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ON THE POWERS OF MAXIMAL IDEALS IN THE MEASURE ALGEBRA

LÁSZLÓ SZÉKELYHIDI

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Abstract. In this paper, we describe the powers of maximal ideals in the measure algebra of some locally compact Abelian groups in terms of the derivatives of the Fourier–Laplace transform of compactly supported measures. We show that if the locally compact Abelian group has sufficiently many real characters, then all derivatives of the Fourier–Laplace transform of a measure at some point of its spectrum completely characterize the measure. We also show that the derivatives of the Fourier–Laplace transform of a measure can be used to describe the powers of the maximal ideals corresponding to the points of the spectrum of the measure on discrete Abelian groups with finite torsion-free rank.

1. Introduction

Spectral analysis and spectral synthesis deal with the description of different varieties. One of the fundamental theorems about spectral synthesis in this field is due to Laurent Schwartz. Recently, several new results on spectral analysis and spectral synthesis have been found on discrete Abelian groups (see [6], [7], [5]) and also in the nondiscrete case (see, e.g., [3], [8]–[10]). In [2], the author formulated problems and proved results concerning spectral synthesis on locally compact Abelian groups. Recently, we introduced a method of studying spectral synthesis problems using annihilators of varieties on locally compact Abelian groups (see [11]). Based on these investigations, it has turned out that the powers of maximal ideals in the group algebra of a discrete Abelian group, or, more generally, in the

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measure algebra of a locally compact Abelian group, play a vital role in the basic problems of spectral analysis and spectral synthesis. In this paper we present a description of these powers of maximal ideals in terms of the derivatives of the Fourier–Laplace transforms of the measures in the measure algebra. We shall use the terminology and results in [11]. Concerning harmonic analysis on groups and Fourier algebras, see [1].

In this paper $\mathbb{C}$ denotes the set of complex numbers. For a locally compact Abelian group G we denote by $\mathcal{C}(G)$ the locally convex topological vector space of all continuous complex-valued functions defined on G, equipped with the pointwise operations and with the topology of uniform convergence on compact sets. The elements of $\mathcal{C}(G)$ will be simply called *functions*. For each function f we define $\check{f}$ by $\check{f}(x) = f(-x)$, whenever x is in G. By a *ring* we always mean a commutative ring with unit.

It is known that the dual of $\mathcal{C}(G)$ can be identified with the space $\mathcal{M}_c(G)$ of all compactly supported complex Borel measures on G which is equipped with the pointwise operations and with the weak*-topology. In what follows we call the elements of $\mathcal{M}_c(G)$ simply *measures*. The pairing between $\mathcal{C}(G)$ and $\mathcal{M}_c(G)$ is given by the formula

$$\langle \mu, f\rangle = \int f\, d\mu.$$

Convolution on $\mathcal{M}_c(G)$ is defined by

$$\mu * \nu(f) = \int f(x+y)\, d\mu(x)\, d\nu(y)$$

for arbitrary measures μ, ν and function f. Convolution converts the linear space $\mathcal{M}_c(G)$ into a commutative topological algebra with unit δ_0, 0 being the zero in G.

We also define convolution of measures with arbitrary functions by the similar formula

$$f * \mu(x) = \int f(x-y)\, d\mu(y)$$

for each measure μ, function f, and x in G. It is easy to see that, equipped with the action $f \mapsto f * \mu$, the space $\mathcal{C}(G)$ is a topological module over $\mathcal{M}_c(G)$. For each subset H in $\mathcal{C}(G)$, the *annihilator* of H in $\mathcal{M}_c(G)$ is the set

$$\operatorname{Ann} H = \{\mu : f * \mu = 0 \text{ for each } f \in H\}.$$

We also define the dual concept: for every subset K in $\mathcal{M}_c(G)$, the *annihilator* of K in $\mathcal{C}(G)$ is the set

$$\operatorname{Ann} K = \{f : f * \mu = 0 \text{ for each } \mu \in K\}.$$

Translation with the element y in G is the operator mapping the function f onto its *translate* $\tau_y f$ defined by $\tau_y f(x) = f(x+y)$ for each x in G. Clearly, we have $\tau_y f = f * \delta_{-y}$. A subset of $\mathcal{C}(G)$ is called *translation invariant* if it contains all translates of its elements. A closed linear subspace of $\mathcal{C}(G)$ is called a *variety* on G if it is translation invariant. Obviously, varieties are exactly the same as closed submodules in $\mathcal{C}(G)$. For each function f the smallest variety containing f

is called the *variety generated by* f or, simply, the *variety of* f, and it is denoted by $\tau(f)$, which is obviously the intersection of all varieties including f.

Theorem 1.1. *For each variety V in $\mathcal{C}(G)$ its annihilator* $\operatorname{Ann} V$ *is a closed ideal in $\mathcal{M}_c(G)$. Similarly, for each ideal I in $\mathcal{M}_c(G)$ its annihilator* $\operatorname{Ann} I$ *is a variety in $\mathcal{C}(G)$.*

Proof. Clearly, $\operatorname{Ann} V$ is a closed subspace in $\mathcal{M}_c(G)$. For each μ in $\operatorname{Ann} V$, ν in $\mathcal{M}_c(G)$, and f in V we have

$$\begin{aligned}(\nu * \mu) * f(x) &= \int f(x-y)\, d(\nu * \mu)(y)\\ &= \int\int f(x-u-v)\, d\mu(v)\, d\nu(u) = \int (f*\mu)(x-u)\, d\nu(u) = 0,\end{aligned}$$

as $f * \mu = 0$. This means $\nu * \mu$ is in $\operatorname{Ann} V$, and $\operatorname{Ann} V$ is a closed ideal in $\mathcal{M}_c(G)$.

