \frac{x - y}{\|x - y\|} - \frac{v_n - w_n}{\|v_n - w_n\|} \right\| < \varepsilon$$

for every $n \in \mathbb{N}$. The function F takes only nonnegative values, and the condition (5.2) implies that $F(v_n) - F(w_n) > 0$, so $v_n \in \text{supp } F$. Since the maximal possible value of F is $\|\tilde{x} - \tilde{y}\| < \varepsilon/4$, the condition (5.2) implies also that

$$\|v_n - w_n\| < \frac{F(v_n) - F(w_n)}{1 - 2\varepsilon} < \frac{\varepsilon}{4(1 - 2\varepsilon)} < \varepsilon.$$

As $v_n \in \text{supp } F$, we have $\|v_n - \tilde{x}\| < \|\tilde{x} - \tilde{y}\| < \varepsilon$, so $\text{dist}(v_n, \text{conv}\{x, y\}) < \varepsilon$.

We have verified all the conditions of Lemma 4.4. The application of that Lemma shows that X has the local directional Bishop–Phelps–Bollobás property for Lipschitz functionals. \square

Acknowledgments. Kadets’s work was partially supported by Junta de Andalucía and FEDER grant FQM-185, by MINECO grant MTM2014-57838-C2-1-P, and by Fundación Séneca/Región de Murcia grant 19368/PI/14. Martín’s work was partially supported by MINECO and FEDER project no. MTM2012-31755 and by Junta de Andalucía and FEDER grant FQM-185. Soloviova’s work was partially supported by a grant from Akhiezer’s Fund, 2015.

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