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SQUARE FUNCTIONS AND SPECTRAL MULTIPLIERS FOR BESSEL OPERATORS IN UMD SPACES

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Abstract. In this paper, we consider square functions (also called Littlewood–Paley g-functions) associated to Hankel convolutions acting on functions in the Bochner–Lebesgue space $L^p((0,\infty),\mathbb{B})$, where $\mathbb{B}$ is a UMD Banach space. As special cases, we study square functions defined by fractional derivatives of the Poisson semigroup for the Bessel operator $\Delta_\lambda = -x^{-\lambda}\frac{d}{dx}x^{2\lambda}\frac{d}{dx}x^{-\lambda}$, $\lambda > 0$. We characterize the UMD property for a Banach space $\mathbb{B}$ by using $L^p((0,\infty),\mathbb{B})$-boundedness properties of g-functions defined by Bessel–Poisson semigroups. As a by-product, we prove that the fact that the imaginary power $\Delta_\lambda^{i\omega}$, $\omega \in \mathbb{R}\setminus\{0\}$, of the Bessel operator Δ_λ is bounded in $L^p((0,\infty),\mathbb{B})$, $1 < p < \infty$, characterizes the UMD property for the Banach space $\mathbb{B}$. As applications of our results for square functions, we establish the boundedness in $L^p((0,\infty),\mathbb{B})$ of spectral multipliers $m(\Delta_\lambda)$ of Bessel operators defined by functions m which are holomorphic in sectors Σ_ϑ.

1. Introduction

Square functions (also called *Littlewood–Paley g-functions*) were considered in the works of Littlewood, Paley, Zygmund, and Marcinkiewicz during the decade of the 1930s (see [34], [55]). These functions were introduced to get new equivalent norms, for example, in L^p-spaces. By using these new equivalent norms the boundedness of some operators (e.g., multipliers) can be established.

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Suppose that (Ω, Σ, μ) is a σ-finite measure space and that $\{T_t\}_{t>0}$ is a symmetric diffusion semigroup of operators in the sense of Stein [45]. For every $k \in \mathbb{N}$, the square function g_k associated to $\{T_t\}_{t>0}$ is defined by

$$g_k\big(\{T_t\}_{t>0}\big)(f)(x) = \Big(\int_0^\infty \big|t^k \partial_t^k T_t(f)(x)\big|^2 \frac{dt}{t}\Big)^{1/2}.$$

In [45, p. 120] it is shown that, for every $k \in \mathbb{N}$ and $1 < p < \infty$, there exists $C > 0$ such that

$$\begin{aligned}\frac{1}{C}\big\|f - E_0(f)\big\|_{L^p(\Omega,\mu)} &\le \big\|g_k\big(\{T_t\}_{t>0}\big)(f)\big\|_{L^p(\Omega,\mu)} \\ &\le C\|f\|_{L^p(\Omega,\mu)}, \quad f \in L^p(\Omega, \mu), \end{aligned} \tag{1.1}$$

where $E_0(f) = \lim_{t\to\infty} T_t(f)$ is the projection onto the fixed-point space of $\{T_t\}_{t>0}$. As an application of (1.1), it can be proved that Laplace transform-type multipliers associated with $\{T_t\}_{t>0}$ are bounded from $L^p(\Omega, \mu)$ into itself, $1 < p < \infty$ (see [45, p. 121]). Note that when $E_0 = 0$, (1.1) says that by defining for every $1 < p < \infty$ and $k \in \mathbb{N}$,

$$|||f|||_k = \big\|g_k\big(\{T_t\}_{t>0}\big)(f)\big\|_{L^p(\Omega,\mu)}, \quad f \in L^p(\Omega, \mu),$$

$|||\cdot|||_k$ is a norm equivalent to the usual one in $L^p(\Omega, \mu)$. Meda in [37] extends the property (1.1) to symmetric contraction semigroups $\{T_t\}_{t>0}$ with $E_0 = 0$ and he applies it to get the boundedness in L^p of spectral multipliers $m(\mathcal{L})$, where the operator $\mathcal{L}$ is the infinitesimal generator of $\{T_t\}_{t>0}$ and m is a holomorphic and bounded function in a sector $\Sigma_\vartheta = \{z \in \mathbb{C} : |\operatorname{Arg} z| < \vartheta\}$. Here $1 < p < \infty$ and $\vartheta \in [0, \pi/2]$ are connected.

We consider the functions

$$\mathbb{W}_t(z) = \frac{e^{-|z|^2/4t}}{(4\pi t)^{n/2}}, \quad z \in \mathbb{R}^n \text{ and } t > 0, \tag{1.2}$$

and

$$\mathbb{P}_t(z) = b_n \frac{t}{(t^2 + |z|^2)^{(n+1)/2}}, \quad z \in \mathbb{R}^n \text{ and } t > 0,$$

where $b_n = \pi^{-(n+1)/2}\Gamma((n+1)/2)$.

As it is well known, the classical heat semigroup $\{\mathbb{W}_t\}_{t>0}$ is defined by

$$\mathbb{W}_t(f)(x) = \int_{\mathbb{R}^n} \mathbb{W}_t(x-y) f(y)\, dy, \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

and the classical Poisson semigroup $\{\mathbb{P}_t\}_{t>0}$ is given by

$$\mathbb{P}_t(f)(x) = \int_{\mathbb{R}^n} \mathbb{P}_t(x-y) f(y)\, dy, \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

for every $f \in L^p(\mathbb{R}^n)$. Note that $\{\mathbb{W}_t\}_{t>0}$ and $\{\mathbb{P}_t\}_{t>0}$ are generated by $-\Delta$ and $-\sqrt{\Delta}$, respectively, where $\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator. The classical heat and Poisson semigroups are the first examples of diffusion semigroups having a trivial fixed-point space. For every measurable function $g : \mathbb{R}^n \to \mathbb{C}$, we define $g_t(x) = t^{-n} g(x/t)$, $x \in \mathbb{R}^n$ and $t > 0$.

Let $k \in \mathbb{N}$. We can write

$$g_k\big(\{\mathbb{W}_t\}_{t>0}\big)(f)(x) = \Big(\int_0^\infty \big|(\varphi_{\sqrt{t}} * f)(x)\big|^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in \mathbb{R}^n,$$

being $\varphi(x) = \partial_t^k G_{\sqrt{t}}(x)_{|t=1}$ and $G(x) = (4\pi)^{-n/2} e^{-|x|^2/4}$, $x \in \mathbb{R}^n$. Also, we have

$$g_k\big(\{\mathbb{P}_t\}_{t>0}\big)(f)(x) = \Big(\int_0^\infty \big|(\phi_t * f)(x)\big|^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in \mathbb{R}^n,$$

where $\phi(x) = \partial_t^k \mathbb{P}_t(x)_{|t=1}$, $x \in \mathbb{R}^n$.

If $\psi \in L^2(\mathbb{R}^n)$, we consider the square function defined by

$$g_\psi(f)(x) = \Big(\int_0^\infty \big|(\psi_t * f)(x)\big|^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in \mathbb{R}^n.$$

Thus, g_ψ includes as special cases $g_k(\{\mathbb{W}_t\}_{t>0})$ and $g_k(\{\mathbb{P}_t\}_{t>0})$.

In the rest of this article, if $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz class, then we denote by $\widehat{f}$ the Fourier transform of f given by

$$\widehat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot y}\, dx, \quad y \in \mathbb{R}^n.$$

As it is well known, the Fourier transform can be extended to $L^2(\mathbb{R}^n)$ as an isometry in $L^2(\mathbb{R}^n)$.

Theorem A. *Suppose that $\psi \in L^2(\mathbb{R}^n)$ satisfies the following properties:*

(i) *if $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0,1\}^n$ and $|\alpha| = \sum_{j=1}^n \alpha_j \le 1 + [n/2]$, then the distributional derivative $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \widehat{\psi}$ is represented by a measurable function and*

$$\sup_{|z|=1} \int_0^\infty t^{2|\alpha|} \Big|\Big(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \widehat{\psi}\Big)(tz)\Big|^2 \frac{dt}{t} < \infty,$$

(ii) $\inf_{|z|=1} \int_0^\infty |\widehat{\psi}(tz)|^2 \frac{dt}{t} > 0$.

Then, for every $1 < p < \infty$, there exists $C > 0$ such that

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n)} \le \big\|g_\psi(f)\big\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

Also, square functions can be defined by using functional calculus for operators (see, for instance, [32], [38]). Note that if A is the infinitesimal generator of the analytic semigroup $\{T_t\}_{t>0}$, we can write, for every $k \in \mathbb{N}$,

$$t^k \partial_t^k T_t = F_k(tA), \quad t > 0,$$

where $F_k(z) = (-z)^k e^{-z}$, $z \in \mathbb{C}$.

Suppose that $\mathbb{B}$ is a Banach space and that T is a linear bounded operator from $L^p(\Omega, \mu)$ into itself, where $1 < p < \infty$. We define $T \otimes I_{\mathbb{B}}$ on $L^p(\Omega, \mu) \otimes \mathbb{B}$ in the usual way. If T is positive, $T \otimes I_{\mathbb{B}}$ can be extended to $L^p(\Omega, \mu, \mathbb{B})$ as a bounded operator from $L^p(\Omega, \mu, \mathbb{B})$ into itself. To simplify, we continue denoting this extension by T.

The objective is to give a definition for the square functions, when we consider functions taking values in a Banach space $\mathbb{B}$, which define equivalent norms in the Bochner–Lebesgue space $L^p(\Omega, \mu, \mathbb{B})$, $1 < p < \infty$.

Let $\{T_t\}_{t>0}$ be a symmetric diffusion semigroup on a σ-finite measure space (Ω, Σ, μ). We denote by $\{P_t\}_{t>0}$ the subordinated semigroup to $\{T_t\}_{t>0}$, that is,

$$P_t(f) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}} T_u(f)\, du, \quad t > 0.$$

Thus, $\{P_t\}_{t>0}$ is also a symmetric diffusion semigroup. The classical Poisson semigroup is the subordinated semigroup of the classical heat semigroup.

In order to define g-functions in a Banach-valued setting, the more natural way is to replace the absolute value in the scalar definition by the norm in $\mathbb{B}$. This is the way followed, for instance, in [36] and [53], where those authors work with square functions defined by subordinated semigroups $\{P_t\}_{t>0}$ as follows:

$$g_{1,\mathbb{B}}\big(\{P_t\}_{t>0}\big)(f)(x) = \Big(\int_0^\infty \big\|t\partial_t P_t(f)(x)\big\|_{\mathbb{B}}^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in \Omega.$$

Actually, in [36] and [53] the generalized square functions are considered where the L^2-norm is replaced by the L^q-norm, $1 < q < \infty$. As a consequence of [36, Theorems 5.2 and 5.3] (see also [30]), we deduce that for a certain $1 < p < \infty$ there exists $C > 0$ such that, for every $f \in L^p(\mathbb{R}^n, \mathbb{B})$,

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \big\|g_{1,\mathbb{B}}\big(\{\mathbb{P}_t\}_{t>0}\big)(f)\big\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})},$$

if and only if $\mathbb{B}$ is isomorphic to a Hilbert space.

In order to get new equivalent norms for $L^p(\Omega, \mu, \mathbb{B})$ by using square functions and for Banach spaces $\mathbb{B}$ which are not isomorphic to Hilbert spaces, stochastic integrals and γ-radonifying operators have been considered. We point out the work of Bourgain [11]; Hytönen [23]; Hytönen, Van Neerven and Portal [25]; Hytönen and Weis [26]; Kaiser [27]; and Kaiser and Weis [28] among others.

In the present article we use *γ-radonifying operators*. We recall some definitions and properties about this kind of operator that will be useful later.