For the dual statement it is clear that $\operatorname{Ann} I$ is a closed subspace in $\mathcal{C}(G)$. Moreover, if f is in $\operatorname{Ann} I$, y is in G, and μ is in I, then $\delta_{-y} * \mu$ is in I. Hence, we have

$$\tau_y f * \mu = (f * \delta_{-y}) * \mu = f * (\delta_{-y} * \mu) = 0,$$

and we infer that $\tau_y f$ is in $\operatorname{Ann} I$; and hence $\operatorname{Ann} I$ is a variety. □

Theorem 1.2. *For each variety $V \subseteq W$ in $\mathcal{C}(G)$ we have* $\operatorname{Ann} V \supseteq \operatorname{Ann} W$ *and for each ideal $I \subseteq J$ in $\mathcal{M}_c(G)$ we have* $\operatorname{Ann} I \supseteq \operatorname{Ann} J$. *In addition, we have* $\operatorname{Ann}(\operatorname{Ann} V) = V$ *and* $\operatorname{Ann}(\operatorname{Ann} I) \supseteq I$. *In particular, $V \neq W$ implies* $\operatorname{Ann} V \neq \operatorname{Ann} W$.

Proof. Let $V \subseteq W$ be varieties in $\mathcal{C}(G)$, and let $I \subseteq J$ be ideals in $\mathcal{M}_c(G)$. For every μ in $\operatorname{Ann} W$ and for each f in V we have that f is in W; hence $f * \mu = 0$. This proves that μ is in $\operatorname{Ann} V$, and $\operatorname{Ann} V \supseteq \operatorname{Ann} W$. Similarly, if f is in $\operatorname{Ann} J$ and μ is in I, then μ is in J; hence, $f * \mu = 0$, which proves that f is in $\operatorname{Ann} I$, and $\operatorname{Ann} I \supseteq \operatorname{Ann} J$.

Assume that f is in V and μ is in $\operatorname{Ann} V$; then, by definition, $f * \mu = 0$, and hence f is in $\operatorname{Ann}(\operatorname{Ann} V)$, which proves $\operatorname{Ann}(\operatorname{Ann} V) \supseteq V$. Similarly, we have $\operatorname{Ann}(\operatorname{Ann} I) \supseteq I$.

Suppose now that $\operatorname{Ann}(\operatorname{Ann} V) \supsetneq V$. Consequently, there is a function f in $\operatorname{Ann}(\operatorname{Ann} V)$ such that f is not in V. By the Hahn–Banach theorem, there is a λ in $\mathcal{M}_c(G)$ such that $\check{\lambda}(f) \neq 0$, and $\check{\lambda}$ vanishes on V. This means

$$(\varphi * \lambda)(0) = \int \varphi(-y)\, d\lambda(y) = \lambda(\check{\varphi}) = \check{\lambda}(\varphi) = 0,$$

whenever φ is in V. As V is a variety, this implies, by the previous theorem, that λ is in $\operatorname{Ann} V$; in particular, $f * \lambda = 0$, a contradiction. This proves $\operatorname{Ann}(\operatorname{Ann} V) = V$, which also implies $\operatorname{Ann} V \neq \operatorname{Ann} W$, whenever $V \neq W$. □

We note that for ideals in $\mathcal{M}_c(G)$ the equality $\operatorname{Ann}(\operatorname{Ann} I) = I$ does not hold in general (see [7]). Nevertheless, the following theorem holds true (see [7]).

Theorem 1.3. *Let G be a discrete Abelian group. Then* $\operatorname{Ann}(\operatorname{Ann} I) = I$ *holds for every ideal I in $\mathcal{M}_c(G)$.*

To understand the nondiscrete case, we need the following lemma.

Lemma 1.4. *Let G be a locally compact group, and let I be an ideal in $\mathcal{M}_c(G)$. Then* $\operatorname{Ann}(\operatorname{Ann}(\operatorname{Ann} I)) = \operatorname{Ann} I$.

Proof. Let $V = \operatorname{Ann} I$, and then V is a variety on G; hence, by Theorem 1.2, we have $\operatorname{Ann}(\operatorname{Ann} V) = V$. It follows $\operatorname{Ann} I = V = \operatorname{Ann}(\operatorname{Ann} V) = \operatorname{Ann}(\operatorname{Ann}(\operatorname{Ann} I))$. □

Now we can prove the following theorem characterizing those ideals in $\mathcal{M}_c(G)$ which coincide with the second annihilator.

Theorem 1.5. *Let G be a locally compact group, and let I be an ideal in $\mathcal{M}_c(G)$. Then we have* $\operatorname{Ann}(\operatorname{Ann} I) = I$ *if and only if I is closed.*

Proof. By Theorem 1.1, the annihilator of each variety is closed; in particular, $J = \operatorname{Ann}(\operatorname{Ann} I)$, as the annihilator of the variety $\operatorname{Ann} I$, is closed, which proves the necessity of our condition.

Conversely, suppose that I is closed, and I is a proper subset of J. By Lemma 1.4, we have $\operatorname{Ann} J = \operatorname{Ann} I$. Let μ be in J such that μ is not in I. As the space $\mathcal{M}_c(G)$ with the weak*-topology is locally convex, by the Hahn–Banach theorem there is a linear functional ξ in $\mathcal{M}_c(G)^*$, such that $\check{\xi}$ vanishes on I and $\check{\xi}(\mu) \neq 0$. It is known that every weak*-continuous linear functional on a dual space arises from an element of the original space; that is, there is an f in $\mathcal{C}(G)$ with $\xi(\nu) = \nu(f)$ for each ν in $\mathcal{M}_c(G)$. We infer $\mu(\check{f}) = \check{\xi}(\mu) \neq 0$, and μ is in J, and hence f is not in $\operatorname{Ann} J$. On the other hand, $\check{\nu}(f) = \nu(\check{f}) = \check{\xi}(\nu) = 0$ for each ν in I, as $\check{\xi}$ vanishes on I, which implies that f is in $\operatorname{Ann} I = \operatorname{Ann} J$, a contradiction. □

Corollary 1.6. *Let G be a locally compact Abelian group. Then the mapping $V \leftrightarrow \operatorname{Ann} V$ sets up one-to-one inclusion-reversing correspondences between the varieties in $\mathcal{M}_c(G)$ and the closed ideals in $\mathcal{M}_c(G)$.*

2. Exponentials

A basic function class is formed by the common eigenfunctions of all translation operators, that is, by those nonzero continuous functions $\varphi : G \to \mathbb{C}$ satisfying

$$\tau_y \varphi = m(y) \cdot \varphi \tag{2.1}$$

with some $m : G \to \mathbb{C}$; that is,

$$\varphi(x + y) = m(y)\varphi(x) \tag{2.2}$$

for all x, y in G. It follows that $\varphi(y) = \varphi(0) \cdot m(y)$, which implies that $\varphi(0) \neq 0$ and, by (2.2),

$$m(x + y) = m(x)m(y) \tag{2.3}$$

for each x, y in G. Nonzero continuous functions $m : G \to \mathbb{C}$ satisfying (2.3) for each x, y in G are called *exponentials*. Clearly, every exponential generates a one-dimensional variety and, conversely, every one-dimensional variety is generated by an exponential. Sometimes exponentials are called *generalized characters*.

