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SZEGÖ-TYPE DECOMPOSITIONS FOR ISOMETRIES

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Abstract. The notion of Szegö-type properties of positive Borel measures is well known and widely exploited. In this paper, we consider a class of orthogonal decompositions of isometries on Hilbert spaces which correspond to Szegö-type properties of their elementary measures. Our decompositions are closely connected with some special families of invariant subspaces. It is shown that this connection holds for the decomposition constructed in the paper. We illustrate our results with several examples. We also give a short proof of Mlak's theorem on the elementary measures of completely nonunitary contractions.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For a given isometry $V \in \mathcal{B}(\mathcal{H})$, denote by $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$ its Wold decomposition and by E the spectral measure of its minimal unitary extension. For every $x \in \mathcal{H}$ the mapping $\mu_x : \mathbb{B}(\mathbb{T}) \ni \omega \mapsto \langle E(\omega)x, x\rangle$ is a positive Borel measure, where $\mathbb{B}(\mathbb{T})$ denotes the σ-algebra of all Borel subsets of the unit circle $\mathbb{T}$. The measure μ_x is called the *elementary measure* of x (and V).

Recall that a unitary operator $U \in \mathcal{B}(\mathcal{K})$ is called a *unitary dilation* of a contraction $T \in \mathcal{B}(\mathcal{H})$ if

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}} \quad \text{for } n \in \mathbb{N},$$

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where $P|_{\mathcal{H}}$ is the orthogonal projection onto $\mathcal{H} \subset \mathcal{K}$. Each contraction has a minimal unitary dilation (see [13]). We can extend the definition of elementary measures to contractions. Indeed, if E is a spectral measure for U, then the elementary measure of $x \in \mathcal{H}$ is the positive Borel measure $\mu_x : \mathbb{B}(\mathbb{T}) \ni \omega \mapsto \langle E(\omega)x, x\rangle$.

Let μ be a nonnegative regular Borel measure on $\mathbb{T}$. We say that μ is a *Szegö measure* if for any $\omega \in \mathbb{B}(\mathbb{T})$ the inclusion $\chi_\omega L^2(\mu) \subset H^2(\mu)$ implies $\mu(\omega) = 0$, where $H^2(\mu)$ denotes the closure in $L^2(\mu)$ of the algebra of all analytic polynomials, and χ_ω denotes the characteristic function of the set ω.

We have the following (see [5], [6]).

Proposition 1.1. *A measure μ on $\mathbb{T}$ is a Szegö measure if and only if*

(1) *μ is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{T}$,*
(2) *$\log \frac{d\mu}{dm}$ is Lebesgue summable.*

We say that μ is *Szegö-singular* if $H^2(\mu) = L^2(\mu)$. Denote by $\mathcal{A}$ the algebra of all analytic polynomials and by $\mathcal{A}_0$ the subalgebra of those members of $\mathcal{A}$ which vanish at 0. Observe the following.

Remark 1.2. μ is *Szegö-singular* if and only if $\inf_{p\in\mathcal{A}_0} \int |1-p|^2\, d\mu = 0$.

By the Szegö theorem (see [6, p. 49]), for μ absolutely continuous with respect to m, we have the formula

$$\inf_{p\in\mathcal{A}_0} \int |1-p|^2\, d\mu = \exp\Big(\int \log \frac{d\mu}{dm}\, dm\Big). \tag{1.1}$$

Each Borel regular measure μ on $\mathbb{T}$ has a unique decomposition

$$\mu = \chi_\omega \mu + \chi_{\mathbb{T}\setminus\omega}\mu,$$

where ω is a μ-essentially unique Borel set, $\chi_\omega\mu$ is a Szegö measure, and $\chi_{\mathbb{T}\setminus\omega}\mu$ is Szegö-singular. The above decomposition is a special case of a more general result shown for natural representations in [5].

Recall that a *decomposition* of an operator $T \in \mathcal{B}(\mathcal{H})$ means $T = T_1 \oplus T_2$ where $T_1 \in \mathcal{B}(\mathcal{H}_1), T_2 \in \mathcal{B}(\mathcal{H}_2)$, and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. This implies that $T_i = T|_{\mathcal{H}_i}$ and $\mathcal{H}_i$ is reducing for T (invariant for T and T^*) for $i = 1, 2$. In known examples of decomposition like Wold decomposition or Lebesgue decomposition, one component has some additional property (e.g., is unitary or its spectral measure is absolutely continuous with respect to the Lebesgue measure) while the second component completely fails to have that property. The aforementioned decompositions are unique because the considered properties are hereditary (which means that the property is inherited by the restriction of the operator to any reducing subspace). In the case considered in this paper, the relevant property of an isometry $V \in \mathcal{B}(\mathcal{H})$ is that $\mathcal{H}$ is spanned by vectors whose elementary measures are Szegö. This property is not hereditary (see Example 2.5). Therefore, we do not obtain a unique decomposition with respect to this property but a family of such decompositions. We call them *Szegö-type decompositions* (see Definition 2.7).

We study two natural Szegö-type decompositions. The first was introduced in [4] and used there in the context of two commuting isometries. The second decomposition is based on unilateral and bilateral shift parts of a given isometry. Both decompositions are based on so-called wandering vectors.

Definition 1.3. Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. A vector $w \in \mathcal{H}$ is called *wandering* for V if

$$\langle V^n w, w \rangle = 0$$

for all $n \in \mathbb{N}_+$.

In Section 3, we give a characterization of the aforementioned decompositions using intersections of some special families of invariant subspaces. In Section 4, closely connected with the results of [4], we give a comparison of our decompositions and the Lebesgue decomposition. In Section 5, we give a simpler proof of Mlak's theorem of [10]. In Section 6, we show connections between the subject of our paper and the invariant subspace problem. We conclude with Problem 7.4, which is important for the construction in Section 7 (see Example 7.1).

2. Szegö-type decompositions for isometries

In this section, we introduce decompositions of isometries which are connected with Szegö measures. First we show connections between wandering vectors and Szegö measures.

Proposition 2.1. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. If $0 \neq w \in \mathcal{H}$ is a wandering vector for V, then its elementary measure μ_w is a Szegö measure.*

Proof. Isometry V restricted to the smallest invariant subspace containing w is a unilateral shift with one-dimensional wandering subspace generated by w. Hence μ_w is equal to the Lebesgue measure, and consequently is a Szegö measure. □

Theorem 2.2. *Let S be a unilateral shift on a Hilbert space $\mathcal{H}$. Then all elementary measures of S and S^* are Szegö.*

Proof. For $x \in \mathcal{H}$ denote by μ_x (resp., ν_x) the elementary measure of S (resp., of S^*) corresponding to x. It is well known that this measure is absolutely continuous. First let us assume that μ_x is Szegö-singular for some $x \in \mathcal{H}$. Then, by Remark 1.2,

$$\begin{aligned} 0 &= \inf_{p \in \mathcal{A}_0} \int |1-p|^2 \, d\mu_x = \inf_{p \in \mathcal{A}_0} \int |\bar{z}|^2 |1-p|^2 \, d\mu_x \\ &= \inf_{p \in \mathcal{A}} \int |\bar{z} - p|^2 \, d\mu_x = \inf_{p \in \mathcal{A}} \left\| \widehat{S}^* x - p(S)x \right\|^2, \end{aligned}$$

where $\widehat{S}$ is the bilateral shift extending S. By the above equality, the minimal S invariant subspace containing x reduces S to a unitary operator, which leads to a contradiction. As a consequence, there is no Szegö-singular elementary measure.