Assume that $\mathbb{B}$ is a Banach space and that H is a Hilbert space. We choose a sequence $\{\gamma_j\}_{j\in\mathbb{N}}$ of independent standard Gaussian variables defined on some probability space $(\Omega, \mathcal{F}, \rho)$. By $\mathbb{E}$ we denote the expectation with respect to ρ. A linear operator $T : H \to \mathbb{B}$ is said to be *γ-summing* ($T \in \gamma^\infty(H, \mathbb{B})$) when

$$\|T\|_{\gamma^\infty(H,\mathbb{B})} = \sup\Big(\mathbb{E}\Big\|\sum_{j=1}^k \gamma_j T(h_j)\Big\|_{\mathbb{B}}^2\Big)^{1/2} < \infty,$$

where the supremum is taken over all the finite family $\{h_j\}_{j=1}^k$ of orthonormal vectors in H. Note that $\gamma^\infty(H, \mathbb{B})$ endowed with the norm $\|\cdot\|_{\gamma^\infty(H,\mathbb{B})}$ is a Banach space. We say that a linear operator $T : H \to \mathbb{B}$ is γ-radonifying (in short, $T \in \gamma(H, \mathbb{B})$) when $T \in \overline{\mathcal{F}(H, \mathbb{B})}^{\gamma^\infty(H,\mathbb{B})}$, where $\mathcal{F}(H, \mathbb{B})$ denotes the span of finite-range operators from H to $\mathbb{B}$. If $\mathbb{B}$ does not contain isomorphic copies of c_0, then $\gamma(H, \mathbb{B}) = \gamma^\infty(H, \mathbb{B})$ (see [22], [31], [50, Theorem 4.3]). Note that if $\mathbb{B}$ is

UMD, $\mathbb{B}$ does not contain isomorphic copies of c_0. If H is separable and $\{h_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H, a linear operator $T: H \to \mathbb{B}$ is γ-radonifying if and only if the series $\sum_{j=1}^{\infty} \gamma_j T(h_j)$ converges in $L^2(\Omega, \mathbb{B})$. In this case, we have

$$\|T\|_{\gamma^\infty(H,\mathbb{B})} = \left(\mathbb{E}\left\|\sum_{j=1}^{\infty} \gamma_j T(h_j)\right\|_{\mathbb{B}}^2\right)^{1/2}.$$

From this point on, we write $\|\cdot\|_{\gamma(H,\mathbb{B})}$ to refer to $\|\cdot\|_{\gamma^\infty(H,\mathbb{B})}$ when operators act on $\gamma(H,\mathbb{B})$.

Throughout this paper we always consider $H = L^2((0,\infty), dt/t)$. Suppose that $f: (0,\infty) \to \mathbb{B}$ is a measurable function such that $S \circ f \in H$, for every $S \in \mathbb{B}^*$, the dual space of $\mathbb{B}$. Then, there exists a bounded linear operator $T_f : H \to \mathbb{B}$ [shortly, $T_f \in L(H,\mathbb{B})$] such that

$$\big\langle S, T_f(h)\big\rangle_{\mathbb{B}^*,\mathbb{B}} = \int_0^\infty \big\langle S, f(t)\big\rangle_{\mathbb{B}^*,\mathbb{B}} h(t)\frac{dt}{t}, \quad h \in H \text{ and } S \in \mathbb{B}^*.$$

We say that $f \in \gamma((0,\infty), dt/t, \mathbb{B})$ provided that $T_f \in \gamma(H,\mathbb{B})$. The space $\{T_f : f \in \gamma((0,\infty), dt/t, \mathbb{B})\}$ is dense in $\gamma(H,\mathbb{B})$. It is usual to identify f and T_f.

Usually, γ-radonifying operators are considered for real Banach $\mathbb{B}$ and real Hilbert space H. However, as has been mentioned in [19], [29], and [50], the main properties of γ-radonifying operators (in particular, the properties we use in this article) also hold for complex Banach space $\mathbb{B}$ and complex Hilbert spaces H.

Banach spaces with the UMD property play an important role in our results. The Hilbert transform $\mathcal{H}(f)$ of $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, is defined by

$$\mathcal{H}(f)(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy, \quad \text{a.e. } x \in \mathbb{R}.$$

As it is well known, the Hilbert transform defines a bounded linear operator from $L^p(\mathbb{R})$ into itself, $1 < p < \infty$, and from $L^1(\mathbb{R})$ into $L^{1,\infty}(\mathbb{R})$; also, $\mathcal{H}$ is defined on $L^p(\mathbb{R}) \otimes \mathbb{B}$, $1 \le p < \infty$, in a natural way. We identify $\mathbb{B}$ as a UMD space when the Hilbert transform can be extended to $L^p(\mathbb{R},\mathbb{B})$ as a bounded operator from $L^p(\mathbb{R},\mathbb{B})$ into itself for some (equivalently, for every) $1 < p < \infty$. There exist a lot of characterizations of UMD Banach spaces. The articles of Bourgain [11] and Burkholder [12] have been fundamental in the development of the theory of Banach spaces with the UMD property. UMD Banach spaces are the suitable setting to analyze vector-valued Littlewood–Paley functions.

Kaiser and Weis [28] (see also [27]) considered, for every $\psi \in L^2(\mathbb{R}^n)$, the operator (usually called a *wavelet transform associated to* ψ) $\mathcal{W}_\psi$ defined by

$$\mathcal{W}_\psi(f)(t,x) = (\psi_t * f)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

for every $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{B})$, the $\mathbb{B}$-valued Schwartz space.

The following result was established by Kaiser and Weis.

Theorem B ([28, Theorem 4.2])**.** *Suppose that $\mathbb{B}$ is a UMD Banach space with Fourier type $r \in (1,2]$ and that $\psi \in L^2(\mathbb{R}^n)$ satisfies the following two conditions:*

(i) *If* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$ *and* $|\alpha| \leq 1 + [n/r]$, *then the distributional derivative* $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \widehat{\psi}$ *is represented by a measurable function and*

$$\sup_{|z|=1} \int_0^\infty t^{2|\alpha|} \Big| \Big(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \widehat{\psi} \Big)(tz) \Big|^2 \frac{dt}{t} < \infty;$$

(ii) $\inf_{|z|=1} \int_0^\infty |\widehat{\psi}(tz)|^2 \frac{dt}{t} > 0$.

Then, for every $1 < p < \infty$ *there exists* $C > 0$ *such that*

$$\frac{1}{C} \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})} \leq \big\| \mathcal{W}_\psi(f) \big\|_{L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))} \leq C \|f\|_{L^p(\mathbb{R}^n, \mathbb{B})}, \quad f \in \mathcal{S}(\mathbb{R}^n, \mathbb{B}).$$

Note that since $\gamma(H, \mathbb{C}) = H$, Theorem B can be seen as a vector-valued generalization of Theorem A. We recall that every UMD Banach space has Fourier type greater than 1 (see [10]) and the complex plane $\mathbb{C}$ has Fourier type 2.

Our objective here is to get new equivalent norms for $L^p((0,\infty), \mathbb{B})$ when $\mathbb{B}$ is a UMD Banach space by using square functions involving Hankel convolutions and Poisson semigroups associated with Bessel operators. These square functions allow us to obtain new characterizations of UMD Banach spaces. We also describe the UMD property by the boundedness in $L^p((0,\infty), \mathbb{B})$, $1 < p < \infty$, of the imaginary power $\Delta_\lambda^{i\omega}$, $\omega \in \mathbb{R} \setminus \{0\}$, of the Bessel operator $\Delta_\lambda = -x^{-\lambda} \frac{d}{dx} x^{2\lambda} \frac{d}{dx} x^{-\lambda}$, on $(0,\infty)$. As a consequence of our results about square functions in the Bessel setting, we obtain $L^p((0,\infty), \mathbb{B})$-boundedness properties for spectral multipliers associated with Bessel operators.

If J_ν denotes the Bessel function of the first kind and order $\nu > -1$, we have

$$\Delta_{\lambda,x}\big(\sqrt{xy} J_{\lambda-1/2}(xy)\big) = y^2 \sqrt{xy} J_{\lambda-1/2}(xy), \quad x, y \in (0,\infty). \tag{1.3}$$

Here and later, unless otherwise stated, we assume that $\lambda > 0$. The Hankel transform $h_\lambda(f)$ of $f \in L^1(0,\infty)$ is defined by

$$h_\lambda(f)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) f(y)\, dy, \quad x \in (0,\infty).$$

This transform plays in the Bessel setting the same role as the Fourier transformation in the classical (Laplacian) setting [see (1.3)].

We consider the space $\mathcal{S}_\lambda(0,\infty)$ of all those smooth functions ϕ on $(0,\infty)$ such that, for every $m, k \in \mathbb{N}$,

$$\eta_{m,k}^\lambda(\phi) = \sup_{x \in (0,\infty)} x^m \Big| \Big(\frac{1}{x} \frac{d}{dx} \Big)^k \big(x^{-\lambda} \phi(x) \big) \Big| < \infty.$$

If $\mathcal{S}_\lambda(0,\infty)$ is endowed with the topology generated by the family $\{\eta_{m,k}^\lambda\}_{m,k \in \mathbb{N}}$ of seminorms, then $\mathcal{S}_\lambda(0,\infty)$ is a Fréchet space and h_λ is an isomorphism on $\mathcal{S}_\lambda(0,\infty)$ (see [54, Lemma 8]). Moreover, $h_\lambda^{-1} = h_\lambda$ on $\mathcal{S}_\lambda(0,\infty)$. The Hankel transformation h_λ can be also extended to $L^2(0,\infty)$ as an isometry (see [48, p. 214 and Theorem 129]).

By adapting the results in [21], we define the Hankel convolution $f\#_\lambda g$ of $f,g\in L^1((0,\infty),x^\lambda dx)$ by

$$(f\#_\lambda g)(x)=\int_0^\infty f(y)\,{}_\lambda\tau_x(g)(y)\,dy,\quad x\in(0,\infty),$$

where the Hankel translation ${}_\lambda\tau_x(g)$ of g is given by

$${}_\lambda\tau_x(g)(y)=\frac{(xy)^\lambda}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\int_0^\pi(\sin\theta)^{2\lambda-1}\frac{g(\sqrt{(x-y)^2+2xy(1-\cos\theta)})}{((x-y)^2+2xy(1-\cos\theta))^{\lambda/2}}\,d\theta,$$

for each $x,y\in(0,\infty)$. Note that there is not a group operation $\circ$ on $(0,\infty)$ for which

$${}_\lambda\tau_x(g)(y)=g(x\circ y),\quad x,y\in(0,\infty).$$

The following interchange formula holds:

$$h_\lambda(f\#_\lambda g)=x^{-\lambda}h_\lambda(f)h_\lambda(g),\quad f,g\in L^1\big((0,\infty),x^\lambda\,dx\big).\tag{1.4}$$

If ψ is a measurable function on $(0,\infty)$, then we define

$$\psi_{(t)}(x)=\psi_{(t)}^\lambda(x)=\frac{1}{t^{\lambda+1}}\psi\Big(\frac{x}{t}\Big),\quad t,x\in(0,\infty).$$

If $\psi\in\mathcal{S}_\lambda(0,\infty)$ and $\mathbb{B}$ is a Banach space, then we define the operator (Hankel wavelet transform) $\mathcal{W}_{\psi,\mathbb{B}}^\lambda$ as

$$\mathcal{W}_{\psi,\mathbb{B}}^\lambda(f)(t,x)=(\psi_{(t)}\#_\lambda f)(x),\quad t,x\in(0,\infty),$$

for every $f\in L^p((0,\infty),\mathbb{B})$, $1<p<\infty$.

We establish in our first result a Hankel version of Theorem B.

Theorem 1.1. *Let $\mathbb{B}$ be a UMD Banach space, $\lambda>0$ and $1<p<\infty$. Suppose that $\psi\in\mathcal{S}_\lambda(0,\infty)$ is not identically zero and that $\int_0^\infty x^\lambda\psi(x)\,dx=0$. Then, there exists $C>0$ such that*

$$\frac{1}{C}\|f\|_{L^p((0,\infty),\mathbb{B})}\leq\big\|\mathcal{W}_{\psi,\mathbb{B}}^\lambda(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}\leq C\|f\|_{L^p((0,\infty),\mathbb{B})}$$

for every $f\in L^p((0,\infty),\mathbb{B})$.

Harmonic analysis associated with Bessel operators was first analyzed by Muckenhoupt and Stein [39]. Recently, that study has been completed (see [2], [5], [7]).

The Poisson semigroup $\{P_t^\lambda\}_{t>0}$ associated to the operator Δ_λ is defined as

$$P_t^\lambda(f)(x)=\int_0^\infty P_t^\lambda(x,y)f(y)\,dy,\quad t,x\in(0,\infty),$$

for every $f\in L^p(0,\infty)$, $1\leq p<\infty$. The Poisson kernel $P_t^\lambda(x,y)$, $t,x,y\in(0,\infty)$, is defined by (see [52])

$$P_t^\lambda(x,y)=\frac{2\lambda(xy)^\lambda t}{\pi}\int_0^\pi\frac{(\sin\theta)^{2\lambda-1}}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\lambda+1}}\,d\theta,$$
$$t,x,y\in(0,\infty).$$

For every $t > 0$, we can write

$$P_t^\lambda(f) = K_{(t)}^\lambda \#_\lambda f, \quad f \in L^p(0,\infty), 1 \leq p < \infty,$$

where

$$K^\lambda(x) = \frac{2^{\lambda+1/2}\Gamma(\lambda+1)}{\sqrt{\pi}} \frac{x^\lambda}{(1+x^2)^{\lambda+1}}, \quad x \in (0,\infty).$$

Then, for every $k \in \mathbb{N}$ and $f \in L^p(0,\infty)$, $1 < p < \infty$,

$$g_k\big(\{P_t^\lambda\}_{t>0}\big)(f)(x) = \Big(\int_0^\infty \big|(t^k \partial_t^k K_{(t)}^\lambda \#_\lambda f)(x)\big|^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in (0,\infty).$$

g-functions in the Bessel setting were studied in [8].