Using translation, one introduces *modified differences* in the following manner: for each continuous function f in $\mathcal{C}(G)$ and y in G we let

$$\Delta_{f;y} = \delta_{-y} - f(y)\delta_0.$$

Hence, $\Delta_{f;y}$ is an element of $\mathcal{M}_c(G)$. For a given f in $\mathcal{C}(G)$, the closed ideal generated by all modified differences of the form $\Delta_{f;y}$ with y in G is denoted by M_f. We have the following theorem.

Theorem 2.1. *Let G be a locally compact Abelian group, and let $f : G \to \mathbb{C}$ be a continuous function. The ideal M_f is proper if and only if f is an exponential. In this case $M_f = \operatorname{Ann}\tau(f)$ is maximal, and $\mathcal{M}_c(G)/M_f$ is topologically isomorphic to the complex field.*

Proof. As M_f is closed, by Theorem 1.5 we have $\operatorname{Ann}(\operatorname{Ann} M_f) = M_f$.

Suppose that f is an exponential. Then $f \neq 0$, and

$$\Delta_{f;y} * f(x) = f(x+y) - f(y)f(x) = 0$$

for each x, y in G; hence f is in $\operatorname{Ann} M_f$. As $\tau(f)$ consists of all constant multiples of f, we infer that $\tau(f) \subseteq \operatorname{Ann} M_f$. Moreover, if φ is in $\operatorname{Ann} M_f$, then we have

$$0 = \Delta_{f;y} * \varphi(x) = \varphi(x+y) - f(y)\varphi(x)$$

for each x, y in G. It follows that $\varphi = \varphi(0)\cdot f$, and hence φ is in $\tau(f)$. We conclude that $\tau(f) = \operatorname{Ann} M_f$ and $M_f = \operatorname{Ann}\tau(f)$.

We define the mapping $\Phi_f : \mathcal{M}_c(G) \to \mathbb{C}$ by

$$\Phi_f(\mu) = \mu(\check{f}) = \int f(-y)\, d\mu(y)$$

for each μ in $\mathcal{M}_c(G)$. Then Φ_f is a linear mapping, $\Phi_f(\delta_0) = 1$, and for each μ, ν in $\mathcal{M}_c(G)$ we have

$$\begin{aligned}\Phi_f(\mu * \nu) &= \int f(-x-y)\, d\mu(x)\, d\nu(y)\\ &= \int f(-x)\, d\mu(x) \int f(-y)\, d\nu(y) = \Phi_f(\mu)\cdot\Phi_f(\nu);\end{aligned}$$

hence Φ_f is an algebra homomorphism. Obviously, Φ_f maps $\mathcal{M}_c(G)$ onto $\mathbb{C}$, and hence it is a multiplicative linear functional. We infer that $\operatorname{Ker}\Phi_f$ is a maximal ideal and that $\mathcal{M}_c(G)/\operatorname{Ker}\Phi_f$ is isomorphic to the complex field $\mathbb{C}$. For each μ in $\operatorname{Ker}\Phi_f$ we have $\mu(\check{f}) = 0$; hence, for each complex number c we have

$$cf * \mu(x) = c\int f(x-y)\, d\mu(y) = cf(x)\mu(\check{f}) = 0,$$

and, consequently, μ is in $\operatorname{Ann}\tau(f) = M_f$. It follows that $\operatorname{Ker}\Phi_f \subseteq M_f$, which implies that M_f is a maximal ideal. We also have that $\operatorname{Ker}\Phi_f$ is closed, and hence Φ_f is continuous. As Φ_f is also open, we have that $\mathcal{M}_c(G)/M_f$ is topologically isomorphic to the complex field.

Finally, if M_f is proper, then $\operatorname{Ann} M_f$ is nonzero. Let $\varphi \neq 0$ be a function in $\operatorname{Ann} M_f$; then we have

$$0 = \Delta_{f;y} * \varphi(x) = \varphi(x+y) - f(y)\varphi(x),$$

and in the same way as above we conclude that f is an exponential. The theorem is proved. □

Given a ring R, we call a maximal ideal M in R an *exponential maximal ideal* if the residue ring R/M is isomorphic to the complex field. If R is a topological ring, then we require the isomorphism to be topological. From the above proof it is clear that if G is a locally compact Abelian group, then each exponential maximal ideal in $\mathcal{M}_c(G)$ is of the form $M_m = \operatorname{Ann} \tau(m)$ with some exponential m. The maximal ideal in $\mathcal{M}_c(G)$ corresponding to the exponential identically 1, that is, the annihilator of all constant functions on G, is called the *augmentation ideal*.

Given a locally compact Abelian group G, the set of all exponentials is denoted by $\widetilde{G}$ and is called the *generalized character group* of G. It is easy to check that, equipped with the compact-open topology, $\widetilde{G}$ is a topological Abelian group. Obviously, $\hat{G}$, the dual of G, is a closed subgroup of $\widetilde{G}$ (see, e.g., [2]).

Let G be a locally compact Abelian group. A continuous homomorphism of G into the additive group of real numbers is called a *real character*. By addition of real characters and multiplication by real numbers being defined pointwise and using the compact-open topology, the set of all real characters of G is a locally convex real topological vector space and it is denoted by $\operatorname{Hom}(G, \mathbb{R})$.