The measure μ_x, as an elementary measure of a unilateral shift, is absolutely continuous with respect to the Lebesgue measure. Therefore, by (1.1), Proposition 1.1, and Remark 1.2, μ_x is a Szëgo measure.

By elementary calculation (see [9, proof of Lemma 2.1]), we can show that ν_x is Szegö if and only if μ_x is Szegö. □

Corollary 2.3. *Let $\{\mu_x : x \in \mathcal{H}\}$ be the set of elementary measures of a bilateral shift U on a Hilbert space $\mathcal{H}$. Then the set $\{x \in \mathcal{H} : \mu_x$ is Szegö$\}$ is dense in $\mathcal{H}$.*

Proof. We have $\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} U^n L$, where L is a wandering subspace for U. The operator U restricted to every subspace $\bigoplus_{n=k}^{\infty} U^n L$ ($k \in \mathbb{Z}$) is a unilateral shift. Hence, by Theorem 2.2, every subspace $\bigoplus_{n=k}^{\infty} U^n L$ consists of vectors with Szegö elementary measures. Consequently, the set $\bigcup_{k=0}^{-\infty}(\bigoplus_{n=k}^{\infty} U^n L)$ which is dense in $\mathcal{H}$ has the same property. □

On the other hand, we have a nice characterization of spaces which consist of vectors having Szegö-singular elementary measures.

Proposition 2.4. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry.*

- *Let $x \in \mathcal{H}$. Then $x \in \bigvee\{V^n x : n \geq 1\}$ if and only if elementary measure μ_x is Szegö-singular.*
- *Let $L \subset \mathcal{H}$ be a V-reducing subspace. Then each V-invariant subspace of L is V-reducing if and only if elementary measures for all vectors in L are Szegö-singular.*

Proof. The first assertion is a consequence of the equality $\inf_{p\in A_0} \|x - p(V)x\|^2 = \inf_{p\in A_0} \int |1-p|^2\, d\mu_x$ and Remark 1.2.

For the proof of the second assertion, assume that w is an arbitrary Laurent polynomial of z. Then for a minimal unitary extension U of V and $x \in L$ we have

$$\left\|w(U)x - p(V)x\right\|^2 = \int |w-p|^2\, d\mu_x \tag{2.1}$$

for every analytic polynomial p. Since every V-invariant subspace of L is reducing, the infimum of the left-hand side in (2.1) taken over all analytic polynomials p is 0. Consequently, $w \in H^2(\mu_x)$. Since w was arbitrary, we have $H^2(\mu_x) = L^2(\mu_x)$, which means that μ_x is Szegö-singular.

Conversely, assume that for each $x \in L$ the measure μ_x is Szegö-singular. If M is a V-invariant subspace of L, then every vector $x \in M \ominus VM$ is wandering. Hence, by Proposition 2.1, we have $x = 0$, and consequently $M = VM$, which finishes the proof. □

Now, let us consider an example where the set of all vectors whose elementary measures are Szegö-singular cannot be a linear space.

Example 2.5. Let $\mathcal{H} = L^2(m)$, where m is the normalized Lebesgue measure on $\mathbb{T}$ and $S \in \mathcal{B}(\mathcal{H})$ is the operator of multiplication by z. Each vector can be approximated by a linear combination of wandering vectors (whose elementary measures are Szegö by Proposition 2.1). On the other hand, if we take a measurable set $\alpha \subset \mathbb{T}$ such that $m(\alpha) < 1$, then by Proposition 2.4 the subspace $\chi_\alpha L^2(m)$ consists of vectors whose elementary measures are Szegö-singular. Hence any vector $f \in \mathcal{H}$ is a sum of two singular elements $\chi_{\mathbb{T}_+} f, \chi_{\mathbb{T}_-} f$, where $\mathbb{T}_+ := \{z \in \mathbb{T} : \Im z \geq 0\}$ and $\mathbb{T}_- := \{z \in \mathbb{T} : \Im z < 0\}$.

We denote by $\mathcal{F}$ the set of all vectors $\in \mathcal{H}$ whose elementary measures are Szegö-singular.

Remark 2.6. The set $\mathcal{F}$ may not be a linear subspace.

In light of the above example, we introduce the following definition.

Definition 2.7. We call an isometry $V \in \mathcal{B}(\mathcal{H})$ a *Szegö isometry* if $\mathcal{H}$ is spanned by vectors whose elementary measures are Szegö. We call an isometry $V \in \mathcal{B}(\mathcal{H})$ *Szegö-singular* if the elementary measure of any vector is Szegö-singular.

We say that a decomposition $V = V_1 \oplus V_2$ is a *Szegö-type decomposition* if V_1 is Szegö-singular and V_2 is a Szegö isometry.

Proposition 2.4 implies the following characterization.

Remark 2.8. An isometry is Szegö-singular if and only if it does not contain any nontrivial wandering vector.

Note that Szegö-singular isometries are unitary operators, but not all unitary isometries are Szegö-singular. A unilateral shift is a Szegö isometry.

Now let us consider another example (see [4]).

Example 2.9. Denote $\alpha := \{z \in \mathbb{T} : \arg z \in [\frac{2}{3}\pi, \frac{4}{3}\pi]\}$. Then $\alpha \cup \alpha^2 = \mathbb{T}$. Let $\mathcal{H} = L^2(\alpha) \oplus L^2(\alpha^2) \oplus L^2(\alpha)$ and $U \in \mathcal{B}(\mathcal{H})$ be multiplication by z. Set $\mathcal{H}_1 := L^2(\alpha)$, and set $\mathcal{H}_2 := L^2(\alpha) \oplus L^2(\alpha^2)$. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is a reducing decomposition such that $\mathcal{H}_1 \subset \mathcal{F}$ and $\mathcal{H}_2$ is spanned by vectors whose elementary measures are Szegö. Unfortunately, such a decomposition is not unique.

The above example shows that, generally, we cannot define a unique Szegö-type decomposition.

Now, for a given isometry $V \in \mathcal{B}(\mathcal{H})$, we introduce two subspaces which naturally generate Szegö-type decompositions.