In [7] it was considered the square function defined by

$$g_{1,\mathbb{B}}\big(\{P_t^\lambda\}_{t>0}\big)(f)(x) = \Big(\int_0^\infty \big\|t\partial_t P_t^\lambda(f)(x)\big\|_{\mathbb{B}}^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in (0,\infty)$$

for every $f \in L^p((0,\infty),\mathbb{B})$, $1 < p < \infty$. According to [7, Theorems 2.4 and 2.5] and [30], we have that $\mathbb{B}$ is isomorphic to a Hilbert space if and only if, for some (equivalently, for every) $1 < p < \infty$, there exists $C > 0$ such that, for every $f \in L^p((0,\infty),\mathbb{B})$,

$$\frac{1}{C}\|f\|_{L^p((0,\infty),\mathbb{B})} \leq \big\|g_{1,\mathbb{B}}\big(\{P_t^\lambda\}_{t>0}\big)(f)\big\|_{L^p(0,\infty)} \leq C\|f\|_{L^p((0,\infty),\mathbb{B})}.$$

Note that the semigroup $\{P_t^\lambda\}_{t>0}$ is not Markovian. Hence, the results in [36] do not imply those in [7]. Also, the theory developed in [23] does not apply for the Bessel–Poisson semigroup.

In [44] Segovia and Wheeden defined a fractional derivative as follows. Suppose that $F : \Omega \times (0,\infty) \to \mathbb{C}$ is a good enough function, where $\Omega \subset \mathbb{R}^n$, and $\beta > 0$. The β-derivative $\partial_t^\beta F$ is defined by

$$\partial_t^\beta F(x,t) = \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m F(x,t+s) s^{m-\beta-1}\, ds, \quad x \in \Omega, t \in (0,\infty),$$

where $m \in \mathbb{N}$ and $m-1 \leq \beta < m$. By using this fractional derivative, Segovia and Wheeden obtained characterizations of Sobolev spaces.

If $\mathbb{B}$ is a Banach space and $\beta > 0$, we define the operator $G_{P,\mathbb{B}}^{\lambda,\beta}$ by

$$G_{P,\mathbb{B}}^{\lambda,\beta}(f)(x) = t^\beta \partial_t^\beta P_t^\lambda(f)(x), \quad t, x \in (0,\infty),$$

for every $f \in L^p((0,\infty),\mathbb{B})$, $1 < p < \infty$.

We now prove that the operators $G_{P,\mathbb{B}}^{\lambda,\beta}$ allow us to get new equivalent norms in $L^p((0,\infty),\mathbb{B})$ provided that $\mathbb{B}$ is a UMD space.

Theorem 1.2. *Let $\mathbb{B}$ be a UMD Banach space, $\lambda, \beta > 0$ and $1 < p < \infty$. Then, there exists $C > 0$ such that*

$$\frac{1}{C}\|f\|_{L^p((0,\infty),\mathbb{B})} \leq \big\|G_{P,\mathbb{B}}^{\lambda,\beta}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p((0,\infty),\mathbb{B})}, \tag{1.5}$$

for every $f \in L^p((0,\infty),\mathbb{B})$.

In [37, Theorem 1] it is assumed that the semigroup is contractive. As was mentioned earlier, the Poisson semigroup $\{P_t^\lambda\}_{t>0}$ associated with Δ_λ is not contractive when $0 < \lambda < 1$. A crucial point in the proof of [37, Theorem 1] is [37, Theorem 2], where the L^p-boundedness properties for the g-functions on contractive semigroups are stated. In order to prove Theorem 1.5 below, we will use Theorem 1.2, which holds in the noncontractive range $0 < \lambda < 1$.

For every $\lambda > 0$, the Poisson semigroup $\{P_t^\lambda\}_{t>0}$ is generated by $-\sqrt{\Delta_\lambda}$ in $L^p(0,\infty)$, $1 < p < \infty$. According to [41, Proposition 6.1], $\{P_t^\lambda\}_{t>0}$ is contractive in $L^p(0,\infty)$, $1 < p < \infty$, provided that $\lambda \geq 1$. Then, by [47, Theorem 6.1], equivalence (1.5) follows from [51, Proposition 2.16] (see also [35, Lemma 2.3]) when $\lambda \geq 1$, $\beta > 0$, $1 < p < \infty$, and $\mathbb{B}$ is a UMD Banach space. In Theorem 1.2 (1.5) is established for every $\lambda > 0$. Our proof (see Section 3) does not use functional calculus arguments. We exploit the fact that the Bessel operator Δ_λ is, in some sense, a nice perturbation of the Laplacian operator $-d^2/dx^2$. We connect the g-function operator $G_{P,\mathbb{B}}^{\lambda,\beta}$ with the corresponding operator associated with the classical Poisson semigroup and then we apply Theorem B.

We also consider square functions associated with Bessel–Poisson semigroups involving a derivative with respect to x. If $\mathbb{B}$ is a Banach space, we define, for every $f \in L^p((0,\infty),\mathbb{B})$, $1 < p < \infty$,

$$\mathcal{G}_{P,\mathbb{B}}^\lambda(f)(t,x) = tD_{\lambda,x}^* P_t^{\lambda+1}(f)(x), \quad x, t \in (0,\infty),$$

where $D_\lambda^* = -x^{-\lambda}\frac{d}{dx}x^\lambda$.

Theorem 1.3. *Let $\mathbb{B}$ be a UMD Banach space, $\lambda > 0$ and $1 < p < \infty$. Then the operator $\mathcal{G}_{P,\mathbb{B}}^\lambda$ is bounded from $L^p((0,\infty),\mathbb{B})$ into $L^p((0,\infty),\gamma(H,\mathbb{B}))$.*

The operators $G_{P,\mathbb{B}}^{\lambda,1}$ and $\mathcal{G}_{P,\mathbb{B}}^\lambda$ are connected by certain Cauchy–Riemann-type equations and Riesz transforms associated with Bessel operators. These relations allow us to get new characterizations of UMD Banach spaces. Also, the equivalence of L^p-norms in Theorem 1.2 characterizes UMD Banach spaces. In order to see this last property, we need first to describe UMD Banach spaces by using L^p-boundedness of the imaginary power $\Delta_\lambda^{i\omega}$, $\omega \in \mathbb{R} \setminus \{0\}$, of Bessel operators (see Proposition 5.1).

Theorem 1.4. *Let $\mathbb{B}$ be a Banach space and $\lambda > 0$. The following assertions are equivalent:*

(i) *$\mathbb{B}$ is UMD;*
(ii) *for some (equivalently, for every) $1 < p < \infty$, there exists $C > 0$ such that*

$$\frac{1}{C}\|f\|_{L^p((0,\infty),\mathbb{B})} \leq \left\|G_{P,\mathbb{B}}^{\lambda,1}(f)\right\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}, \quad f \in L^p(0,\infty) \otimes \mathbb{B}, \tag{1.6}$$

and

$$\left\|\mathcal{G}_{P,\mathbb{B}}^\lambda(f)\right\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p((0,\infty),\mathbb{B})}, \quad f \in L^p(0,\infty) \otimes \mathbb{B}; \tag{1.7}$$

(iii) *for some (equivalently, for every)* $1 < p < \infty$ *and* $\beta > 0$, *there exists* $C > 0$ *such that, for* $\delta = \beta$ *and* $\delta = \beta + 1$,

$$\frac{1}{C}\|f\|_{L^p((0,\infty),\mathbb{B})} \le \left\|G^{\lambda,\delta}_{P,\mathbb{B}}(f)\right\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))} \le C\|f\|_{L^p((0,\infty),\mathbb{B})}, \quad f \in L^p(0,\infty) \otimes \mathbb{B}. \tag{1.8}$$

Inspired in [37, Theorem 1] as an application of the result in Theorem 1.2, we give sufficient conditions in order that spectral multipliers associated with Bessel operators are bounded in $L^p((0,\infty),\mathbb{B})$, $1 < p < \infty$.

If $f \in \mathcal{S}_\lambda(0,\infty)$, then from (1.3) we deduce that

$$h_\lambda(\Delta_\lambda f)(x) = x^2 h_\lambda(f)(x), \quad x \in (0,\infty).$$

We define

$$\Delta_\lambda f = h_\lambda\big(x^2 h_\lambda(f)\big), \quad f \in D(\Delta_\lambda),$$

where the domain $D(\Delta_\lambda)$ of Δ_λ is

$$D(\Delta_\lambda) = \big\{f \in L^2(0,\infty) : x^2 h_\lambda(f) \in L^2(0,\infty)\big\}.$$

Suppose that $m \in L^\infty(0,\infty)$. The spectral multiplier $m(\Delta_\lambda)$ is defined by

$$m(\Delta_\lambda)(f) = h_\lambda\big(m(x^2)h_\lambda\big), \quad f \in L^2(0,\infty). \tag{1.9}$$

Since h_λ is bounded in $L^2(0,\infty)$, it is clear that $m(\Delta_\lambda)$ is bounded from $L^2(0,\infty)$ into itself. At this point, the question is to give conditions on the function m which imply that the operator $m(\Delta_\lambda)$ can be extended from $L^2(0,\infty) \cap L^p(0,\infty)$ to $L^p(0,\infty)$ as a bounded operator from $L^p(0,\infty)$ into itself for some $p \in (1,\infty) \setminus \{2\}$.

In [3] and [9] Laplace transform-type Hankel multipliers were investigated. A function m is said to be of *Laplace transform type* when

$$m(y) = y\int_0^\infty e^{-yt}\psi(t)\,dt, \quad y \in (0,\infty),$$

for some $\psi \in L^\infty(0,\infty)$. If m is of Laplace transform type, then the operator $m(\Delta_\lambda)$ defined in (1.9) can be extended to $L^p(0,\infty)$ as a bounded operator from $L^p(0,\infty)$ into itself, $1 < p < \infty$, and from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$ (see [3], [9], [45, p. 121]).

Let $\omega \in \mathbb{R} \setminus \{0\}$. The imaginary power $\Delta_\lambda^{i\omega}$ of Δ_λ is defined by

$$\Delta_\lambda^{i\omega}(f) = h_\lambda\big(y^{2i\omega}h_\lambda(f)\big), \quad f \in L^2(0,\infty).$$

Since

$$y^{i\omega} = y\int_0^\infty e^{-yt}\frac{t^{-i\omega}}{\Gamma(1-i\omega)}\,dt, \quad y \in (0,\infty),$$

the operator $\Delta_\lambda^{i\omega}$ is a Laplace transform-type Hankel multiplier.

In Proposition 5.1 (Section 5) we show that a Banach space $\mathbb{B}$ is UMD if and only if the operator $\Delta_\lambda^{i\omega}$, $\omega \in \mathbb{R}$, is a bounded operator from $L^p((0,\infty),\mathbb{B})$ into itself, for some (equivalently, for every) $1 < p < \infty$. This is a Bessel version of [18, Theorem, p. 402].

In the following theorem we establish a Banach-valued version of [37, Theorem 1] for the Bessel operator.

If $m \in L^\infty(0,\infty)$, we define, for every $n \in \mathbb{N}$,

$$m_n(t,y) = (ty)^n e^{-ty/2} m(y^2), \quad t, y \in (0,\infty),$$

and $\mathcal{M}_n(t,u)$, $t \in (0,\infty)$, $u \in \mathbb{R}$, represents the Mellin transform of m_n with respect to the variable y, that is,

$$\mathcal{M}_n(t,u) = \int_0^\infty m_n(t,y) y^{-iu-1}\, dy, \quad u \in \mathbb{R} \text{ and } t > 0.$$

Theorem 1.5. *Let $\mathbb{B}$ be a UMD Banach space, $\lambda > 0$ and $m \in L^\infty(0,\infty)$. Suppose that for some $1 < p < \infty$ and $n \in \mathbb{N}$ the following property holds:*

$$\int_{\mathbb{R}} \sup_{t>0} \big|\mathcal{M}_n(t,u)\big| \|\Delta_\lambda^{iu/2}\|_{L^p((0,\infty),\mathbb{B}) \to L^p((0,\infty),\mathbb{B})}\, du < \infty. \tag{1.10}$$

Then, $m(\Delta_\lambda)$ can be extended from $\mathcal{S}_\lambda(0,\infty) \otimes \mathbb{B}$ to $L^p((0,\infty),\mathbb{B})$ as a bounded operator from $L^p((0,\infty),\mathbb{B})$ into itself.