Theorem 2.2. *Let G be a locally compact Abelian group. Then $\widetilde{G}$ is topologically isomorphic to $\hat{G} \times \operatorname{Hom}(G, \mathbb{R})$.*

Proof. Let m be in $\widetilde{G}$; then $\chi : G \to \mathbb{C}$ defined by $\chi(g) = m(g) \cdot |m(g)|^{-1}$ for g in G is obviously a character of G. On the other hand, $a : g \mapsto \ln |m(g)|$ is a real character; hence the mapping

$$m \mapsto (\chi, a)$$

maps $\widetilde{G}$ into $\hat{G} \times \operatorname{Hom}(G, \mathbb{R})$. It is easy to see that this mapping is open and continuous. As $m = \chi \cdot \exp a$, it is injective, and as the function $g \mapsto \chi \cdot \exp a$ is an exponential for each character χ and real character a, it is also surjective. □

Theorem 2.3. *Let G be a locally compact Abelian group. If G is compact, then $\hat{G}$ is topologically isomorphic to $\widetilde{G}$. If G is compactly generated and $\hat{G}$ is topologically isomorphic to $\widetilde{G}$, then G is compact.*

Proof. If G is compact, then there is a nonzero real character on G. Indeed, if $a : G \to \mathbb{R}$ is a real character, then $a(G)$ is a compact subgroup of the additive group of $\mathbb{R}$; hence it is $\{0\}$. It follows that $\operatorname{Hom}(G, \mathbb{R}) = \{0\}$ and $\widetilde{G} = \hat{G}$ by the previous theorem. If G is a compactly generated locally compact Abelian group, then, by the structure theorem, $G = \mathbb{R}^n \times \mathbb{Z}^k \times K$, where K is a compact Abelian group and n, k are natural numbers. Hence, $\widetilde{G}$ can be topologically isomorphic to $\hat{G}$ only if $n = k = 0$ and G is compact. □

3. Powers of maximal ideals in the ring of Laurent polynomials

The motivation of this paper is the characterization of powers of maximal ideals in the group algebra of $\mathbb{Z}^n$. By the recent results in [11], powers of maximal ideals play a fundamental role in characterizing varieties which possess spectral synthesis on discrete Abelian groups. A well-known particular case for a characterization of this type is the polynomial ring $\mathbb{C}[x]$, where every maximal ideal M has the form

$$M_\lambda = \{p : p(\lambda) = 0\}$$

with some complex number λ. In this case it is known that for every natural number k we have

$$M_\lambda^{k+1} = \{p : p^{(j)}(\lambda) = 0 \text{ for } j = 0, 1, \dots, k\}.$$

An immediate generalization is given in the theorem below. First we introduce some notation. Given a positive integer n, addition, subtraction, and inequalities in $\mathbb{Z}^n$ are defined componentwise; further, we write $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ whenever $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is in $\mathbb{Z}^n$. Let $\mathbb{C}_0^n$ denote the set of all vectors in $\mathbb{C}^n$ whose every component is different from zero. For λ in $\mathbb{C}_0^n$ and α in $\mathbb{Z}^n$ we write

$$\lambda^\alpha = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}.$$

We adopt this notation for differential operators. Let $\mathbb{C}[z, z^{-1}]$ denote the ring of Laurent polynomials in n variables, where $z = (z_1, z_2, \dots, z_n)$. Then $\partial_j : \mathbb{C}[z, z^{-1}] \to \mathbb{C}[z, z^{-1}]$ is the partial differential operator with the usual meaning, and for every α with $\alpha \geq 0$ we write

$$\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}.$$

More generally, if $P : \mathbb{C}^n \to \mathbb{C}$ is a complex polynomial in n variables, then $P(\partial)$ has the obvious meaning: formally, the variable z_j in P is replaced by ∂_j, and the constant $P(0)$ in P is replaced by $P(0)$-times the identity operator. Using this notation, the Taylor formula for polynomials in n variables has the form

$$P(z) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial^\alpha P(z_0)(z - z_0)^\alpha. \tag{3.1}$$

Here z, z_0 are arbitrary in $\mathbb{C}^n$ and we use the notation $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$.

We need the following simple lemma.

Lemma 3.1. *Let k be a positive integer, let α an element in $\mathbb{N}^n$, and let $r_1, r_2, \dots, r_k$ be arbitrary in $\mathbb{C}[z, z^{-1}]$. Then we have*

$$\partial^\alpha(r_1 \cdot r_2 \cdot \dots \cdot r_k) = \sum_{\beta_1+\beta_2+\dots+\beta_k=\alpha} \frac{\alpha!}{\beta_1!\beta_2!\cdots\beta_k!}\partial^{\beta_1}r_1\partial^{\beta_2}r_2\cdots\partial^{\beta_k}r_k. \tag{3.2}$$

Proof. We prove by induction on k and the statement obviously holds for $k=1$. To prove it for $k+1$, we proceed as

$$\begin{aligned}
&\partial^\alpha(r_1\cdot r_2\cdot\dots\cdot r_k\cdot r_{k+1})\\
&\quad=\partial^\alpha(q\cdot r_{k+1})=\sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\partial^\beta q\cdot\partial^{\alpha-\beta}r_{k+1}\\
&\quad=\sum_{\beta_1+\beta_2+\dots+\beta_k=\beta}\sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\frac{\beta!}{\beta_1!\beta_2!\cdots\beta_k!}\partial^{\beta_1}r_1\partial^{\beta_2}r_2\cdots\partial^{\beta_k}r_k\partial^{\alpha-\beta}r_{k+1}\\
&\quad=\sum_{\beta_1+\beta_2+\dots+\beta_{k+1}=\alpha}\frac{\alpha!}{\beta_1!\beta_2!\cdots\beta_{k+1}!}\partial^{\beta_1}r_1\partial^{\beta_2}r_2\cdots\partial^{\beta_k}r_k\partial^{\beta_{k+1}}r_{k+1},
\end{aligned}$$

which was to be proved. □

Now we have the following result.