First, let us consider the reducing subspace $\mathcal{H}_0 := \mathcal{H} \ominus \mathcal{H}_w$, where $\mathcal{H}_w$ is the subspace spanned by all wandering vectors for V. In [4], we gave a precise description of $\mathcal{H}_w$ and $\mathcal{H}_0$. From the definition of $\mathcal{H}_w$ and Proposition 2.1 we conclude that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_w$ is a Szegö-type decomposition. Moreover, the subspace spanned by all vectors whose elementary measures are Szegö is maximal. Such a decomposition will be called a *Szegö-type I decomposition*.

Second, if we consider the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\mathcal{H}_2$ reduces V to a direct sum of unilateral and bilateral shifts, and $\mathcal{H}_1$ does not contain any wandering vector, then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ generates a Szegö-type decomposition of V. Such a decomposition is not unique. Thus we define the subspace

$$\mathcal{H}_{ns} := \bigcap \{\mathcal{H}_1 : \mathcal{H}_1^\perp \text{ reduces } V \text{ to a direct sum of unilateral and bilateral shifts}\}.$$

For the orthogonal complement we have

$$\mathcal{H}_{ns}^\perp = \bigvee \{\mathcal{H}_2 : \mathcal{H}_2 \text{ reduces } V \text{ to a direct sum of unilateral and bilateral shifts}\}.$$

Hence, by Propositions 2.1 and 2.4, the decomposition $\mathcal{H} = \mathcal{H}_{ns} \oplus (\mathcal{H}_{ns})^\perp$ is a Szegö-type decomposition of V. It will be called a *Szegö-type II decomposition*.

3. Relations between Szegö-type decompositions

In this section, we consider two families of subspaces defined for a given isometry $V \in \mathcal{B}(\mathcal{H})$:

$$\widehat{\mathcal{M}} := \Big\{ M \subset \mathcal{H} : V(M) \subset M, \bigvee_{n \geq 0} \widehat{V}^{*n}(M) = \widehat{\mathcal{H}} \Big\},$$

where $\widehat{V} \in \mathcal{B}(\widehat{\mathcal{H}})$ is the minimal unitary extension of V, and

$$\mathcal{M} := \Big\{ M \subset \mathcal{H} : V(M) \subset M, \bigvee_{n \geq 0} V^{*n}(M) = \mathcal{H} \Big\}.$$

These families generate two subspaces $\bigcap \widehat{\mathcal{M}} := \bigcap \{M : M \in \widehat{\mathcal{M}}\}$ and $\bigcap \mathcal{M} := \bigcap \{M : M \in \mathcal{M}\}$. Note that the subspaces $M_n = \mathcal{H}_u \oplus \bigoplus_{k \geq n} V^k(\ker V^*)$ belong to $\widehat{\mathcal{M}}$ and $\mathcal{M}$. On the other hand, $\bigcap_{n \geq 0} M_n = \mathcal{H}_u$. It follows that $\bigcap \widehat{\mathcal{M}}$ and $\bigcap \mathcal{M}$ are subspaces of $\mathcal{H}_u$.

Theorem 3.1. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. Then*

$$\mathcal{H}_0 = \bigcap \mathcal{M}.$$

Proof. First we show that $\bigcap \mathcal{M} \subset \mathcal{H}_0$. We may assume that $\mathcal{H}_0 \neq \mathcal{H}$. Fix a wandering vector $v \in \mathcal{H}$. Take $M := \mathcal{H} \ominus \{v, V^*v, V^{2*}v, V^{3*}v, \dots\}$. Since $\mathcal{H} \ominus M$ is invariant for V^*, the subspace M is invariant for V. Moreover, $Vv \in M$ because v is wandering. Hence $V^{*k}v = V^{*(k+1)}Vv \in V^{*(k+1)}(M)$. Thus $\bigvee_{k \geq 0} V^{*k}(M) = \mathcal{H}$. Consequently, $M \in \mathcal{M}$ and $v \perp M$. Therefore, $\bigcap \mathcal{M}$ is orthogonal to all wandering vectors for V, and consequently $\bigcap \mathcal{M} \subset \mathcal{H}_0$.

Now we show that $\mathcal{H}_0 \subset \bigcap \mathcal{M}$. Fix $M \in \mathcal{M}$. Denote by $V|_{M_u} \oplus V|_{M_s}$ the Wold decomposition of the isometry $V|_M$. Since $V|_{M_u}$ is unitary, for every $x \in M_u$ we have $\|x\| = \|(V|_M)^*x\| = \|P_M V^* x\| \leq \|V^*x\| \leq \|x\|$. Hence $P_M V^* x = V^* x$. On the other hand, $P_M V^* x = (V|_M)^* x$. Consequently, $(V|_M)^*|_{M_u} = V^*|_{M_u}$ and since M_u reduces $V|_M$, it reduces V as well. Thus $\mathcal{H} = \bigvee_{n \geq 0} V^{*n} M = M_u \oplus \bigvee_{n \geq 0} V^{*n} M_s$. The isometry $V|_{M_s}$ is a unilateral shift. As a consequence, $\bigvee_{n \geq 0} V^{*n} M_s$ is spanned by wandering vectors for V. Hence $\mathcal{H}_0 \subset M_u \subset M$. □

Theorem 3.2. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. The subspace $\bigcap \widehat{\mathcal{M}}$ is reducing for V and generates a Szegö-type decomposition.*

Moreover, $\mathcal{H}_0 \subset \bigcap \widehat{\mathcal{M}}$.

Proof. For every $M \in \widehat{\mathcal{M}}$ we have $VM \in \widehat{\mathcal{M}}$ and $VM \subset M$. Hence $\bigcap \widehat{\mathcal{M}} \subset \bigcap_{M \in \widehat{\mathcal{M}}} VM \subset \bigcap \widehat{\mathcal{M}}$. Consequently, $V(\bigcap \widehat{\mathcal{M}}) = \bigcap_{M \in \widehat{\mathcal{M}}} VM = \bigcap \widehat{\mathcal{M}}$ since V is injective. Thus $\bigcap \widehat{\mathcal{M}}$ is reducing for V.

Now we show that each subspace of $\bigcap \widehat{\mathcal{M}}$ which is invariant for V is reducing for V. Since $\bigcap \widehat{\mathcal{M}} \subset \bigcap_{n \in \mathbb{N}} V^n M$ for $M \in \widehat{\mathcal{M}}$, it follows that $\bigcap \widehat{\mathcal{M}}$ reduces V to a unitary operator. Take an invariant subspace $L \subset \bigcap \widehat{\mathcal{M}}$ and a vector $x \in L \ominus VL$. Consider the space $L_x := \bigvee_{n \in \mathbb{Z}} \{V^n x\}$, where $V^n = V^{*|n|}$ for $n < 0$. Then L_x reduces V. Since all vectors in $L \ominus VL$ are wandering, $V|_{L_x}$ is a bilateral shift.