We now specify some conditions over the function m and the UMD Banach space $\mathbb{B}$ for which (1.10) is satisfied. As in [37, Theorem 3], we consider $m \in L^\infty(0,\infty)$ that extends to a bounded analytic function in a sector $\Sigma_\vartheta = \{z \in \mathbb{C} : |\operatorname{Arg} z| < \vartheta\}$. In this case, we have

$$\sup_{t>0} \big|\mathcal{M}_n(t,u)\big| \le C e^{\pi|u|/2}\big(1 + |u|\big), \quad u \in \mathbb{R}.$$

By [13, Corollary 1] (see also [3, Corollary 1.2]), we can obtain, for every $1 < p < \infty$,

$$\begin{aligned} &\|\Delta_\lambda^{iu}\|_{L^p(0,\infty)\to L^p(0,\infty)} \\ &\quad \le C\big(1 + |u|^3 \log |u|\big)^{|1/p-1/2|} \exp\big(\pi|1/p - 1/2||u|\big), \quad u \in \mathbb{R}, \end{aligned} \tag{1.11}$$

where $C > 0$ depends on p but does not depend on u.

Even when we consider the usual Laplacian operator instead of the Bessel operator Δ_λ, it is not known if (1.11) holds when the functions take values in a UMD Banach space (see, for instance, [46, Corollary 2.5.3]). In order to get an estimate as (1.11), replacing $L^p(0,\infty)$ by $L^p((0,\infty),\mathbb{B})$, we need to strengthen the property of the Banach spaces as follows. $\mathbb{B}$ must be isomorphic to a closed subspace of a complex interpolation space $[\mathbb{H}, X]_\theta$, where $0 < \theta < 1$, $\mathbb{H}$ is a Hilbert space, and X is a UMD Banach space. When $\mathbb{B}$ satisfies this property for some $\theta \in (0,1)$, we write $\mathbb{B} \in I_\theta(\mathfrak{H}, \mathit{UMD})$. The class $\bigcup_{\theta\in(0,1)} I_\theta(\mathfrak{H}, \mathit{UMD})$ includes all UMD lattices ([42, Corollary on p. 216]) and it also includes the Schatten ideals C_p, $p \in (1,\infty)$ (see [14]). It is clear that $\mathbb{B}$ is UMD provided that $\mathbb{B} \in \bigcup_{\theta\in(0,1)} I_\theta(\mathfrak{H}, \mathit{UMD})$.

As far as it is known, it is an open problem whether every UMD Banach space is in $\bigcup_{\theta\in(0,1)} I_\theta(\mathfrak{H}, \mathit{UMD})$ ([42, Problem 4 on p. 220]). This class of Banach spaces has been used, for instance, in [23], [36], and [46], and also it plays a central role in the vector-valued version of Carleson's theorem recently established in [24].

Theorem 1.6. *Let $\lambda > 0$. Suppose that m is a bounded holomorphic function in Σ_ϑ, for certain $\vartheta \in (0,\pi)$, and suppose that the Banach space $\mathbb{B}$ is in $I_\theta(\mathfrak{H}, UMD)$, for some $\theta \in (0, \vartheta/\pi)$. Then, the spectral multiplier $m(\Delta_\lambda)$ can be extended to $L^q((0,\infty),\mathbb{B})$ as a bounded operator from $L^q((0,\infty),\mathbb{B})$ into itself, for every $q \in [2/(1+\theta), 2/(1-\theta)]$.*

In the following sections, we present proofs for our theorems. Throughout this paper C and c always denote positive constants, not necessarily the same in each occurrence.

2. Proof of Theorem 1.1

2.1. First we prove that there exists $C > 0$ such that

$$\big\|\mathcal{W}^\lambda_{\psi,\mathbb{B}}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))} \le C\|f\|_{L^p((0,\infty),\mathbb{B})}, \tag{2.1}$$

for every $f \in L^p((0,\infty),\mathbb{B})$.

We choose $\phi \in \mathcal{S}(\mathbb{R})$ such that $\phi(x^2) = x^{-\lambda}\psi(x)$, $x \in (0,\infty)$ (see [49, p. 85]). Then, we can write, for each $t, x, y \in (0,\infty)$,

$$\begin{aligned}
{}_\lambda\tau_x(\psi_{(t)})(y) &= \frac{(xy)^\lambda t^{-\lambda-1}}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\int_0^\pi \psi\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\\
&\quad\times\big((x-y)^2+2xy(1-\cos\theta)\big)^{-\lambda/2}(\sin\theta)^{2\lambda-1}\,d\theta\\
&= \frac{(xy)^\lambda t^{-2\lambda-1}}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\int_0^\pi (\sin\theta)^{2\lambda-1}\phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\Big)\,d\theta.
\end{aligned}$$

We define the function Φ as follows:

$$\Phi(x) = \frac{1}{\sqrt{\pi}2^{\lambda+1/2}\Gamma(\lambda)}\int_0^\infty u^{\lambda-1}\phi(x^2+u)\,du, \quad x \in \mathbb{R}.$$

It is not hard to see that $\Phi \in \mathcal{S}(\mathbb{R})$. Hence, since $\widehat{\Phi}(0) = 0$ (see [5, (17)]), Φ satisfies conditions (C1) and (C2) in [28, p. 111] [(i) and (ii) in Theorem A].

We consider the operator

$$\mathcal{W}_{\Phi,\mathbb{B}}(f)(t,x) = (\Phi_t * f)(x), \quad f \in L^p(\mathbb{R},\mathbb{B}), t \in (0,\infty), \text{ and } x \in \mathbb{R}.$$

According to [28, Theorem 4.2] (Theorem B), we have that, for every $f \in \mathcal{S}(\mathbb{R}) \otimes \mathbb{B}$,

$$\big\|\mathcal{W}_{\Phi,\mathbb{B}}(f)\big\|_{L^p(\mathbb{R},\gamma(H,\mathbb{B}))} \le C\|f\|_{L^p(\mathbb{R},\mathbb{B})}. \tag{2.2}$$

We are going too see that the inequality (2.2) holds for every $f \in L^p(\mathbb{R},\mathbb{B})$. Let $f \in L^p(\mathbb{R},\mathbb{B})$. We choose a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}) \otimes \mathbb{B}$ such that $f_n \to f$, as $n \to \infty$, in $L^p(\mathbb{R},\mathbb{B})$. According to (2.2), by defining

$$\widetilde{\mathcal{W}}_{\Phi,\mathbb{B}}(f) = \lim_{n\to\infty}\mathcal{W}_{\Phi,\mathbb{B}}(f_n),$$

where the limit is understood in $L^p(\mathbb{R},\gamma(H,\mathbb{B}))$, we have that

$$\big\|\widetilde{\mathcal{W}}_{\Phi,\mathbb{B}}(f)\big\|_{L^p(\mathbb{R},\gamma(H,\mathbb{B}))} \le C\|f\|_{L^p(\mathbb{R},\mathbb{B})}.$$

Also, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and a subset Ω of $\mathbb{R}$ such that

$$\widetilde{\mathcal{W}}_{\Phi,\mathbb{B}}(f)(x) = \lim_{k\to\infty} \mathcal{W}_{\Phi,\mathbb{B}}(f_{n_k})(\cdot,x), \quad x \in \Omega,$$

where the limit is understood in $\gamma(H,\mathbb{B})$, and $|\mathbb{R}\setminus\Omega| = 0$.

For every $\varepsilon > 0$,

$$\mathcal{W}_{\Phi,\mathbb{B}}(f_n)(\cdot,x) \to \mathcal{W}_{\Phi,\mathbb{B}}(f)(\cdot,x), \quad \text{as } n\to\infty, \text{ in } L^2\big((\varepsilon,\infty), dt/t, \mathbb{B}\big),$$

uniformly in $x\in\mathbb{R}$. Indeed, let $\varepsilon > 0$. By using Minkowski's inequality we get

$$\begin{aligned}
&\Big(\int_\varepsilon^\infty \big\|\mathcal{W}_{\Phi,\mathbb{B}}(f_n)(t,x) - \mathcal{W}_{\Phi,\mathbb{B}}(f)(t,x)\big\|_{\mathbb{B}}^2 \frac{dt}{t}\Big)^{1/2} \\
&\quad\le \int_{\mathbb{R}} \big\|f_n(y) - f(y)\big\|_{\mathbb{B}} \Big(\int_\varepsilon^\infty \frac{1}{t^3}\Big|\Phi\Big(\frac{|x-y|}{t}\Big)\Big|^2 dt\Big)^{1/2} dy \\
&\quad\le C\int_{\mathbb{R}} \big\|f_n(y) - f(y)\big\|_{\mathbb{B}} \Big(\int_\varepsilon^\infty \frac{1}{(t+|x-y|)^3}\, dt\Big)^{1/2} dy \\
&\quad\le C\int_{\mathbb{R}} \frac{\|f_n(y)-f(y)\|_{\mathbb{B}}}{\varepsilon+|x-y|}\, dy \le C\|f_n - f\|_{L^p(\mathbb{R},\mathbb{B})}\Big(\int_{\mathbb{R}} \frac{dy}{(\varepsilon+|y|)^{p'}}\Big)^{1/p'} \\
&\quad\le C\varepsilon^{-1/p}\|f_n - f\|_{L^p(\mathbb{R},\mathbb{B})}, \quad n\in\mathbb{N} \text{ and } x\in\mathbb{R},
\end{aligned}$$

where p' is the conjugated exponent of p, that is, $p' = p/(p-1)$.

Let $S\in\mathbb{B}^*$. Since $\gamma(H,\mathbb{B})$ is continuously contained in the space $L(H,\mathbb{B})$ of linear bounded operators from H into $\mathbb{B}$, for every $x\in\Omega$ and $h\in L^2((0,\infty),dt/t)$ with $\operatorname{supp}(h)\subset(0,\infty)$, we have that

$$\begin{aligned}
\big\langle S, \big[\widetilde{\mathcal{W}}_{\Phi,\mathbb{B}}(f)(x)\big](h)\big\rangle_{\mathbb{B}^*,\mathbb{B}} &= \lim_{k\to\infty}\big\langle S, \big[\mathcal{W}_{\Phi,\mathbb{B}}(f_{n_k})(\cdot,x)\big](h)\big\rangle_{\mathbb{B}^*,\mathbb{B}} \\
&= \lim_{k\to\infty}\int_0^\infty \big\langle S, \mathcal{W}_{\Phi,\mathbb{B}}(f_{n_k})(t,x)\big\rangle_{\mathbb{B}^*,\mathbb{B}} h(t)\frac{dt}{t} \\
&= \int_0^\infty \big\langle S, \mathcal{W}_{\Phi,\mathbb{B}}(f)(t,x)\big\rangle_{\mathbb{B}^*,\mathbb{B}} h(t)\frac{dt}{t}.
\end{aligned}$$

Hence, for every $x\in\Omega$, $\langle S, \mathcal{W}_{\Phi,\mathbb{B}}(f)(\cdot,x)\rangle_{\mathbb{B}^*,\mathbb{B}} \in L^2((0,\infty),dt/t)$ and

$$\big\langle S, \big[\widetilde{\mathcal{W}}_{\Phi,\mathbb{B}}(f)(x)\big](h)\big\rangle_{\mathbb{B}^*,\mathbb{B}} = \int_0^\infty \big\langle S, \mathcal{W}_{\Phi,\mathbb{B}}(f)(t,x)\big\rangle_{\mathbb{B}^*,\mathbb{B}} h(t)\frac{dt}{t}, \quad h\in H.$$

We conclude that $\widetilde{\mathcal{W}}_{\Phi,\mathbb{B}}(f)(x) = \mathcal{W}_{\Phi,\mathbb{B}}(f)(\cdot,x)$, $x\in\Omega$, as elements of $\gamma(H,\mathbb{B})$. Hence, (2.2) holds for every $f\in L^p(\mathbb{R},\mathbb{B})$.