Theorem 3.2. *Let n be a positive integer. Then for every maximal ideal M in $\mathbb{C}[z,z^{-1}]$ there exists a complex vector λ in $\mathbb{C}_0^n$ such that for each natural number k we have*

$$M^{k+1}=\big\{r:\partial^\alpha r(\lambda)=0 \textit{ for every } \alpha\in\mathbb{N}^n \textit{ with } |\alpha|\le k\big\}. \tag{3.3}$$

Proof. It is well known that for every maximal ideal M in $\mathbb{C}[z,z^{-1}]$ there exists a unique λ in $\mathbb{C}_0^n$ such that

$$M=M_\lambda=\big\{r:r(\lambda)=0\big\}.$$

Let k be a positive integer, and let

$$I_{k+1}=\big\{r:\partial^\alpha r(\lambda)=0 \text{ for every } \alpha\in\mathbb{N}^n \text{ with } |\alpha|\le k\big\}.$$

We will show that $M^{k+1}=I_{k+1}$. Let $r_1,r_2,\dots,r_{k+1}$ be arbitrary in M. Then, by the previous lemma, we have, for each α in $\mathbb{N}^n$ with $|\alpha|\le k$,

$$\partial^\alpha(r_1r_2\cdots r_{k+1})=\sum_{\beta_1+\beta_2+\dots+\beta_{k+1}=\alpha}\frac{\alpha!}{\beta_1!\beta_2!\cdots\beta_{k+1}!}\partial^{\beta_1}r_1\partial^{\beta_2}r_2\cdots\partial^{\beta_{k+1}}r_{k+1}.$$

In each term of the sum we have $|\beta_1|+|\beta_2|+\dots+|\beta_{k+1}|=|\alpha|\le k$, which implies that $|\beta_j|=0$ for some j with $j=1,2,\dots,k+1$. It follows that each term in the sum has a factor of the form $\partial^{\beta_j}r_j=\partial^0 r_j=r_j$, which vanishes at λ. It follows that $\partial^\alpha(r_1r_2\cdots r_{k+1})(\lambda)=0$ for each α with $|\alpha|\le k$, and hence $r_1r_2\cdots r_{k+1}$ belongs to I_{k+1}. As every element in M^{k+1} is a sum of functions of the form $r_1r_2\cdots r_{k+1}$, we have $M^{k+1}\subseteq I_{k+1}$.

For the converse, we suppose that r is in I_{k+1}; that is, it has the property that

$$\partial^\alpha r(\lambda)=0$$

for every α in $\mathbb{N}^n$ with $|\alpha|\le k$. There exists a monomial q such that $p=r\cdot q$ is a polynomial. Obviously, by the above consideration, we have

$$\partial^\alpha p(\lambda)=0$$

for every α in $\mathbb{N}^n$ with $|\alpha| \leq k$. By Taylor's formula, we have that

$$p(z) = \sum_{|\alpha| \geq k+1} \frac{1}{\alpha!} \partial^\alpha p(\lambda)(z-\lambda)^\alpha.$$

On the right-hand side, every term has a factor which is a multiple of $(z-\lambda)^\alpha$, where $|\alpha| = k+1$. This implies that $\frac{(z-\lambda)^\alpha}{q(z)}$ is in M^{k+1}; hence, $r = \frac{p}{q}$ is in M^{k+1} and the proof is complete. □

4. Fourier–Laplace transformation

Given the locally compact Abelian group G for every μ in $\mathcal{M}_c(G)$, we define the function $\hat{\mu} : \widetilde{G} \to \mathbb{C}$ by

$$\hat{\mu}(m) = \mu(\check{m}) = \int m(-y)\, d\mu(y)$$

whenever m is in $\check{G}$. Obviously, $\hat{\mu}(m) = m * \mu(0)$. Also, we have $\hat{\mu}(m) = \Phi_m(\mu)$, where Φ_m is defined in Theorem 2.1 with $m = f$. The function $\hat{\mu}$ is called the *Fourier–Laplace transform* of μ and the mapping $\mu \mapsto \hat{\mu}$ is called the *Fourier–Laplace transformation*. The following result is well known.

Theorem 4.1. *Let G be a locally compact Abelian group. Then for each measure μ in $\mathcal{M}_c(G)$ its Fourier–Laplace transform $\hat{\mu}$ is a continuous function on $\widetilde{G}$. The Fourier–Laplace transformation $\mu \to \hat{\mu}$ is a continuous injective algebra homomorphism of $\mathcal{M}_c(G)$ into $\mathcal{C}(\widetilde{G})$, the latter equipped with the pointwise linear operations and multiplication, and with the topology of pointwise convergence.*

Proof. Let $(m_i)_{i\in I}$ be a generalized sequence in $\widetilde{G}$ converging to the exponential m in $\widetilde{G}$. Then $\check{\mu}_i \to \check{\mu}$ uniformly on the compact set $\operatorname{supp}\mu$; hence we have $\hat{\mu}_i(m) \to \hat{\mu}$ proving that $\hat{\mu}$ is continuous.

We introduce the notation

$$\mathcal{F}(\mu) = \hat{\mu}$$

for each μ in $\mathcal{M}_c(G)$. Obviously, $\mathcal{F} : \mathcal{M}_c(G) \to \mathcal{C}(\widetilde{G})$ is a linear mapping. Suppose that $(\mu_\alpha)_{\alpha\in A}$ is a generalized sequence in $\mathcal{M}_c(G)$ converging to μ in the weak*-topology. Then for each m in $\widetilde{G}$ we have $\mu_\alpha(\check{m}) \to \mu(\check{m})$; that is, $\hat{\mu}_\alpha(m) \to \hat{\mu}(m)$, which gives the continuity of $\mathcal{F}$. Finally, for μ, ν in $\mathcal{M}_c(G)$ and m in $\widetilde{G}$ we have

$$\begin{aligned}\mathcal{F}(\mu * \nu)(m) = (\mu * \nu)(\check{m}) &= \int m(-x-y)\, d\mu(x)\, d\nu(y) \\ &= \int m(-x)\, d\mu(x) \int m(-y)\, d\nu(y) = \mu(\check{m}) \cdot \nu(\check{m}) \\ &= \mathcal{F}(\mu)(m) \cdot \mathcal{F}(\nu)(m);\end{aligned}$$

hence $\mathcal{F}$ is an algebra homomorphism.

The injectivity of the Fourier–Laplace transformation follows from the linearity and injectivity of the Fourier transformation. □

The range of the Fourier–Laplace transformation in $\mathcal{C}(\widetilde{G})$, that is, the set of all Fourier–Laplace transforms, will be denoted by $\mathcal{A}(G)$. This is a subalgebra of $\mathcal{C}(\widetilde{G})$, isomorphic to $\mathcal{M}_c(G)$, which is sometimes called the *Fourier algebra* of G.