Let $M_n := (\mathcal{H} \ominus L_x) \oplus \bigvee_{k \in \mathbb{N}} \{V^{n+k}x\} \in \widehat{\mathcal{M}}$. Then $x \in \bigcap \widehat{\mathcal{M}} \subset \bigcap_{n \in \mathbb{N}} M_n = \mathcal{H} \ominus L_x$, and so $x = 0$. Hence $L \ominus VL = \{0\}$. Finally, the subspace L is reducing for V.

Before we prove that $\bigcap \widehat{\mathcal{M}}$ generates a Szegö-type decomposition, we will show that $\mathcal{H}_0 \subset \bigcap \widehat{\mathcal{M}}$. Pick $x \in \mathcal{H}_0$ and consider the invariant subspace $L_x^+ := \bigvee_{n \in \mathbb{N}} \{V^n x\}$. We have $L_x^+ \subset \mathcal{H}_0$. Since each invariant subspace of $\mathcal{H}_0$ is reducing, L_x^+ is reducing for V. Moreover, for $M \in \widehat{\mathcal{M}}$ consider the subspace $H_M := \bigcap_{n \in \mathbb{N}} V^n M \subset M$. Since $V(H_M) = H_M$, the subspace H_M is reducing for V. Thus, denoting by $\widehat{V} \in \mathcal{B}(\widehat{\mathcal{H}})$ the minimal unitary extension of V, and applying Wold decomposition for $V|_M$, we have

$$\widehat{V}^{*k}(M) = H_M \oplus \bigoplus_{n \in \mathbb{N}} \widehat{V}^n\big(M \ominus \widehat{V}(M)\big) \oplus \bigoplus_{0 \le n \le k} \widehat{V}^{*n}\big(M \ominus \widehat{V}(M)\big)$$

for all $k \in \mathbb{N}$. Note that $L_x^+ \subset \widehat{\mathcal{H}} = \bigvee_{k \ge 0} \widehat{V}^{*k}(M)$. From [4, Theorem 3.10] we know that each vector which is orthogonal to all wandering vectors for V is also orthogonal to all wandering vectors for $\widehat{V}$. Thus the subspace $L_x^+ \subset \mathcal{H}_0$ is orthogonal to $\bigoplus_{n \in \mathbb{N}} \widehat{V}^n(M \ominus \widehat{V}(M)) \oplus \bigoplus_{0 \le n \le k} \widehat{V}^{*n}(M \ominus \widehat{V}(M))$. Hence $x \in L_x^+ \subset H_M \subset M$. But $M \in \widehat{\mathcal{M}}$ was arbitrary, and so $x \in \bigcap \widehat{\mathcal{M}}$. Finally, $\mathcal{H}_0 \subset \bigcap \widehat{\mathcal{M}}$.

As a consequence, $(\bigcap \widehat{\mathcal{M}})^\perp \subset \mathcal{H}_w$. Thus, by Propositions 2.1 and 2.4, we conclude that $\mathcal{H} = \bigcap \widehat{\mathcal{M}} \oplus (\bigcap \widehat{\mathcal{M}})^\perp$ is a Szegö-type decomposition. □

Let us describe relations between $\mathcal{H}_0$ and $\bigcap \widehat{\mathcal{M}}$ more precisely.

Theorem 3.3. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry.*

If there are wandering vectors for the unitary part $V|_{\mathcal{H}_u}$, then

$$\bigcap \widehat{\mathcal{M}} = \mathcal{H}_0.$$

If there is no nontrivial wandering vector for $V|_{\mathcal{H}_u}$ and $\dim \mathcal{N}(V^*) < \infty$, then

$$\bigcap \widehat{\mathcal{M}} = \mathcal{H}_u.$$

Proof. First assume that there exists a wandering vector $v \in \mathcal{H}_u$. Thus the subspace $M := (\mathcal{H} \ominus \{\dots, V^{*2}v, V^*v, v, Vv, V^2v, \dots\}) \oplus \bigvee \{V^n v : n \in \mathbb{N}_+\}$ belongs to $\widehat{\mathcal{M}}$ and $v \perp M$. Since $\mathcal{H}_s \perp \bigcap \widehat{\mathcal{M}}$, by [4, Theorem 3.10] we have $\mathcal{H}_w \perp \bigcap \widehat{\mathcal{M}}$. Hence $\bigcap \widehat{\mathcal{M}} \subset \mathcal{H}_0$. By Theorem 3.2 we get the first statement.

Now, assume that V does not have any wandering vector in $\mathcal{H}_u$, and the unilateral shift $V|_{\mathcal{H}_s}$ has a finite multiplicity. Choose $M \in \widehat{\mathcal{M}}$ and consider two Wold decompositions: $V = U \oplus S$ and $V|_M = U' \oplus S'$, where U, U' are unitary operators and S, S' are unilateral shifts. The subspace which reduces an isometry $V|_M$ to a unitary operator U' also reduces V. Thus $U = U' \oplus U''$ for some unitary operator U''. By the definition of $\widehat{\mathcal{M}}$ we know that the minimal unitary extension of V, denoted as $\widehat{V}$, is also the minimal unitary extension of $V|_M$. Thus $U \oplus \widehat{S} = \widehat{V} = \widehat{V|_M} = U' \oplus \widehat{S'}$, where $\widehat{S}, \widehat{S'}$ are minimal bilateral shifts which extend the unilateral shifts S and S'. Hence $U'' \oplus \widehat{S} = \widehat{S'}$. The unilateral shifts S and S' have finite multiplicities. Thus the spectral multiplicity functions of $\widehat{S}$

and $\widehat{S}'$ are constant on the unit circle. The difference of these two functions is the spectral multiplicity function of U''. Thus the spectral multiplicity function of U'' is constant on the unit circle. Hence U'' is a bilateral shift or the zero operator (see [11]). Since $\mathcal{H}_u$ does not contain any wandering vector, we cannot reduce U to a bilateral shift. Hence $U'' = 0$. As a consequence, $\mathcal{H}_u \subset M$ for each $M \in \widehat{\mathcal{M}}$. Finally, $\bigcap \widehat{\mathcal{M}} = \mathcal{H}_u$. □

Proposition 3.4. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. Then*

$$\bigcap \widehat{\mathcal{M}} \subset \mathcal{H}_{ns}.$$

Proof. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be a decomposition such that $V|_{\mathcal{H}_2}$ is a direct sum of a unilateral and a bilateral shift. Set $H_2 := (\mathcal{H}_2)_s \oplus \bigoplus_{n\geq 0} V^n(W)$, where $(\mathcal{H}_2)_s$ is the unilateral shift subspace of $V|_{\mathcal{H}_2}$ and W is a wandering subspace of the remaining bilateral shift. Then $M_n := \mathcal{H}_1 \oplus V^n(H_2)$ belongs to $\widehat{\mathcal{M}}$. Thus $\bigcap \widehat{\mathcal{M}} \subset \bigcap_{n\in\mathbb{N}} M_n = \mathcal{H}_1$. This shows that $\bigcap \widehat{\mathcal{M}} \subset \mathcal{H}_{ns}$. □

From Theorem 3.2 and Proposition 3.4 we get the following.