Suppose now that $f\in L^p((0,\infty),\mathbb{B})$. By defining the function f_o as the odd extension of f to $\mathbb{R}$, we have that

$$\begin{aligned}
\mathcal{W}_{\Phi,\mathbb{B}}(f_o)(t,x) &= \frac{1}{t}\int_{-\infty}^{+\infty} \Phi\Big(\frac{x-y}{t}\Big) f_o(y)\, dy \\
&= -\frac{1}{t}\int_0^\infty \Phi\Big(\frac{x+y}{t}\Big) f(y)\, dy + \frac{1}{t}\int_0^\infty \Phi\Big(\frac{x-y}{t}\Big) f(y)\, dy \\
&= L^1_{\Phi,\mathbb{B}}(f)(t,x) + L^2_{\Phi,\mathbb{B}}(f)(t,x), \quad x\in\mathbb{R} \text{ and } t\in(0,\infty).
\end{aligned}$$

Since $\mathbb{B}$ is UMD, by [50, Theorem 4.3], we can write, for each $x \in (0,\infty)$,

$$\big\|L^1_{\Phi,\mathbb{B}}(f)(\cdot,x)\big\|_{\gamma(H,\mathbb{B})} = \sup\Big(E\Big\|\sum_{j=1}^k \gamma_j \int_0^\infty L^1_{\Phi,\mathbb{B}}(f)(t,x)h_j(t)\frac{dt}{t}\Big\|_{\mathbb{B}}^2\Big)^{1/2},$$

where the supremum is taken over all the finite family $\{h_j\}_{j=1}^k$ of orthonormal elements of H. Let $x \in (0,\infty)$. Assume that $\{h_j\}_{j=1}^k$ is a set of orthonormal functions in H. We have that

$$\begin{aligned}\sum_{j=1}^k &\gamma_j \int_0^\infty L^1_{\Phi,\mathbb{B}}(f)(t,x)h_j(t)\frac{dt}{t}\\ &= \sum_{j=1}^k \gamma_j \int_0^\infty \frac{1}{t}\int_0^\infty \Phi\Big(\frac{x+y}{t}\Big)f(y)\,dy h_j(t)\frac{dt}{t}\\ &= \int_0^\infty f(y)\sum_{j=1}^k \gamma_j \int_0^\infty \frac{1}{t}\Phi\Big(\frac{x+y}{t}\Big)h_j(t)\frac{dt}{t}\,dy.\end{aligned}$$

The interchange of the order of integration is justified because

$$\begin{aligned}\int_0^\infty &\|f(y)\|_{\mathbb{B}}\int_0^\infty \Big|\frac{1}{t}\Phi\Big(\frac{x+y}{t}\Big)h_j(t)\Big|\frac{dt}{t}\,dy\\ &\leq \int_0^\infty \|f(y)\|_{\mathbb{B}}\Big(\int_0^\infty \frac{1}{t^3}\Big|\Phi\Big(\frac{x+y}{t}\Big)\Big|\Big)\,dt)^{1/2}\,dy\\ &\leq C\int_0^\infty \frac{\|f(y)\|_{\mathbb{B}}}{x+y}\,dy \leq \|f\|_{L^p((0,\infty),\mathbb{B})}\Big(\int_0^\infty \frac{dy}{(x+y)^{p'}}\,dy\Big)^{1/p'} < \infty,\end{aligned}$$

where $p' = p/(p-1)$.

Then, we obtain, by using Minkowski's inequality,

$$\begin{aligned}\Big(E\Big\|&\sum_{j=1}^k \gamma_j \int_0^\infty L^1_{\Phi,\mathbb{B}}(f)(t,x)h_j(t)\frac{dt}{t}\Big\|_{\mathbb{B}}^2\Big)^{1/2}\\ &= \Big(E\Big\|\int_0^\infty f(y)\sum_{j=1}^k \gamma_j \int_0^\infty \frac{1}{t}\Phi\Big(\frac{x+y}{t}\Big)h_j(t)\frac{dt}{t}\,dy\Big\|_{\mathbb{B}}^2\Big)^{1/2}\\ &\leq \Big(E\Big(\int_0^\infty \|f(y)\|_{\mathbb{B}}\Big|\sum_{j=1}^k \gamma_j \int_0^\infty \frac{1}{t}\Phi\Big(\frac{x+y}{t}\Big)h_j(t)\frac{dt}{t}\Big|dy\Big)^2\Big)^{1/2}\\ &\leq \int_0^\infty \|f(y)\|_{\mathbb{B}}\Big(E\Big|\sum_{j=1}^k \gamma_j \int_0^\infty \frac{1}{t}\Phi\Big(\frac{x+y}{t}\Big)h_j(t)\frac{dt}{t}\Big|^2\Big)^{1/2}dy\\ &\leq \int_0^\infty \|f(y)\|_{\mathbb{B}}\Big(\int_0^\infty \frac{1}{t^3}\Big|\Phi\Big(\frac{x+y}{t}\Big)\Big|^2\,dt\Big)^{1/2}\,dy.\end{aligned}$$

In the last inequality, we have taken into account that $\gamma(H,\mathbb{C}) = H$.

Hence, it follows that

$$\begin{aligned}\big\|L^1_{\Phi,\mathbb{B}}(f)(\cdot,x)\big\|_{\gamma(H,\mathbb{B})} &\leq C\big\|f(y)\big\|_{\mathbb{B}}\Big(\int_0^\infty \frac{1}{t^3}\Big|\Phi\Big(\frac{x+y}{t}\Big)\Big|^2\,dt\Big)^{1/2}\,dy\\ &\leq C\int_0^\infty \frac{\|f(y)\|_{\mathbb{B}}}{x+y}\,dy \leq C\big(H_0\big(\|f\|_{\mathbb{B}}\big)(x)+H_\infty\big(\|f\|_{\mathbb{B}}\big)(x)\big),\end{aligned}$$

where H_0 and H_∞ denote the Hardy operators defined by

$$H_0(g)(z)=\frac{1}{z}\int_0^z g(y)\,dy,\quad z\in(0,\infty),$$

and

$$H_\infty(g)(z)=\int_z^\infty \frac{g(y)}{y}\,dy,\quad z\in(0,\infty).$$

Since H_0 and H_∞ are bounded operators from $L^p(0,\infty)$ into itself (see [20, p. 244, (9.9.1) and (9.9.2)]), $L^1_{\Phi,\mathbb{B}}$ is a bounded operator from $L^p((0,\infty),\mathbb{B})$ into the space $L^p((0,\infty),\gamma(H,\mathbb{B}))$.

Inequality (2.1) will be proved once we establish that

$$\big\|[\mathcal{W}^\lambda_{\psi,\mathbb{B}}-L^2_{\Phi,\mathbb{B}}](f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}\leq C\|f\|_{L^p((0,\infty),\mathbb{B})},\quad f\in L^p\big((0,\infty),\mathbb{B}\big). \tag{2.3}$$

In order to do this, we study the function

$$K_\lambda(t,x,y)={}_\lambda\tau_x(\psi_{(t)})(y)-\Phi_t(x-y),\quad t,x,y\in(0,\infty).$$

First we write

$${}_\lambda\tau_x(\psi_{(t)})(y)=H_{\lambda,1}(t,x,y)+H_{\lambda,2}(t,x,y),\quad t,x,y\in(0,\infty),$$

where, for every $t,x,y\in(0,\infty)$,

$$H_{\lambda,1}(t,x,y)=\frac{(xy)^\lambda t^{-2\lambda-1}}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\int_0^{\pi/2}(\sin\theta)^{2\lambda-1}\phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\Big)\,d\theta.$$

We have that, for every $x,y\in(0,\infty)$,

$$\begin{aligned}&\big\|H_{\lambda,2}(\cdot,x,y)\big\|_H\\ &\quad\leq C(xy)^\lambda\\ &\qquad\times\Big(\int_0^\infty t^{-4\lambda-3}\Big(\int_{\pi/2}^\pi(\sin\theta)^{2\lambda-1}\Big|\phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\Big)\Big|\,d\theta\Big)^2\,dt\Big)^{1/2}\\ &\quad\leq C(xy)^\lambda\begin{cases}\frac{1}{|x-y|^{2\lambda+1}}(\int_0^\infty u^{-4\lambda-3}(\int_{\pi/2}^\pi(\sin\theta)^{2\lambda-1}\\ \quad\times|\phi(\frac{(x-y)^2+2xy(1-\cos\theta)}{(x-y)^2u^2})|\,d\theta)^2\,du)^{1/2},\quad y\notin(\frac{x}{2},2x),\\ \frac{1}{(xy)^{\lambda+1/2}}(\int_0^\infty u^{-4\lambda-3}(\int_{\pi/2}^\pi(\sin\theta)^{2\lambda-1}\\ \quad\times|\phi(\frac{(x-y)^2+2xy(1-\cos\theta)}{xyu^2})|\,d\theta)^2\,du)^{1/2},\quad y\in(\frac{x}{2},2x).\end{cases}\end{aligned}$$

Then, since $\phi \in \mathcal{S}(\mathbb{R})$, it follows that

$$\begin{aligned}\big\|H_{\lambda,2}(\cdot,x,y)\big\|_H &\leq C\frac{(xy)^\lambda}{|x-y|^{2\lambda+1}}\Big(\int_1^\infty u^{-4\lambda-3}\,du+\int_0^1 du\Big)^{1/2}\\ &\leq C\begin{cases}\frac{1}{x}, & 0<y<x/2,\\ \frac{1}{y}, & y>2x>0,\end{cases}\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}\big\|H_{\lambda,2}(\cdot,x,y)\big\|_H &\leq C\frac{1}{(xy)^{1/2}}\Big(\int_1^\infty u^{-4\lambda-3}\,du+\int_0^1 du\Big)^{1/2}\\ &\leq \frac{C}{x},\quad y\in\Big(\frac{x}{2},2x\Big).\end{aligned}\tag{2.5}$$

By proceeding in a similar way we can see that

$$\big\|H_{\lambda,1}(\cdot,x,y)\big\|_H \leq C\begin{cases}\frac{1}{x}, & 0<y<\frac{x}{2},\\ \frac{1}{y}, & y>2x>0,\end{cases}\tag{2.6}$$

and also that

$$\big\|\Phi_t(x-y)\big\|_H \leq \frac{C}{|x-y|}\leq C\begin{cases}\frac{1}{x}, & 0<y<x/2,\\ \frac{1}{y}, & y>2x,\end{cases}\qquad x\in(0,\infty).\tag{2.7}$$

Suppose now that $x\in(0,\infty)$ and that $x/2<y<2x$. We split the difference $H_{\lambda,1}(t,x,y)-\Phi_t(x-y)$, $t\in(0,\infty)$, as follows:

$$\begin{aligned}&H_{\lambda,1}(t,x,y)-\Phi_t(x-y)\\ &\quad=\frac{(xy)^\lambda t^{-2\lambda-1}}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\int_0^{\pi/2}\big[(\sin\theta)^{2\lambda-1}-\theta^{2\lambda-1}\big]\phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\Big)\,d\theta\\ &\qquad+\frac{(xy)^\lambda t^{-2\lambda-1}}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\\ &\qquad\times\int_0^{\pi/2}\theta^{2\lambda-1}\Big[\phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\Big)-\phi\Big(\frac{(x-y)^2+xy\theta^2}{t^2}\Big)\Big]\,d\theta\\ &\qquad+\frac{(xy)^\lambda t^{-2\lambda-1}}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\int_0^{\pi/2}\theta^{2\lambda-1}\phi\Big(\frac{(x-y)^2+xy\theta^2}{t^2}\Big)\,d\theta-\Phi_t(x-y)\\ &\quad=H_{\lambda,1,1}(t,x,y)+H_{\lambda,1,2}(t,x,y)+H_{\lambda,1,3}(t,x,y),\quad t,x,y\in(0,\infty).\end{aligned}$$