5. Derivations

Let G be a locally compact Abelian group. The continuous function $a : G \to \mathbb{C}$ is called an *additive function* if it is a homomorphism of G into the additive topological group of complex numbers. The set of all additive functions on G is denoted by $\mathrm{Hom}(G, \mathbb{C})$. Equipped with the pointwise addition and with the compact-open topology, it is a topological Abelian group, which is topologically isomorphic to $\mathrm{Hom}(G, \mathbb{R}) \times \mathrm{Hom}(G, \mathbb{R})$. The function $X : \mathbb{C} \to G$ is called a *one-parameter subgroup* in G if it is a homomorphism of the topological group $\mathbb{C}$ into G. The function $X : \mathbb{R} \to G$ is called a *real one-parameter subgroup* in G if it is a homomorphism of the topological group $\mathbb{R}$ into G. We have the following theorem (see [2]).

Theorem 5.1. *Let G be a locally compact Abelian group. For every additive function $a : G \to \mathbb{C}$ the function $X_a : \mathbb{C} \to \widetilde{G}$ defined by*

$$X_a(z)(g) = \exp za(g) \tag{5.1}$$

for g in G and z in $\mathbb{C}$ is a one-parameter subgroup in $\widetilde{G}$. If a is a real character, then the restriction of X_a to $\mathbb{R}$ is a real one-parameter subgroup in $\widetilde{G}$.

Proof. The continuity of X_a is obvious. For g, h in G and z in $\mathbb{R}$ we have

$$X_a(z)(g+h) = \exp ra(g+h) = \exp za(g) \cdot \exp ra(h) = X_a(z)(g) \cdot X_a(z)(h);$$

that is, $X_a(z)$ is an exponential on G for each z in $\mathbb{C}$. Moreover, for z, w in $\mathbb{C}$ and g in G we have

$$X_a(z+w)(g) = \exp(z+w)a(g) = \exp za(g) \cdot \exp wa(g) = X_a(z)(g) \cdot X_a(w)(g);$$

that is,

$$X_a(z+w) = X_a(z)X_a(w),$$

and hence X_a is a one-parameter subgroup in $\widetilde{G}$. Clearly, if a is real valued, then the restriction of X_a to $\mathbb{R}$ is a real one-parameter subgroup. □

Theorem 5.2. *Let G be a locally compact Abelian group. For every one-parameter subgroup $X : \mathbb{C} \to \widetilde{G}$ there exists a unique additive function $a : G \to \mathbb{C}$ such that $X = X_a$. If X is a real one-parameter subgroup, then a is a real character.*

Proof. Let $X : \mathbb{C} \to \widetilde{G}$ be a one-parameter subgroup, and let g be an element of G. We define, for each w in $\mathbb{C}$,

$$\varphi_g(w) = X(w)(g).$$

Then $\varphi_g : \mathbb{C} \to \mathbb{C} - \{0\}$ is continuous and satisfies

$$\varphi_g(z+w) = \varphi_g(z)\varphi_g(w)$$

for every z, w in $\mathbb{C}$. It follows that φ_g is an exponential function of the form

$$\varphi_g(w) = \exp \lambda(g) w,$$

where $\lambda : G \to \mathbb{C}$ is a continuous function. On the other hand, for g, h in G we have

$$\begin{aligned}\exp \lambda(g+h)w &= \varphi_{g+h}(w) = X(w)(g+h) = X(w)(g) \cdot X(w)(h)\\ &= \varphi_g(w) \cdot \varphi_h(w) = \exp\big[\lambda(g) + \lambda(h)\big] w,\end{aligned}$$

which implies

$$\exp\big[\lambda(g+h) - \lambda(g) - \lambda(h)\big] w = 1$$

for every g, h in G and w in $\mathbb{C}$. We infer that $[\lambda(g+h) - \lambda(g) - \lambda(h)]w$ is an integer multiple of $2\pi i$ for any choice of g, h in G and w in $\mathbb{C}$, which is possible only if $\lambda(g+h) - \lambda(g) - \lambda(h) = 0$, that is, if $\lambda : G \to \mathbb{C}$ is additive. Hence, we have

$$X(w)(g) = \varphi_g(w) = \exp \lambda(g) w$$

for each g in G and w in $\mathbb{C}$, which implies immediately that λ is real-valued, if X is a real one-parameter subgroup. $\square$

Let G be a locally compact Abelian group, let $f : \widetilde{G} \to \mathbb{C}$ be a function, let $X : \mathbb{C} \to \widetilde{G}$ be a one-parameter subgroup in $\widetilde{G}$, and let m be an exponential on G. We define the *derivative of f along X at m* as the limit

$$\lim_{z\to 0} \lim_{w\to 0} \frac{1}{w}\Big(f\big(m \cdot X(z+w)\big) - f\big(m \cdot X(z)\big)\Big) \tag{5.2}$$

whenever it exists and is finite (see, e.g., [2]). We denote it by $\partial_X f(m)$. If, in addition, the one-parameter subgroup X has the form $X = X_a$ with some additive function a, then $\partial_X f(m)$ is called the *derivative of f along the additive function a*, and it is denoted by $\partial_a f(m)$. If $\partial_X f(m)$ is defined for each m in $\widetilde{G}$, then its derivative along the one-parameter subgroup Y is denoted by $\partial_Y \partial_X f(m)$, and so on. Repeating this process, we can define $\mathcal{C}^\infty$-functions in the obvious way: the function f is called $\mathcal{C}^\infty$ if $P(\partial_{X_1}, \partial_{X_2}, \dots, \partial_{X_k}) f(m)$ exists for every positive integer k, for every choice of the one-parameter subgroups X_i in $\widetilde{G}$, for every polynomial P, and for every exponential m on G. Here $P(\partial_{X_1}, \partial_{X_2}, \dots, \partial_{X_k})$ denotes the usual polynomial differential operator: P is a complex polynomial in k variables and the variables are formally replaced by the differential operators $\partial_{X_1}, \partial_{X_2}, \dots, \partial_{X_k}$. In particular, for every nonnegative integer $\alpha_1, \alpha_2, \dots, \alpha_k$ the meaning of the notation $\partial_{X_1}^{\alpha_1} \partial_{X_2}^{\alpha_2} \cdots \partial_{X_k}^{\alpha_k}$ is obvious, too. If the one-parameter subgroup X has the form X_a, then we use the notation ∂_a instead of ∂_{X_a}. In light of Theorem 5.2, we can keep this notation in any case, as every one-parameter subgroup X in $\widetilde{G}$ has the form $X = X_a$.