Corollary 3.5. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. Then*

$$\mathcal{H}_0 \subset \bigcap \widehat{\mathcal{M}} \subset \mathcal{H}_{ns}.$$

Remark 3.6. If the isometry V is unitary, then $\widehat{\mathcal{M}} = \mathcal{M}$. Moreover, Szegö-type I and II decompositions are equal; that is, $\mathcal{H}_0 = \bigcap \widehat{\mathcal{M}} = \mathcal{H}_{ns}$.

In some cases the family $\widehat{\mathcal{M}}$ defines Szegö-type II decomposition. Indeed, by Theorem 3.3 and Proposition 3.4, we get the following.

Corollary 3.7. *For any isometry V such that $\dim \mathcal{N}(V^*) < \infty$ we have*

$$\bigcap \widehat{\mathcal{M}} = \mathcal{H}_{ns}.$$

4. Szegö-type I and II decompositions via Lebesgue decomposition

Any isometry V acting on a Hilbert space $\mathcal{H}$ has Lebesgue decomposition, which combined with Wold decomposition gives us the following decomposition:

$$\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}, \tag{4.1}$$

where the subspaces $\mathcal{H}_s$, $\mathcal{H}_{ac}$, $\mathcal{H}_{sing}$ reduce V, the operator $V|_{\mathcal{H}_s}$ is a unilateral shift, the operator $V|_{\mathcal{H}_{sing}}$ is unitary singular (i.e., its spectral measure is singular to the Lebesgue measure on the unit circle), and $V|_{\mathcal{H}_{ac}}$ is unitary absolutely continuous (i.e., its spectral measure is absolutely continuous with respect to the Lebesgue measure on the unit circle).

As a direct consequence of [4, Theorem 3.10], we can compare a Szegö-type I decomposition with decomposition (4.1).

Theorem 4.1. *For any isometry $V \in \mathcal{B}(\mathcal{H})$ we have the following:*

- *if V has no wandering vectors, then $\mathcal{H}_0 = \mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$;*
- *if V has wandering vectors, then $\mathcal{H}_0 = \mathcal{H}_{sing}$, $\mathcal{H}_w = \mathcal{H}_{ac} \oplus \mathcal{H}_s$.*

Comparison of a Szegö-type II decomposition and decomposition (4.1) is as follows.

Theorem 4.2. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. We have the following:*

- *if $\mathcal{H}_u$ contains no wandering vector, then $\mathcal{H}_{ns} = \mathcal{H}_u = \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$;*
- *if $\mathcal{H}_u$ contains a wandering vector, then $\mathcal{H}_{ns} = \mathcal{H}_0 = \mathcal{H}_{sing}$.*

Proof. Assume that $\mathcal{H}_u$ does not contain any wandering vector. Consider a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\mathcal{H}_2$ reduces V to a direct sum of unilateral and bilateral shifts. Actually, $\mathcal{H}_2$ reduces V to a unilateral shift. Hence $\mathcal{H}_u \subset \mathcal{H}_{ns}$. Thus $\mathcal{H}_{ns} = \mathcal{H}_u$.

Now we consider the second case: $\mathcal{H}_u$ contains a wandering vector. For the unitary operator $U := V|_{\mathcal{H}_u}$, by Remark 3.6 we have $\bigcap \widehat{\mathcal{M}}_U = \bigcap \mathcal{M}_U$, where $\widehat{\mathcal{M}}_U$ and $\mathcal{M}_U$ denote the corresponding families of subspaces for U. If we take $M_U \in \widehat{\mathcal{M}}_U$, then $M_n := M_U \oplus V^n \mathcal{H}_s \in \widehat{\mathcal{M}}$ and $\bigcap_{n\in\mathbb{N}} M_n = M_U$. Thus, by Theorem 4.1, for U we get

$$\bigcap \widehat{\mathcal{M}} \subset \bigcap_{M_U \in \widehat{\mathcal{M}}_U} \bigcap_{n\in\mathbb{N}} M_U \oplus V^n \mathcal{H}_s = \bigcap \widehat{\mathcal{M}}_U = \bigcap \mathcal{M}_U = \mathcal{H}_{sing}.$$

On the other hand, again by Theorem 4.1 and Corollary 3.5, we have $\mathcal{H}_{sing} = \mathcal{H}_0 \subset \mathcal{H}_{ns}$. Finally, $\mathcal{H}_{ns} = \mathcal{H}_{sing}$. □

The above theorems have an immediate consequence.

Corollary 4.3. *For a nonunitary isometry V the subspaces $\mathcal{H}_{ns}$ and $\mathcal{H}_0$ are different if and only if the unitary part of V is not singular and it does not have any wandering vectors.*

The above corollary can be illustrated by the following example.

Example 4.4. Denote $\mathbb{T}_+ := \{z \in \mathbb{T} : \Im z \geq 0\}$ and μ the Lebesgue measure on $\mathbb{T}_+$. Let $\mathcal{H} = L^2(\mu) \oplus H^2(\mu)$, and denote by $V \in \mathcal{B}(\mathcal{H})$ the operator of multiplication by z. Then it is easy to see that $\mathcal{H}_{ns} = L^2(\mu)$. By Theorem 4.1 we get $\mathcal{H}_w = \mathcal{H}$ and $\mathcal{H}_0 = \{0\}$.

5. A simpler proof of Mlak's theorem

Now we give a simpler proof of Mlak's theorem of [10].

Theorem 5.1. *Let T be a completely nonunitary contraction on a Hilbert space $\mathcal{H}$. Then for each $x \in \mathcal{H}$ its elementary measure μ_x is Szegö.*

Proof. By [8] we can construct a superspace $\mathcal{K} = M \oplus N$ and a contractive extension $\widetilde{T}$ of T such that $\mathcal{H} \subset \mathcal{K}$, M and N reduce $\widetilde{T}$, $\widetilde{T}|_M$ is an isometry, and $\widetilde{T}|_N$ is a $C_{0\cdot}$ contraction. It is well known that a $C_{0\cdot}$ contraction can be extended to a backward shift (see, e.g., [2], [13]), and so we can assume that, from the start, $\mathcal{K}$ is constructed in such a way that $\widetilde{T}|_N$ is a backward shift.