By using the mean-value theorem, we get

$$\begin{aligned}&\big\|H_{\lambda,1,1}(\cdot,x,y)\big\|_H\\ &\quad\leq C(xy)^\lambda\Big\{\int_0^\infty\Big(\int_0^{\pi/2}\theta^{2\lambda+1}\Big|\phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\Big)\Big|\,d\theta\Big)^2\frac{dt}{t^{4\lambda+3}}\Big\}^{1/2}\\ &\quad\leq\frac{C}{(xy)^{1/2}}\Big\{\int_0^\infty\Big(\int_0^{\pi/2}\theta^{2\lambda+1}\Big|\phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{u^2xy}\Big)\Big|\,d\theta\Big)^2\frac{du}{u^{4\lambda+3}}\Big\}^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{(xy)^{1/2}}\Big\{\int_1^\infty \frac{du}{u^{4\lambda+3}}\\
&\quad+\int_0^1\Big(\int_0^{\pi/2}\theta^{2\lambda+1}\Big(\frac{u^2xy}{(x-y)^2+xy\theta^2}\Big)^{\lambda+3/4}d\theta\Big)^2\frac{du}{u^{4\lambda+3}}\Big\}^{1/2}\\
&\leq \frac{C}{x},
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|H_{\lambda,1,2}(\cdot,x,y)\big\|_H\\
&\quad\leq \frac{C}{(xy)^{1/2}}\Big\{\int_0^\infty\Big(\int_0^{\pi/2}\theta^{2\lambda-1}\Big|\int_{1-\cos\theta}^{\theta^2/2}\phi'\Big(\frac{(x-y)^2+2xyz}{u^2xy}\Big)\,dz\Big|\,d\theta\Big)^2\frac{du}{u^{4\lambda+7}}\Big\}^{1/2}\\
&\quad\leq \frac{C}{(xy)^{1/2}}\Big\{\int_1^\infty\frac{du}{u^{4\lambda+7}}\\
&\qquad+\int_0^1\Big(\int_0^{\pi/2}\theta^{2\lambda-1}\int_{1-\cos\theta}^{\theta^2/2}\Big(\frac{u^2xy}{(x-y)^2+2xyz}\Big)^{\lambda+7/4}dz\,d\theta\Big)^2\frac{du}{u^{4\lambda+7}}\Big\}^{1/2}\\
&\quad\leq \frac{C}{(xy)^{1/2}}\Big\{1+\int_0^1\Big(\int_0^{\pi/2}\theta^{2\lambda-1}\Big|\int_{1-\cos\theta}^{\theta^2/2}\frac{dz}{z^{\lambda+7/4}}\Big|\,d\theta\Big)^2du\Big\}^{1/2}\leq\frac{C}{x}.
\end{aligned}
$$

On the other hand, a suitable change of variables allows us to write

$$
\begin{aligned}
H_{\lambda,1,3}(t,x,y)&=\frac{(xy)^\lambda t^{-2\lambda-1}}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\int_0^{\pi/2}\theta^{2\lambda-1}\phi\Big(\frac{(x-y)^2+xy\theta^2}{t^2}\Big)\,d\theta\\
&\qquad-\frac{1}{t\sqrt{\pi}2^{\lambda+1/2}\Gamma(\lambda)}\int_0^\infty u^{\lambda-1}\phi\Big(\Big(\frac{x-y}{t}\Big)^2+u\Big)\,du\\
&=-\frac{(xy)^\lambda t^{-2\lambda-1}}{\sqrt{\pi}2^{\lambda-1/2}\Gamma(\lambda)}\int_{\pi/2}^\infty\theta^{2\lambda-1}\phi\Big(\frac{(x-y)^2+xy\theta^2}{t^2}\Big)\,d\theta,\quad t>0.
\end{aligned}
$$

Hence, we deduce that

$$
\begin{aligned}
&\big\|H_{\lambda,1,3}(\cdot,x,y)\big\|_H\\
&\quad\leq \frac{C}{(xy)^{1/2}}\Big\{\int_0^\infty\Big(\int_{\pi/2}^\infty\theta^{2\lambda-1}\Big|\phi\Big(\frac{(x-y)^2+xy\theta^2}{xyu^2}\Big)\Big|\,d\theta\Big)^2\frac{du}{u^{4\lambda+3}}\Big\}^{1/2}\\
&\quad\leq \frac{C}{(xy)^{1/2}}\Big\{\int_1^\infty\Big(\int_{\pi/2}^\infty\theta^{2\lambda-1}\Big(\frac{xyu^2}{(x-y)^2+xy\theta^2}\Big)^{\lambda+1/4}d\theta\Big)^2\frac{du}{u^{4\lambda+3}}\\
&\qquad+\int_0^1\Big(\int_{\pi/2}^\infty\theta^{2\lambda-1}\Big(\frac{xyu^2}{(x-y)^2+xy\theta^2}\Big)^{\lambda+3/4}d\theta\Big)^2\frac{du}{u^{4\lambda+3}}\Big\}^{1/2}\\
&\quad\leq\frac{C}{x}.
\end{aligned}
$$

By putting together the above estimates we obtain

$$
\big\|H_{\lambda,1}(\cdot,x,y)-\Phi_t(x-y)\big\|_H\leq\frac{C}{x},\quad 0<\frac{x}{2}<y<2x. \tag{2.8}
$$

From (2.4)–(2.8) we deduce that

$$\left\|K_\lambda(\cdot, x, y)\right\|_H \leq \frac{C}{\max\{x, y\}}, \quad x, y \in (0, \infty). \tag{2.9}$$

By proceeding as in the case of $L^1_{\Phi,\mathbb{B}}$, since $\gamma(H, \mathbb{C}) = H$, we infer from (2.9) that the operator $\mathcal{W}^\lambda_{\psi,\mathbb{B}} - L^2_{\Phi,\mathbb{B}}$ is bounded from $L^p((0,\infty), \mathbb{B})$ into $L^p((0,\infty), \gamma(H, \mathbb{B}))$. Thus, (2.3) is established.

2.2. Our next objective is to show that there exists $C > 0$ such that

$$\|f\|_{L^p((0,\infty),\mathbb{B})} \leq C\left\|\mathcal{W}^\lambda_{\psi,\mathbb{B}}(f)\right\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}, \tag{2.10}$$

for every $f \in L^p((0,\infty), \mathbb{B})$. It is enough to see (2.10) for every $f \in \mathcal{S}_\lambda(0,\infty) \otimes \mathbb{B}$. Indeed, suppose that (2.10) is true for every $f \in \mathcal{S}_\lambda(0,\infty) \otimes \mathbb{B}$. Let $f \in L^p((0,\infty), \mathbb{B})$. We choose a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}_\lambda(0,\infty) \otimes \mathbb{B}$ such that $f_n \to f$, as $n \to \infty$, in $L^p((0,\infty), \mathbb{B})$. Then, by (2.10)

$$\|f_n\|_{L^p((0,\infty),\mathbb{B})} \leq C\left\|\mathcal{W}^\lambda_{\psi,\mathbb{B}}(f_n)\right\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}, \quad n \in \mathbb{N}. \tag{2.11}$$

Since, as it was proved in Section 2.1, $\mathcal{W}^\lambda_{\psi,\mathbb{B}}$ is a bounded operator from $L^p((0,\infty), \mathbb{B})$ into $L^p((0,\infty), \gamma(H, \mathbb{B}))$, by letting $n \to \infty$ in (2.11) we conclude that

$$\|f\|_{L^p((0,\infty),\mathbb{B})} \leq C\left\|\mathcal{W}^\lambda_{\psi,\mathbb{B}}(f)\right\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}.$$

The following result was established in [5, after Lemma 2.4].

Lemma 2.1. *Let $\lambda > 0$. If $\psi \in \mathcal{S}_\lambda(0,\infty)$ is not identically zero, then there exists $\phi \in \mathcal{S}_\lambda(0,\infty)$ such that*

$$\int_0^\infty h_\lambda(\psi)(y) h_\lambda(\phi)(y) y^{-2\lambda-1}\, dy = 1, \tag{2.12}$$

where the last integral is absolutely convergent.

In order to see (2.10), we need to show the next result.

Lemma 2.2. *Let $\lambda > 0$. Suppose that $\psi, \phi \in \mathcal{S}_\lambda(0,\infty)$ satisfy (2.12), with the integral absolutely convergent. If $f, g \in \mathcal{S}_\lambda(0,\infty)$, then*

$$\int_0^\infty f(x) g(x)\, dx = \int_0^\infty \int_0^\infty (f \#_\lambda \psi_{(t)})(y)(g \#_\lambda \phi_{(t)})(y) \frac{dy\, dt}{t}. \tag{2.13}$$

Proof. Let $f, g \in \mathcal{S}_\lambda(0,\infty)$. Note first that the integral in the right-hand side of (2.13) is absolutely convergent. Indeed, according to (2.1), we get

$$\begin{aligned}&\int_0^\infty \int_0^\infty \left|(f \#_\lambda \psi_{(t)})(y)\right| \left|(g \#_\lambda \phi_{(t)})(y)\right| \frac{dy\, dt}{t} \\ &\quad\leq \left\|\mathcal{W}^\lambda_{\psi,\mathbb{C}}(f)\right\|_{L^p((0,\infty),H)} \left\|\mathcal{W}^\lambda_{\phi,\mathbb{C}}(g)\right\|_{L^{p'}((0,\infty),H)} \leq C\|f\|_{L^p(0,\infty)} \|g\|_{L^{p'}(0,\infty)}.\end{aligned}$$

Plancherel equality and the interchange formula for Hankel transforms (1.4) lead to

$$\begin{aligned}\int_0^\infty &(f\#_\lambda\psi_{(t)})(y)(g\#_\lambda\phi_{(t)})(y)\,dy\\ &=\int_0^\infty h_\lambda(f\#_\lambda\psi_{(t)})(y)h_\lambda(g\#_\lambda\phi_{(t)})(y)\,dy\\ &=\int_0^\infty h_\lambda(f)(y)h_\lambda(g)(y)(ty)^{-2\lambda}h_\lambda(\psi)(ty)h_\lambda(\phi)(ty)\,dy,\quad t\in(0,\infty).\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\int_0^\infty\int_0^\infty &(f\#_\lambda\psi_{(t)})(y)(g\#_\lambda\phi_{(t)})(y)\frac{dy\,dt}{t}\\ &=\int_0^\infty\int_0^\infty h_\lambda(f)(y)h_\lambda(g)(y)(ty)^{-2\lambda}h_\lambda(\psi)(ty)h_\lambda(\phi)(ty)\frac{dy\,dt}{t}\\ &=\int_0^\infty h_\lambda(f)(y)h_\lambda(g)(y)\int_0^\infty h_\lambda(\psi)(ty)h_\lambda(\phi)(ty)(ty)^{-2\lambda}\frac{dt\,dy}{t}\\ &=\int_0^\infty h_\lambda(f)(y)h_\lambda(g)(y)\,dy\\ &=\int_0^\infty f(x)g(x)\,dx.\end{aligned}$$

□

An immediate consequence of Lemma 2.2 is the following.

Lemma 2.3. *Let $\mathbb{B}$ be a Banach space and $\lambda>0$. Suppose that $\psi,\phi\in\mathcal{S}_\lambda(0,\infty)$ satisfy (2.12), with the integral absolutely convergent. If $f\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$ and $g\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}^*$, then*

$$\int_0^\infty\big\langle g(x),f(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\,dx=\int_0^\infty\int_0^\infty\big\langle(g\#_\lambda\phi_{(t)})(x),(f\#_\lambda\psi_{(t)})(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\frac{dx\,dt}{t}.$$

Let $f\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$. Since $\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}^*$ is dense in $L^{p'}((0,\infty),\mathbb{B}^*)$, according to [17, Lemma 2.3], we have

$$\|f\|_{L^p((0,\infty),\mathbb{B})}=\sup_{\substack{g\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}^*\\ \|g\|_{L^{p'}((0,\infty),\mathbb{B}^*)}\le 1}}\Big|\int_0^\infty\big\langle g(x),f(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\,dx\Big|.$$

By Lemma 2.1, we choose $\psi,\phi\in\mathcal{S}_\lambda(0,\infty)$ such that (2.12) holds, with the integral absolutely convergent. Since $\mathbb{B}^*$ is UMD, it was proved in Section 2.1 that the operator $\mathcal{W}^\lambda_{\phi,\mathbb{B}^*}$ is bounded from $L^{p'}((0,\infty),\mathbb{B}^*)$ into $L^{p'}((0,\infty),\gamma(H,\mathbb{B}^*))$. According to Lemma 2.3 and [26, Proposition 2.2], we get, for every $g\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}^*$,

$$\begin{aligned}\Big|\int_0^\infty\big\langle g(x),f(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\,dx\Big|&=\Big|\int_0^\infty\int_0^\infty\big\langle(g\#_\lambda\phi_{(t)})(x),(f\#_\lambda\psi_{(t)})(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\frac{dx\,dt}{t}\Big|\\ &\le\int_0^\infty\int_0^\infty\big|\big\langle(g\#_\lambda\phi_{(t)})(x),(f\#_\lambda\psi_{(t)})(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\big|\frac{dx\,dt}{t}\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^\infty \big\|\mathcal{W}^\lambda_{\phi,\mathbb{B}^*}(g)(\cdot,x)\big\|_{\gamma(H,\mathbb{B}^*)}\big\|\mathcal{W}^\lambda_{\psi,\mathbb{B}}(f)(\cdot,x)\big\|_{\gamma(H,\mathbb{B})}\,dx\\
&\leq \big\|\mathcal{W}^\lambda_{\phi,\mathbb{B}^*}(g)\big\|_{L^{p'}((0,\infty),\gamma(H,\mathbb{B}^*))}\big\|\mathcal{W}^\lambda_{\psi,\mathbb{B}}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}\\
&\leq C\|g\|_{L^{p'}((0,\infty),\mathbb{B}^*)}\big\|\mathcal{W}^\lambda_{\psi,\mathbb{B}}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}.
\end{aligned}$$

Hence,

$$\|f\|_{L^p((0,\infty),\mathbb{B})} \leq C\big\|\mathcal{W}^\lambda_{\psi,\mathbb{B}}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}.$$

Thus, the proof of Theorem 1.1 is finished.