Theorem 5.3. *Let G be a locally compact Abelian group, and let μ be a measure in $\mathcal{M}_c(G)$. Then the Fourier–Laplace transform of μ is a $\mathcal{C}^\infty$-function, and for every* *complex polynomial* $P:\mathbb{C}^k\to\mathbb{C}$ *and for arbitrary additive functions* $a_j:G\to\mathbb{C}$ $(j=1,2,\dots,k)$ *we have*

$$P(\partial_{a_1},\partial_{a_2},\dots,\partial_{a_k})\hat{\mu}(m)=\int \check{m}P(\check{a}_1,\check{a}_2,\dots,\check{a}_k)\,d\mu. \tag{5.3}$$

Proof. It is enough to prove the following particular case:

$$\partial_a\hat{\mu}(m)=\int \check{m}\check{a}\,d\mu \tag{5.4}$$

for each additive function $a:G\to\mathbb{C}$; the general case follows then by iteration. We proceed as follows:

$$\begin{aligned}\partial_a\hat{\mu}(m)&=\lim_{z\to0}\lim_{w\to0}\frac{1}{w}\big(\hat{\mu}\big(m\cdot X(z+w)\big)-\hat{\mu}\big(m\cdot X(z)\big)\big)\\&=\lim_{z\to0}\lim_{w\to0}\int m(-g)\exp za(-g)\frac{\exp wa(-g)-1}{w}\,d\mu(g)\\&=\lim_{z\to0}\int m(-g)\exp za(-g)\lim_{w\to0}\frac{\exp wa(-g)-1}{a(-g)w}a(-g)\,d\mu(g)\\&=\int m(-g)a(-g)\,d\mu(g).\end{aligned}$$

□

The linearity of ∂_a and the obvious property

$$\partial_a(f\cdot g)(m)=\partial_af(m)\cdot g(m)+f\cdot\partial_ag(m)$$

show that the mapping $\partial_a:\mathcal{A}(G)\to\mathcal{A}(G)$ is closely related to the derivations of the Fourier algebra. We recall that, given a commutative algebra A, the linear mapping $D:A\to A$ is called a *derivation* if for every x,y in A we have

$$D(xy)=D(x)y+xD(y). \tag{5.5}$$

Then, obviously, ∂_a is a derivation on the Fourier algebra for every additive function a.

The element g in the locally compact Abelian group G is called a *compact element*, if the intersection of all closed subgroups in G including g is compact. The set of all compact elements B in a locally compact Abelian group G is a closed subgroup (see [4, Theorem 9.10, p. 92]), and G/B has no compact elements except zero (see [4, Theorem 24.34, p. 390]). We say that the locally compact Abelian group has *sufficiently many real characters* if for each g in G, different from the identity, there exists a real additive function $a:G\to\mathbb{R}$ such that $a(g)\neq0$. This is the case if and only if $B=\{0\}$ (see [4, Theorem 24.34, p. 390]). This can be reformulated in two other ways: the dual of G is connected or G is topologically isomorphic to $\mathbb{R}^n\times F$, where n is a natural number and F is a discrete torsion-free Abelian group (see [4, Corollary 24.35, p. 390]). We have the following theorem.

Theorem 5.4. *Let* G *be a locally compact Abelian group, and let* $H\subseteq G$ *be a closed subgroup with no compact elements except zero. If* μ *is a measure in* $\mathcal{M}_c(G)$ *such that for some exponential* m *we have* $\partial_{a_1}^{\alpha_1}\partial_{a_2}^{\alpha_2}\cdots\partial_{a_k}^{\alpha_k}\hat{\mu}(m)=0$ *for every choice of the natural numbers* $k,\alpha_1,\alpha_2,\dots,\alpha_k$ *and the additive functions* $a_1,a_2,\dots,a_k$, *then* $\operatorname{supp}\mu\cap H=\emptyset$.

Proof. We remark that the statement of this theorem is that for every continuous function $f : G \to \mathbb{C}$ with $\operatorname{supp} f \subseteq H$ we have $\mu(f) = 0$. First of all, we note that the condition on μ implies that

$$P(\partial_{a_1}, \partial_{a_2}, \dots, \partial_{a_k})\hat{\mu}(m) = 0$$

for every natural number k, for every choice of the additive functions $a_1, a_2, \dots, a_k$, and for every polynomial $P : \mathbb{C}^k \to \mathbb{C}$. Hence, by the previous theorem, we have

$$\int \check{m} P(\check{a}_1, \check{a}_2, \dots, \check{a}_k)\, d\mu = 0 \tag{5.6}$$

for every natural number k, for every choice of the additive functions $a_1, a_2, \dots, a_k$, and for every polynomial $P : \mathbb{C}^k \to \mathbb{C}$. By the assumption on H, for every element $g \neq h$ in H there exists an additive function $a : H \to \mathbb{C}$ such that $a(g) \neq a(h)$. It follows that the algebra of functions of the form

$$g \mapsto P\big(a_1(-g), a_2(-g), \dots, a_k(-g)\big)$$

satisfies the conditions of the Stone–Weierstrass theorem on the compact set H. It follows that, by (5.6), we have

$$\int_H \check{m} f\, d\mu = 0$$

for every continuous function $f : H \to \mathbb{C}$. In particular, taking $f = \overline{\chi}$, where χ is any character of H, we obtain

$$\int_H \overline{\chi}\check{m}\, d\mu = 0.$$

This means that the Fourier–Stieltjes transform of the restriction of the measure $\check{m}\mu$ to H is zero, which implies that this measure is zero; hence $\mu = 0$ on H. By Tietze's extension theorem, every continuous function on H is the restriction of some continuous function on G. In other words, if for the function $f : G \to \mathbb{C}$ we have $\operatorname{supp} f \subseteq H$, then $\mu(f) = 0$. This implies $\operatorname{supp} \mu \cap H = \emptyset$. $\square$