Take $x \in \mathcal{H}$, and denote by y its projection on M and denote by z its projection on N. Then

$$\int p\,d\mu_x = \langle p(T)x, x\rangle$$
$$= \langle p(\widetilde{T})x, x\rangle = \langle p(\widetilde{T})y, y\rangle + \langle p(\widetilde{T})z, z\rangle = \int p\,d\mu_y + \int p\,d\mu_z,$$

where p is an arbitrary analytic polynomial and μ_x, μ_y, μ_z are the elementary measures of the vectors x, y, z respectively. Hence the measure $\mu_x - \mu_y - \mu_z$ annihilates the disk algebra. Since the disk algebra is Dirichlet, this real measure must be 0, and so $\mu_x = \mu_y + \mu_z$. Since T is completely nonunitary, μ_x is absolutely continuous with respect to the Lebesgue measure, and so is μ_z as an elementary measure of a backward shift. Hence μ_y is also absolutely continuous. We have

$$\frac{d\mu_x}{dm} = \frac{d\mu_y}{dm} + \frac{d\mu_z}{dm},$$

where m denotes the Lebesgue measure on the unit circle. Since μ_y is nonnegative, and consequently $\frac{d\mu_y}{dm} \geq 0$, by Theorem 2.2 and Proposition 1.1, we get

$$\int \log\Big(\frac{d\mu_x}{dm}\Big)\,dm = \int \log\Big(\frac{d\mu_y}{dm} + \frac{d\mu_z}{dm}\Big)\,dm \geq \int \log\Big(\frac{d\mu_z}{dm}\Big)\,dm > -\infty,$$

which means that μ_x is a Szegö measure. □

6. Connection with the invariant subspace problem

One of the motivations for considering Szegö-type decompositions is their connection with the invariant subspace problem.

Problem 6.1 (Invariant subspace problem). If $T \in \mathcal{B}(\mathcal{H})$ is a bounded linear operator, then does it have a nontrivial closed invariant subspace?

This question is interesting only for operators on infinite-dimensional separable Hilbert spaces. It is easy to see that answering this question for contractions solves the problem. Moreover, for any contraction T the subspaces $\{x \in \mathcal{H} : T^n x \to 0\}$ and $\mathcal{H} \ominus \{x \in \mathcal{H} : (T^*)^n x \to 0\}$ are closed and invariant for T. Hence Problem 6.1 is interesting only if these subspaces are trivial. A contraction T such that $\{x \in \mathcal{H} : T^n x \to 0\} = \mathcal{H}$ and $\{x \in \mathcal{H} : (T^*)^n x \to 0\} = \mathcal{H}$ (called a C_{11} *contraction*) has a nontrivial closed invariant subspace because it is quasi-similar to a unitary operator (see [13]). The roles of T and T^* are symmetric, and so the only two interesting cases are C_{00} operators (when $\{x \in \mathcal{H} : T^n x \to 0\} = \{0\}$ and $\{x \in \mathcal{H} : (T^*)^n x \to 0\} = \{0\}$) and C_{10} operators (when $\{x \in \mathcal{H} : T^n x \to 0\} = \mathcal{H}$ and $\{x \in \mathcal{H} : (T^*)^n x \to 0\} = \{0\}$).

Before we show how the idea of Szegö-type decomposition (and wandering vectors) can be used to reduce the invariant subspace problem in the case of C_{10} operators, we have to recall the idea of isometric asymptote that comes from Sz.-Nagy [12].

For a given contraction $T \in \mathcal{B}(\mathcal{H})$, $\{T^{*n}T^n\}_{n=1}^{\infty}$ is a decreasing sequence of positive operators. Thus it has a strong limit A which satisfies

$$T^*AT = A.$$

Hence $\|A^{\frac{1}{2}}Tx\| = \|A^{\frac{1}{2}}x\|$ for $x \in \mathcal{H}$. Therefore, there exists an isometry V such that

$$A^{\frac{1}{2}}T = VA^{\frac{1}{2}}.$$

That isometry is called the *isometric asymptote* of T.

Theorem 6.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a C_{10} contraction class. If the isometric asymptote of T is not Szegö-singular, then T has a nontrivial invariant subspace.*

Proof. Denote by V the isometric asymptote of T. Every C_{10} contraction is a completely nonunitary operator. Hence, by [1, Proposition XII.2.1], the singular part of V has to be zero.

Assume that V is not Szegö-singular. Then, by Proposition 2.4, T has a wandering vector. The isometry V contains a unilateral shift or it is a unitary operator. If V is a unitary operator, then any wandering vector w generates a subspace $\bigvee\{V^n : n \in \mathbb{Z}\}$ which reduces V to a bilateral shift. Hence V contains a unilateral or bilateral shift. As a consequence, $\mathbb{T} \subset \sigma(V)$. By [7, Theorem 4] we get $\sigma(V) \subset \sigma(T)$. Since every contraction whose spectrum contains the unit circle has a nontrivial closed invariant subspace (see [2], [3]), the proof is finished. □

7. Questions and final remarks

Theorem 4.1 shows that, for any nonunitary isometry such that $\mathcal{H}_{ac} \neq \{0\}$, there are wandering vectors which do not belong to $\mathcal{H}_s$.

Below we show an explicit method of constructing a wandering vector whose projections onto $\mathcal{H}_u$ and $\mathcal{H}_s$ are both nontrivial.

Example 7.1. Let $\mathbb{T}_+ := \{z \in \mathbb{C} : |z| = 1, \Im z \geq 0\}$, and denote by μ the Lebesgue measure on $\mathbb{T}_+$. Consider the space $\mathcal{H} := L^2(\mu) \oplus \bigoplus_{n=0}^{\infty} l^2$ and the isometry $V := U \oplus \bigoplus_{n=0}^{\infty} S$, where S is a unilateral shift on l^2, and $U \in \mathcal{B}(L^2(\mathbb{T}_+, \mu))$ is the unitary operator of multiplication by z.