3. Proof of Theorem 1.2

3.1. In this section we prove that

$$\big\|G^{\lambda,\beta}_{P,\mathbb{B}}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p((0,\infty),\mathbb{B})}, \quad f\in L^p\big((0,\infty),\mathbb{B}\big), \tag{3.1}$$

for some $C>0$ independent of f.

We define the g-operator associated with the classical Poisson semigroup on $\mathbb{R}$ as

$$G^\beta_{P,\mathbb{B}}(f)(t,x) = t^\beta\partial_t^\beta\mathbb{P}_t(f)(x), \quad x\in\mathbb{R} \text{ and } t\in(0,\infty)$$

for every $f\in L^p(\mathbb{R},\mathbb{B})$.

By [4, Proposition 1] there exists $C>0$ such that

$$\big\|G^\beta_{P,\mathbb{B}}(f)\big\|_{L^p(\mathbb{R},\gamma(H,\mathbb{B}))} \leq C\|f\|_{L^p(\mathbb{R},\mathbb{B})}, \quad f\in\mathcal{S}(\mathbb{R})\otimes\mathbb{B}.$$

The arguments developed in the proof of Theorem 1.1 allow us to show that

$$\big\|G^\beta_{P,\mathbb{B}}(f)\big\|_{L^p(\mathbb{R},\gamma(H,\mathbb{B}))} \leq \|f\|_{L^p(\mathbb{R},\mathbb{B})}, \quad f\in L^p(\mathbb{R},\mathbb{B}). \tag{3.2}$$

In [4, Lemma 1] it was established that

$$t^\beta\partial_t^\beta\mathbb{P}_t(z) = \sum_{k=0}^{(m+1)/2}\frac{c_k}{t}\varphi^k\Big(\frac{z}{t}\Big), \quad z\in\mathbb{R} \text{ and } t\in(0,\infty),$$

where $m\in\mathbb{N}$ is such that $m-1\leq\beta<m$, and, for every $k\in\mathbb{N}$, $0\leq k\leq(m+1)/2$, $c_k\in\mathbb{C}$ and

$$\varphi^k(z) = \int_0^\infty \frac{(1+v)^{m+1-2k}v^{m-\beta-1}}{((1+v)^2+z^2)^{m-k+1}}\,dv, \quad z\in\mathbb{R}.$$

By proceeding as in the proof of [4, Lemma 1], we can obtain the analogous identity in the Bessel setting:

$$\begin{aligned}
t^\beta\partial_t^\beta P^\lambda_t(x,y) = &\sum_{k=0}^{(m+1)/2}\frac{b^\lambda_k}{t^{2\lambda+1}}(xy)^\lambda\\
&\times\int_0^\pi(\sin\theta)^{2\lambda-1}\varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\,d\theta,
\end{aligned} \tag{3.3}$$

where $m \in \mathbb{N}$ is such that $m-1 \leq \beta < m$, and, for every $k \in \mathbb{N}$, $0 \leq k \leq (m+1)/2$,

$$\varphi^{\lambda,k}(z) = \int_0^\infty \frac{(1+v)^{m+1-2k} v^{m-\beta-1}}{((1+v)^2 + z^2)^{\lambda+m-k+1}} \, dv, \quad z \in (0,\infty),$$

and

$$b_k^\lambda = \frac{2\lambda(\lambda+1)\cdots(\lambda+m-k)}{(m-k)!} c_k.$$

Let $k \in \mathbb{N}$, $0 \leq k \leq (m+1)/2$. We define, for every $f \in L^p(\mathbb{R}, \mathbb{B})$,

$$\mathbb{P}_k(f)(t,x) = \int_{\mathbb{R}} \varphi_t^k(x-y) f(y) \, dy, \quad t \in (0,\infty) \text{ and } x \in \mathbb{R}.$$

Let $f \in L^p((0,\infty), \mathbb{B})$. If f_o denotes the odd extension of f to $\mathbb{R}$, we write

$$\begin{aligned}\mathbb{P}_k(f_o)(t,x) &= \int_0^\infty \varphi_t^k(x-y) f(y) \, dy - \int_0^\infty \varphi_t^k(x+y) f(y) \, dy \\ &= \mathbb{P}_{k,1}(f)(t,x) - \mathbb{P}_{k,2}(f)(t,x), \quad t, x \in (0,\infty).\end{aligned}$$

We have that, for every $x, y \in (0,\infty)$,

$$\begin{aligned}\left\| \varphi_t^k(x+y) \right\|_H &\leq \int_0^\infty (1+v)^{m+1-2k} v^{m-\beta-1} \\ &\qquad \times \Big(\int_0^\infty \frac{1}{t^3} \frac{t^{4(m-k+1)}}{((1+v^2)t^2 + (x+y)^2)^{2(m-k+1)}} \, dt \Big)^{1/2} dv \\ &\leq \frac{C}{x+y}. \end{aligned} \tag{3.4}$$

Since $\gamma(H, \mathbb{C}) = H$, we deduce that

$$\begin{aligned}\left\| \mathbb{P}_{k,2}(f)(\cdot, x) \right\|_{\gamma(H,\mathbb{B})} &\leq \int_0^\infty \left\| f(y) \right\|_{\mathbb{B}} \Big\| \frac{1}{t} \varphi^k \Big(\frac{x+y}{t} \Big) \Big\|_H \, dt \\ &\leq C \int_0^\infty \frac{\|f(y)\|_{\mathbb{B}}}{x+y} \, dy \\ &\leq C \big[H_0 \big(\|f\|_{\mathbb{B}} \big)(x) + H_\infty \big(\|f\|_{\mathbb{B}} \big)(x) \big], \quad x \in (0,\infty).\end{aligned}$$

Thus, according to [20, p. 244, (9.9.1) and (9.9.2)], $\mathbb{P}_{k,2}$ is a bounded operator from $L^p((0,\infty), \mathbb{B})$ into $L^p((0,\infty), \gamma(H, \mathbb{B}))$.

We define, for every $f \in L^p((0,\infty), \mathbb{B})$,

$$G_{P,\mathbb{B}}^{\beta,-}(f)(t,x) = \int_0^\infty t^\beta \partial_t^\beta \mathbb{P}_t(x+y) f(y) \, dy, \quad t, x \in (0,\infty).$$

Since $G_{P,\mathbb{B}}^{\beta,-} = \sum_{k=0}^{(m+1)/2} c_k \mathbb{P}_{k,2}$, we conclude that $G_{P,\mathbb{B}}^{\beta,-}$ is a bounded operator from $L^p((0,\infty), \mathbb{B})$ into $L^p((0,\infty), \gamma(H, \mathbb{B}))$. Then, according to (3.2), if for every $f \in L^p((0,\infty), \mathbb{B})$, we define

$$G_{P,\mathbb{B}}^{\beta,+}(f)(t,x) = \int_0^\infty t^\beta \partial_t^\beta \mathbb{P}_t(x-y) f(y) \, dy, \quad t, x \in (0,\infty),$$

the operator $G_{P,\mathbb{B}}^{\beta,+}$ is also bounded from $L^p((0,\infty), \mathbb{B})$ into $L^p((0,\infty), \gamma(H, \mathbb{B}))$.

In order to prove (3.1), it is enough to show that the difference $G^{\lambda,\beta}_{P,\mathbb{B}} - G^{\beta,+}_{P,\mathbb{B}}$ is bounded from $L^p((0,\infty),\mathbb{B})$ into $L^p((0,\infty),\gamma(H,\mathbb{B}))$.

By proceeding as in (3.4), we get, for every $x \in (0,\infty)$,

$$\left\| t^\beta \partial_t^\beta \mathbb{P}_t(x-y) \right\|_H \le \frac{C}{|x-y|} \le C \begin{cases} \frac{1}{x}, & 0 < y < \frac{x}{2}, \\ \frac{1}{y}, & y > 2x. \end{cases} \tag{3.5}$$

We split $P_t^\lambda(x,y)$, $t,x,y \in (0,\infty)$, as follows:

$$\begin{aligned} P_t^\lambda(x,y) &= \frac{2\lambda(xy)^\lambda t}{\pi} \int_0^{\pi/2} \frac{(\sin\theta)^{2\lambda-1}}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\lambda+1}}\, d\theta \\ &\quad + \frac{2\lambda(xy)^\lambda t}{\pi} \int_{\pi/2}^{\pi} \frac{(\sin\theta)^{2\lambda-1}}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\lambda+1}}\, d\theta \\ &= P_t^{\lambda,1}(x,y) + P_t^{\lambda,2}(x,y). \end{aligned}$$

From (3.3) we have

$$\begin{aligned} \left\| t^\beta \partial_t^\beta P_t^{\lambda,2}(x,y) \right\|_H &\le C (xy)^\lambda \int_{\pi/2}^{\pi} (\sin\theta)^{2\lambda-1} \\ &\quad \times \sum_{k=0}^{(m+1)/2} \left\| \frac{1}{t^{2\lambda+1}} \varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t} \Big) \right\|_H d\theta, \end{aligned}$$

and, for every $k \in \mathbb{N}$, $0 \le k \le (m+1)/2$,

$$\begin{aligned} &\left\| \frac{1}{t^{2\lambda+1}} \varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t} \Big) \right\|_H \\ &\le C \int_0^\infty (1+v)^{m+1-2k} v^{m-\beta-1} \Big(\int_0^\infty \frac{t^{4(\lambda+1+m-k)-4\lambda-3}}{((1+v^2)t^2+(x+y)^2)^{2(\lambda+m-k+1)}}\, dt \Big)^{1/2} dv \\ &\le \frac{C}{(x+y)^{2\lambda+1}} \int_0^\infty \frac{v^{m-\beta-1}}{(1+v)^m}\, dv \Big(\int_0^\infty \frac{u^{4(m-k)+1}}{(1+u)^{4(\lambda+m-k+1)}}\, du \Big)^{1/2} \\ &\le \frac{C}{(x+y)^{2\lambda+1}}, \quad x,y \in (0,\infty) \text{ and } \theta \in \Big(\frac{\pi}{2}, \pi \Big). \end{aligned}$$

Hence,

$$\begin{aligned} \left\| t^\beta \partial_t^\beta P_t^{\lambda,2}(x,y) \right\|_H &\le C \frac{(xy)^\lambda}{(x+y)^{2\lambda+1}} \\ &\le \frac{C}{x+y}, \quad x,y \in (0,\infty). \end{aligned} \tag{3.6}$$

Similar manipulations lead to

$$\left\| t^\beta \partial_t^\beta P_t^{\lambda,1}(x,y) \right\|_H \le \frac{C}{|x-y|} \le C \begin{cases} \frac{1}{x}, & 0 < y < x/2, \\ \frac{1}{y}, & y > 2x > 0. \end{cases} \tag{3.7}$$

We decompose $t^\beta \partial_t^\beta P_t^{\lambda,1}(x,y)$, $t,x,y \in (0,\infty)$, as follows:

$$\begin{aligned}
&t^\beta \partial_t^\beta P_t^{\lambda,1}(x,y)\\
&\quad= \sum_{k=0}^{(m+1)/2} \frac{b_k^\lambda}{t^{2\lambda+1}}(xy)^\lambda\\
&\qquad\times \Big\{ \int_0^{\pi/2} \big[(\sin\theta)^{2\lambda-1} - \theta^{2\lambda-1}\big] \varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\,d\theta\\
&\qquad+ \int_0^{\pi/2} \theta^{2\lambda-1}\Big[\varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\\
&\qquad- \varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+xy\theta^2}}{t}\Big)\Big]\,d\theta\\
&\qquad- \int_{\pi/2}^\infty \theta^{2\lambda-1}\varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+xy\theta^2}}{t}\Big)\,d\theta\\
&\qquad+ \int_0^\infty \theta^{2\lambda-1}\varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+xy\theta^2}}{t}\Big)\,d\theta\Big\}\\
&\quad= \sum_{k=0}^{(m+1)/2} b_k^\lambda \big[R_1^{\lambda,k}(t,x,y) + R_2^{\lambda,k}(t,x,y) + R_3^{\lambda,k}(t,x,y) + R_4^{\lambda,k}(t,x,y)\big].
\end{aligned}$$