Corollary 5.5. *Let G be a locally compact Abelian group having sufficiently many real characters. If μ is a measure in $\mathcal{M}_c(G)$ such that for some exponential m we have $\partial_{a_1}^{\alpha_1}\partial_{a_2}^{\alpha_2}\cdots\partial_{a_k}^{\alpha_k}\hat{\mu}(m) = 0$ for every choice of the natural numbers $k, \alpha_1, \alpha_2, \dots, \alpha_k$ and the additive functions $a_1, a_2, \dots, a_k$, then $\mu = 0$.*

Proof. This follows from the previous theorem, since in this case H can be taken as G. $\square$

Theorem 5.6. *Let G be a discrete Abelian group with finite torsion-free rank, and let m be an exponential on G. Then for each natural number k the ideal M_m^{k+1} is the set I_{k+1} of all measures μ in $\mathcal{M}_c(G)$ for which*

$$\partial_{a_1}^{\alpha_1}\partial_{a_2}^{\alpha_2}\cdots\partial_{a_l}^{\alpha_l}\hat{\mu}(m) = 0$$

for every positive integer l, for every α in $\mathbb{N}^l$ with $|\alpha| \leq k$, and for every choice of the additive functions $a_1, a_2, \dots, a_l$. If the torsion-free rank of G is infinite, then $M_1^3 \neq I_3$.

We recall that M_1 is the augmentation ideal.

Proof. Suppose that μ is in M_m^{k+1}. Then μ annihilates $\operatorname{Ann} M_m^{k+1}$. On the other hand, every function $\varphi : G \to \mathbb{C}$ of the form

$$\varphi(g) = a_1(g)^{\alpha_1} a_2(g)^{\alpha_2} \cdots a_l(g)^{\alpha_l} m(g)$$

belongs to $\operatorname{Ann} M_m^{k+1}$, assuming that $\alpha_1 + \alpha_2 + \cdots + \alpha_l \le k$ and $a_1, a_2, \dots, a_l$ are additive functions (see, e.g., [11]). Hence, by assumption, we have

$$\begin{aligned} 0 &= \varphi * \mu(g) \\ &= \int m(g-h) \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} a_1(g)^{\alpha_1-\beta_1} a_1(-h)^{\beta_1} \cdots a_l(g)^{\alpha_l-\beta_l} a_l(-h)^{\beta_l}\, d\mu(h) \\ &= \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} m(g) a_1(g)^{\alpha_1-\beta_1} \cdots a_l(g)^{\alpha_l-\beta_l} \\ &\quad \times \int m(-h) a_1(-h)^{\beta_1} \cdots a_l(-h)^{\beta_l}\, d\mu(h). \end{aligned}$$

As the functions

$$g \mapsto \frac{\alpha!}{\beta!(\alpha-\beta)!} m(g) a_1(g)^{\alpha_1-\beta_1} \cdots a_l(g)^{\alpha_l-\beta_l}$$

are linearly independent for different choices of $\alpha_j - \beta_j$, we infer

$$\int m(-h) a_1(-h)^{\beta_1} \cdots a_l(-h)^{\beta_l}\, d\mu(h) = 0$$

for each $\beta \le \alpha$. By Theorem 5.3, this implies

$$\partial_{a_1}^{\alpha_1} \partial_{a_2}^{\alpha_2} \cdots \partial_{a_l}^{\alpha_l} \hat{\mu}(m) = 0$$

for every positive integer l, for every α in $\mathbb{N}^l$ with $|\alpha| \le k$, and for every choice of the additive functions $a_1, a_2, \dots, a_l$. This means $M_m^{k+1} \subseteq I_{k+1}$.

To prove the reverse inclusion, we use the following result: if the torsion-free rank of G is finite, then the variety $\operatorname{Ann} M_m^{k+1}$ is the set of all exponential monomials of degree at most k corresponding to the exponential m; that is, the set of all functions of the form

$$\varphi(g) = P\big(a_1(g), a_2(g), \dots, a_l(g)\big) m(g),$$

where l is a positive integer, $P : \mathbb{C}^l \to \mathbb{C}$ is a polynomial in l variables and of degree at most k, and $a_1, a_2, \dots, a_l$ are additive functions (see [11]). In particular, by the condition

$$\int m(-h) a_1(-h)^{\beta_1} \cdots a_l(-h)^{\beta_l}\, d\mu(h) = \partial_{a_1}^{\alpha_1} \partial_{a_2}^{\alpha_2} \cdots \partial_{a_l}^{\alpha_l} \hat{\mu}(m) = 0$$

on μ, which is satisfied for every positive integer l, for every α in $\mathbb{N}^l$ with $|\alpha| \le k$, and for every choice of the additive functions $a_1, a_2, \dots, a_l$, we infer that μ annihilates $\operatorname{Ann} M_m^{k+1}$. On the other hand, by Theorem 1.3 we have $\operatorname{Ann} \operatorname{Ann} M_m^{k+1} = M_m^{k+1}$; consequently, μ is in M_m^{k+1}, and hence $I_{k+1} \subseteq M_m^{k+1}$.

To prove the converse statement, suppose that the torsion-free rank of G is infinite. By the results in [11, Chapters 13, 15], there exists a generalized polynomial p of degree 2 on G which is not a polynomial. It means p is in $\operatorname{Ann} M_1^3$. Also, p is not in the closure of all functions of the form $g \mapsto P(a_1, a_2, \dots, a_k)$, where k is a positive integer, $P : \mathbb{C}^k \to \mathbb{C}$ is a polynomial of degree at most 2, and $a_j : G \to \mathbb{C}$ is an additive function. Hence, there exists a measure μ for which $P(a_1, a_2, \dots, a_k) * \mu = 0$ for each positive integer k, for every additive function $a_1, a_2, \dots, a_k$ on G, and for every polynomial $P : \mathbb{C}^k \to \mathbb{C}$ of degree at most 2, but it is not in M_1^3, where M_1 is the augmentation ideal. It follows that μ is in I_3 but it is not in M_1^3. □

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Department of Mathematics, University of Botswana, Gaborone, Private Bag 0022, Botswana and Institute of Mathematics, University of Debrecen, Debrecen, Pf. 12., Hungary.

E-mail address: lszekelyhidi@gmail.com