For a fixed $k \in \mathbb{N}_+$ set $f_k(z) := 1 - \frac{1}{k}(z^2 + z^4 + \cdots + z^{2k})$. We are going to show that

$$c_n^k := \langle U^n f_k, f_k \rangle = \int_{\mathbb{T}_+} z^n \Big| 1 - \frac{1}{k}(z^2 + z^4 + \cdots + z^{2k}) \Big|^2 \, dz = \mathcal{O}\Big(\frac{1}{n^3}\Big).$$

Indeed,

$$\int_{\mathbb{T}_+} z^n \Big|1 - \frac{1}{k}(z^2 + z^4 + \cdots + z^{2k})\Big|^2 dz = \int_{\mathbb{T}_+} z^n \Big(1 + \frac{1}{k} - \sum_{j=1}^{k} \frac{j}{k^2}(z^{2j} + \overline{z}^{2j})\Big) \, dz. \quad (7.1)$$

Since

$$\int_{\mathbb{T}_+} z^n \, dz = \begin{cases} -\frac{2}{1+n} & \text{for even } n, \\ 0 & \text{for odd } n, \end{cases} \quad (7.2)$$

we see that $c_n^k = 0$ for odd n and

$$c_n^k = \frac{-2(1+\frac{1}{k})}{1+2m} + \sum_{j=1}^{k} \frac{j}{k^2}\Big(\frac{2}{1+2(m-j)} + \frac{2}{1+2(m+j)}\Big)$$

for $n = 2m$, where $m \in \mathbb{N}$. Further, we calculate that

$$c_{2m}^k = \sum_{j=1}^{k} \frac{j}{k^2}\Big(\frac{2}{1+2(m-j)} + \frac{2}{1+2(m+j)} - \frac{4}{1+2m}\Big),$$

and, finally,

$$c_{2m}^k = \sum_{j=1}^{k} \frac{j}{k^2}\frac{16j^2}{(1+2(2m-j))(1+2(2m+j))(1+4m)} = \mathcal{O}\Big(\frac{1}{m^3}\Big) = \mathcal{O}\Big(\frac{1}{n^3}\Big).$$

Thus $\sum_{n\in\mathbb{N}} |c_n^k| < \infty$. As a consequence, the following vector is well defined:

$$b^k := \begin{bmatrix} \sqrt{c_1^k} & -\sqrt{c_1^k} & 0 & 0 & \cdots \\ \sqrt{c_2^k} & 0 & -\sqrt{c_2^k} & 0 & \cdots \\ \sqrt{c_3^k} & 0 & 0 & -\sqrt{c_3^k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \bigoplus_{n=0}^{\infty} l^2.$$

Moreover, we can easily compute the mth moment $\langle \bigoplus_{n=0}^{\infty} S^m(b^k), b^k\rangle$ of b^k:

$$\left\langle \begin{bmatrix} \overbrace{0\dots0}^{m} & \sqrt{c_1^k} & -\sqrt{c_1^k} & 0 & \cdots \\ 0\dots0 & \sqrt{c_2^k} & 0 & -\sqrt{c_2^k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \begin{bmatrix} \sqrt{c_1^k} & -\sqrt{c_1^k} & 0 & \cdots \\ \sqrt{c_2^k} & 0 & -\sqrt{c_2^k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \right\rangle = -c_m^k.$$

Hence, the vector $v := f_k \oplus b^k$ is wandering and has a nontrivial decomposition corresponding to the Wold decomposition of V.

Moreover, the vector $v := z^n f_k \oplus b^k$ is wandering for any $n, k \in \mathbb{N}$. Using (7.2), we can compute

$$\|1 - f_k\|^2 = \int_{\mathbb{T}_+} \Big(\frac{1}{k} + \sum_{j=1}^{k-1}(z^{2j} + \overline{z}^{2j})\frac{k-j}{k^2}\Big)\, dz = -\frac{2}{k} + 4\sum_{j=1}^{k-1} \frac{k-j}{k^2}\frac{1}{4j^2-1}.$$

Thus $f_k \to 1$ in $L^2(\mathbb{T}_+)$. Hence $z^n f_k \to z^n$ $(k \to \infty)$. Consequently, the set of all projections of wandering vectors onto $\mathcal{H}_u$ is linearly dense in $\mathcal{H}_u$. □

Using the previous construction, we can show the following result.

Proposition 7.2. *Let $U \in \mathcal{B}(\mathcal{H})$ be a singular unitary operator. Then*

$$\sum_{n=0}^{\infty} \big|\langle U^n x, x\rangle\big| = \infty$$

for all $x \in \mathcal{H}$.

Proof. Assume that, on the contrary, there is a vector $f \in \mathcal{H}$ such that $\sum_{n=0}^{\infty} |c_n| < \infty$, where $c_n := \langle U^n f, f\rangle$ for $n \in \mathbb{N}_+$. As in the previous example we define

$$b := \begin{bmatrix} \sqrt{c_1} & -\sqrt{c_1} & 0 & 0 & \cdots \\ \sqrt{c_2} & 0 & -\sqrt{c_2} & 0 & \cdots \\ \sqrt{c_3} & 0 & 0 & -\sqrt{c_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \bigoplus_{n=0}^{\infty} l^2.$$

Then the vector $v := f + b \in \mathcal{H} \oplus \bigoplus_{n=1}^{\infty} l^2$ is wandering for $V = U \oplus \bigoplus_{n=1}^{\infty} S$, where S is a unilateral shift of multiplicity 1.

The minimal unitary extension $\widehat{V}$ of V can be decomposed as $\widehat{V} = U \oplus \bigoplus_{n=1}^{\infty} \widehat{S}$, where U is the singular part of $\widehat{V}$, and $\widehat{S}$ is a bilateral shift extending S. Thus, by Theorem 4.1, all wandering vectors of $\widehat{V}$ are orthogonal to $\mathcal{H}$. In particular, $v \in \bigoplus_{n=1}^{\infty} l^2$, and so $f = 0$, which leads to the contradiction. □

By Theorem 4.1 it is clear that the set of wandering vectors for an isometry with an absolutely continuous unitary part is dense or trivial. If the answer to the question below is affirmative, then we will be able to prove this fact (and Theorem 4.1) in the elementary way (using construction from Example 7.1).

Problem 7.3. Let $U \in \mathcal{B}(\mathcal{H})$ be an absolutely continuous unitary operator. Is the set $\{x \in \mathcal{H} : \sum_{n=0}^{\infty} |\langle U^n x, x\rangle| < \infty\}$ dense?

The set $F_U := \{x \in \mathcal{H} : \sum_{n=1}^{+\infty} |\langle U^n x, x\rangle| < \infty\}$ is U reducing, and so the space $\mathcal{H} \ominus F_U$ reduces U to an absolutely continuous unitary operator. Hence, to answer Problem 7.3, it is enough to show that $\mathcal{H} \ominus F_U = \{0\}$. Thus Problem 7.3 reduces to the following.

Problem 7.4. Let $U \in \mathcal{B}(\mathcal{H})$ be an absolutely continuous unitary operator. Is it true that $F_U \neq \{0\}$?

In Section 3, we gave a relation between the spaces $\mathcal{H}_{ns}$ and $\bigcap \widehat{\mathcal{M}}$. In the proof of Theorem 3.3 we used an additional assumption. Thus there is a natural question.

Problem 7.5. Can we omit the assumption $\dim \mathcal{N}(V^*) < \infty$ in Theorem 3.3?

For an isometry V without wandering vectors in its unitary part, the inclusion $\mathcal{H}_{ns} \subset \widehat{\mathcal{M}}$ is equivalent to $\mathcal{H}_u \subset M$ for all $M \in \widehat{\mathcal{M}}$.

Using our previous considerations, we can show that $\mathcal{H}_u \subset M$ for all $M \in \widehat{\mathcal{M}}$ such that $M \cap \mathcal{H}_s = \{0\}$. We need the following lemma.

Lemma 7.6. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry, let $N \subset \mathcal{H}$ be a subspace reducing V, and let $M \in \widehat{\mathcal{M}}$. Then*

- *$M \vee N \in \widehat{\mathcal{M}}$;*
- *if $N \subset M$, then $M \ominus N \in \widehat{\mathcal{M}}_{\mathcal{H}\ominus N}$, where $\widehat{\mathcal{M}}_{\mathcal{H}\ominus N}$ denotes the relevant family for the operator $V|_{\mathcal{H}\ominus N}$.*

Proof. Denote by $U \in \mathcal{B}(\mathcal{K})$ the minimal unitary extension of V.

The subspace $M \vee N$ is V-invariant as a linear span of such subspaces. Obviously $\mathcal{K} = \bigvee_{n\geq 0} U^* M \subset \bigvee_{n\geq 0} U^*(M \vee N) \subset \mathcal{K}$, which finishes the proof of the first part.

For the second part, note that $M \ominus N = (I - P_N)M$. Since N reduces V, P_N commutes with V. Therefore, $M \ominus N$ is V-invariant by the following calculation: $V(M \ominus N) = V(I - P_N)M = (I - P_N)VM \subset (I - P_N)M = M \ominus N$.

It remains to show that $K_1 := \bigvee_{n\geq 0} U^{*n}(M \ominus N)$ is the domain of the minimal unitary extension of $V|_{\mathcal{H}\ominus N}$. Since $M \in \widehat{\mathcal{M}}$, we have $\mathcal{K} = K_1 \vee K_2$, where $K_2 := \bigvee_{n\geq 0} U^{*n}N$. Let us show that K_1 is orthogonal to K_2. Note that $U^{*k}N = U^{*(k+1)}UN \subset U^{*(k+1)}N$, and similarly $U^{*k}(M \ominus N) \subset U^{*(k+1)}(M \ominus N)$. Thus for any integers k, l we have $U^{*k}N \subset U^{*\max\{k,l\}}N$ and $U^{*l}(M \ominus N) \subset U^{*\max\{k,l\}}(M \ominus N)$. On the other hand, since U^* is isometry, we have $U^{*\max\{k,l\}}N \perp U^{*\max\{k,l\}}(\mathcal{H} \ominus N)$. Thus $U^{*k}N$ is orthogonal to $U^{*l}(\mathcal{H} \ominus N)$ for all k, l, and we get $\mathcal{K} = K_1 \oplus K_2$.

Next, note that for every $n \geq 0$ we have $P_{\mathcal{H}\ominus N}U^{*n}N = P_{\mathcal{H}\ominus N}P_{\mathcal{H}}U^{*n}N = P_{\mathcal{H}\ominus N}V^{*n}N \subset P_{\mathcal{H}\ominus N}N = \{0\}$. Hence $\mathcal{H} \ominus N$ is orthogonal to K_2. Consequently, $\mathcal{H} \ominus N \subset \mathcal{K} \ominus K_2 = K_1$. Note that K_1 is a minimal U-reducing subspace containing $M \ominus N$. Thus, by the inclusions $M \ominus N \subset \mathcal{H} \ominus N \subset K_1$, it is also a minimal U-reducing subspace containing $\mathcal{H} \ominus N$. In other words, $U|_{K_1}$ is the minimal unitary extension of $V|_{\mathcal{H}\ominus N}$, which finishes the proof. □

Proposition 7.7. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry without wandering vectors in the unitary part, and let $V = V|_{\mathcal{H}_u} \oplus V|_{\mathcal{H}_s}$ be its Wold decomposition. If $M \in \widehat{\mathcal{M}}$ is a subspace such that $M \cap \mathcal{H}_s = \{0\}$, then $\mathcal{H}_u \subset M$.*

Proof. Decompose $\mathcal{H}_s = \bigoplus_{n\geq 0} \mathcal{H}_n$, where $V|_{\mathcal{H}_n}$ is a unilateral shift of multiplicity 1. For each $M \in \widehat{\mathcal{M}}$ define $M_k := M \vee \bigoplus_{n\geq k} \mathcal{H}_n$ for any $k \geq 0$ and $M'_k := M_k \ominus \bigoplus_{n\geq k} \mathcal{H}_n$. Fix k. Note that $M'_k = P_{\mathcal{H}_u \oplus \bigoplus_{n=0}^{k-1} \mathcal{H}_n} M$, which means that M'_k need not be a subspace of M. Let $M_k \in \widehat{\mathcal{M}}$. By Lemma 7.6 we have $M'_k \in \widehat{\mathcal{M}}_{\mathcal{H}_u \oplus \bigoplus_{n=0}^{k-1} \mathcal{H}_n}$, where $\widehat{\mathcal{M}}_{\mathcal{H}_u \oplus \bigoplus_{n=0}^{k-1} \mathcal{H}_n}$ is the relevant family for the operator $V|_{\mathcal{H}_u \oplus \bigoplus_{n=0}^{k-1} \mathcal{H}_n}$. It is important that $\mathcal{H}_u$ equals the unitary subspace of the restriction $V|_{\mathcal{H}_u \oplus \bigoplus_{n=0}^{k-1} \mathcal{H}_n}$. Since for $V|_{\mathcal{H}_u \oplus \bigoplus_{n=0}^{k-1} \mathcal{H}_n}$ the unilateral shift part has finite multiplicity, by Theorem 3.3 we get $\mathcal{H}_u \subset M'_k$. Since $M'_k \subset M_k$, we get $\mathcal{H}_u \subset M_k$ and, consequently, $\mathcal{H}_u \subset \bigcap_{k\geq 0} M_k$. Obviously $M \subset \bigcap_{k\geq 0} M_k$. We will finish the proof by showing that $M = \bigcap_{k\geq 0} M_k$.

Let $x \in \bigcap_{k\geq 0} M_k$. Then for any $k \geq 0$ there are $x_k \in M$ and $y_k \in \bigoplus_{n\geq k} \mathcal{H}_n$ such that $x = x_k + y_k$. Fix k, and consider an arbitrarily large l. From $0 = x - x = x_k - x_l + y_k - y_l$ we get $\mathcal{H}_s \ni y_k - y_l = x_l - x_k \in M$. Since by assumption $M \cap \mathcal{H}_s = \{0\}$, it follows that $y_k = y_l$. Thus $y_k \in \bigoplus_{n\geq l} \mathcal{H}_n$ for arbitrarily large l, which means that $y_k = 0$. Thus $x = x_k \in M$. □

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