Let $k \in \mathbb{N}$, $0 \le k \le (m+1)/2$. By using the mean-value theorem we obtain, when $0 < x/2 < y < 2x$,

$$\begin{aligned}
&\big\|R_1^{\lambda,k}(\cdot,x,y)\big\|_H\\
&\quad\le C(xy)^\lambda \int_0^{\pi/2} \theta^{2\lambda+1} \int_0^\infty (1+v)^{m+1-2k} v^{m-\beta-1}\\
&\qquad\times \Big(\int_0^\infty \frac{t^{4(\lambda+1+m-k)-4\lambda-3}}{((1+v)^2t^2+(x-y)^2+xy\theta^2)^{2(\lambda+m-k+1)}}\,dt\Big)^{1/2} dv\,d\theta\\
&\quad\le C(xy)^\lambda \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{((x-y)^2+xy\theta^2)^{\lambda+1/2}}\,d\theta \le \frac{C}{x},
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\big\|R_2^{\lambda,k}(\cdot,x,y)\big\|_H\\
&\quad\le C(xy)^\lambda \int_0^{\pi/2} \theta^{2\lambda-1} \int_0^\infty (1+v)^{m+1-2k} v^{m-\beta-1}\\
&\qquad\times \Big\{\int_0^\infty t^{4(m-k)+1}\Big|\frac{1}{((1+v)^2t^2+(x-y)^2+2xy(1-\cos\theta))^{\lambda+m-k+1}}\\
&\qquad- \frac{1}{((1+v)^2t^2+(x-y)^2+xy\theta^2)^{\lambda+m-k+1}}\Big|^2 dt\Big\}^{1/2} dv\,d\theta\\
&\quad\le C(xy)^\lambda \int_0^{\pi/2} \theta^{2\lambda-1} \int_0^\infty (1+v)^{m+1-2k} v^{m-\beta-1}
\end{aligned}$$

$$\times \Big\{\int_0^\infty t^{4(m-k)+1}\Big(\frac{xy\theta^4}{((1+v)^2t^2+(x-y)^2+xy\theta^2)^{\lambda+m-k+2}}\Big)^2 dt\Big\}^{1/2} dv\, d\theta$$

$$\leq C(xy)^{\lambda+1}\int_0^{\pi/2}\frac{\theta^{2\lambda+3}}{((x-y)^2+xy\theta^2)^{\lambda+3/2}}\,d\theta \leq \frac{C}{x}, \quad 0<\frac{x}{2}<y<2x. \tag{3.9}$$

We have also that, when $0<x/2<y<2x$,

$$\begin{aligned}
&\big\|R_3^{\lambda,k}(\cdot,x,y)\big\|_H \\
&\quad \leq C(xy)^\lambda\int_{\pi/2}^\infty \theta^{2\lambda-1}\int_0^\infty (1+v)^{m+1-2k}v^{m-\beta-1} \\
&\qquad \times \Big(\int_0^\infty \frac{t^{4(m-k)+1}}{((1+v)^2t^2+(x-y)^2+xy\theta^2)^{2(\lambda+m-k+1)}}\,dt\Big)^{1/2} dv\, d\theta \\
&\quad \leq C(xy)^\lambda \int_{\pi/2}^\infty \frac{\theta^{2\lambda-1}}{((x-y)^2+xy\theta^2)^{\lambda+1/2}}\,d\theta \leq \frac{C}{x}.
\end{aligned}\tag{3.10}$$

Finally, we get that

$$\begin{aligned}
&\int_0^\infty \theta^{2\lambda-1}\varphi^{\lambda,k}\Big(\frac{\sqrt{(x-y)^2+xy\theta^2}}{t}\Big)\,d\theta \\
&\quad = \int_0^\infty \theta^{2\lambda-1}\int_0^\infty \frac{(1+v)^{m+1-2k}v^{m-\beta-1}}{((1+v)^2+[(x-y)^2+xy\theta^2]/t^2)^{\lambda+m-k+1}}\,dv\,d\theta \\
&\quad = t^{2(\lambda+m-k+1)}\int_0^\infty (1+v)^{m+1-2k}v^{m-\beta-1} \\
&\qquad \times \int_0^\infty \frac{\theta^{2\lambda-1}}{((1+v)^2t^2+(x-y)^2+xy\theta^2)^{\lambda+m-k+1}}\,d\theta\,dv \\
&\quad = \frac{t^{2(\lambda+m-k+1)}}{(xy)^\lambda}\int_0^\infty \frac{(1+v)^{m+1-2k}v^{m-\beta-1}}{((1+v)^2t^2+(x-y)^2)^{m-k+1}}\,dv\int_0^\infty \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+m-k+1}}\,du \\
&\quad = \frac{(m-k)!}{2\lambda(\lambda+1)\cdots(\lambda+m-k)}\frac{t^{2\lambda}}{(xy)^\lambda}\varphi^k\Big(\frac{x-y}{t}\Big).
\end{aligned}$$

Then

$$\sum_{k=0}^{(m+1)/2} b_k^\lambda R_4^{\lambda,k}(t,x,y) = t^\beta\partial_t^\beta \mathbb{P}_t(x-y), \quad t,x,y\in(0,\infty). \tag{3.11}$$

By putting together (3.5)–(3.11) we conclude that $G_{P,\mathbb{B}}^{\lambda,\beta}-G_{P,\mathbb{B}}^{\beta,+}$ is bounded from $L^p((0,\infty),\mathbb{B})$ into $L^p((0,\infty),\gamma(H,\mathbb{B}))$, and hence $G_{P,\mathbb{B}}^{\lambda,\beta}$ is a bounded operator from $L^p((0,\infty),\mathbb{B})$ into $L^p((0,\infty),\gamma(H,\mathbb{B}))$.

3.2. We are going to show that there exists $C>0$ such that, for every $f\in L^p((0,\infty),\mathbb{B})$,

$$\|f\|_{L^p((0,\infty),\mathbb{B})} \leq C\big\|G_{P,\mathbb{B}}^{\lambda,\beta}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}. \tag{3.12}$$

Since $G_{P,\mathbb{B}}^{\lambda,\beta}$ is bounded from $L^p((0,\infty),\mathbb{B})$ into $L^p((0,\infty),\gamma(H,\mathbb{B}))$ and $\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$ is a dense subspace of $L^p((0,\infty),\mathbb{B})$, (3.12) holds for every $f\in L^p((0,\infty),\mathbb{B})$ whenever it is true for every $f\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$.

By proceeding as in Section 2.2, the inequality in (3.12) can be proved as a consequence of a polarization identity involving the operator $G^{\lambda,\beta}_{P,\mathbb{B}}$. To show this equality, we first need to establish the following.

Lemma 3.1. *Let λ, $\beta > 0$. Then, for every $f \in \mathcal{S}_\lambda(0,\infty)$,*

$$h_\lambda(t^\beta \partial_t^\beta P_t^\lambda f)(x) = e^{i\pi\beta}(tx)^\beta e^{-xt} h_\lambda(f)(x), \quad t, x \in (0,\infty).$$

Proof. Let $f \in \mathcal{S}_\lambda(0,\infty)$. We have that (see [15, Section 8.5 (19)])

$$h_\lambda(P_t^\lambda f)(x) = e^{-xt} h_\lambda(f)(x), \quad t, x \in (0,\infty).$$

We choose $m \in \mathbb{N}$ such that $m-1 \le \beta < m$. It is not hard to see that $\partial_t^\beta e^{-xt} = e^{i\pi\beta} x^\beta e^{-xt}$, $t, x \in (0,\infty)$. Then

$$\partial_t^\beta h_\lambda(P_t^\lambda f)(x) = e^{i\pi\beta} x^\beta e^{-xt} h_\lambda(f)(x), \quad t, x \in (0,\infty).$$

According to [16, (4.6)], we can write, for every $t, x, y \in (0,\infty)$ and $\theta \in (0,\pi)$,

$$\begin{aligned}
&\partial_t^m \Big[\frac{t}{[(x-y)^2 + 2xy(1-\cos\theta) + t^2]^{\lambda+1}}\Big] \\
&\quad = -\frac{1}{2\lambda}\partial_t^{m+1}\Big[\frac{1}{[(x-y)^2 + 2xy(1-\cos\theta) + t^2]^{\lambda}}\Big] \\
&\quad = \frac{1}{2}\sum_{k=0}^{(m+1)/2} (-1)^{m-k} E_{m+1,k} t^{m+1-2k} \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+m-k)}{[(x-y)^2 + 2xy(1-\cos\theta) + t^2]^{\lambda+m-k+1}},
\end{aligned}$$

where

$$E_{m+1,k} = \frac{2^{m+1-2k}(m+1)!}{k!(m+1-2k)!}, \quad 0 \le k \le \frac{m+1}{2}.$$

Hence, $\partial_t^m[t/[(x-y)^2 + 2xy(1-\cos\theta) + t^2]^{\lambda+1}]$ is continuous in $(t,x,y,\theta) \in (0,\infty)^3 \times (0,\pi)$. Moreover, for each $t, x, y \in (0,\infty)$ and $\theta \in (0,\pi)$,

$$\Big|\partial_t^m \Big[\frac{t}{[(x-y)^2 + 2xy(1-\cos\theta) + t^2]^{\lambda+1}}\Big]\Big| \le \frac{C}{[(x-y)^2 + t^2]^{\lambda+(m+1)/2}}.$$

Then

$$\big|\partial_t^m P_{t+s}^\lambda(f)(x)\big| \le C\int_0^\infty \big|f(y)\big| \frac{(xy)^\lambda}{[(x-y)^2 + (t+s)^2]^{\lambda+(m+1)/2}}\, dy, \quad t, x \in (0,\infty),$$

and $\partial_t^\beta P_t^\lambda(f) \in L^1(0,\infty)$, $t > 0$. Since the function $\sqrt{z}J_\nu(z)$ is bounded on $(0,\infty)$ when $\nu > -1/2$, the derivation under the integral sign is justified and we get

$$h_\lambda(\partial_t^\beta P_t^\lambda f)(x) = \partial_t^\beta h_\lambda\big(P_t^\lambda(f)\big)(x) = e^{i\pi\beta} x^\beta e^{-xt} h_\lambda(f)(x), \quad t, x \in (0,\infty). \quad \square$$

Lemma 3.2. *Let $\mathbb{B}$ be a UMD Banach space and λ, $\beta > 0$. If $f \in \mathcal{S}_\lambda(0,\infty) \otimes \mathbb{B}$ and $g \in \mathcal{S}_\lambda(0,\infty) \otimes \mathbb{B}^*$, then*

$$\begin{aligned}
&\int_0^\infty \big\langle g(x), f(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\, dx \\
&\quad = \frac{e^{i2\pi\beta}2^{2\beta}}{\Gamma(2\beta)}\int_0^\infty\int_0^\infty \big\langle t^\beta \partial_t^\beta P_t^\lambda(g)(x), t^\beta \partial_t^\beta P_t^\lambda(f)(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}} \frac{dt\, dx}{t}. \qquad (3.13)
\end{aligned}$$

Proof. It is enough to show (3.13) when $f, g \in \mathcal{S}_\lambda(0,\infty)$ and $\mathbb{B} = \mathbb{C}$. Let $f, g \in \mathcal{S}_\lambda(0,\infty)$.

Since $\mathbb{C}$ is a UMD Banach space, as it was proved in Section 3.1, the operator $G^{\lambda,\beta}_{P,\mathbb{C}}$ is bounded from $L^p(0,\infty)$ into $L^p((0,\infty), H)$, $1 < p < \infty$. Hence, the integral in the right-hand side of (3.13) is absolutely convergent.

As h_λ is an isometry in $L^2(0,\infty)$ (see [48, p. 214 and Theorem 129]), Lemma 3.1 implies that $t^\beta \partial_t^\beta P_t^\lambda(f) \in L^2(0,\infty)$ and $t^\beta \partial_t^\beta P_t^\lambda(g) \in L^2(0,\infty)$ for every $t > 0$. The Plancherel equality for Hankel transforms and Lemma 3.1 lead to

$$\begin{aligned}
&\int_0^\infty \int_0^\infty t^\beta \partial_t^\beta P_t^\lambda(f)(x) t^\beta \partial_t^\beta P_t^\lambda(g)(x) \frac{dt\,dx}{t} \\
&\quad= \int_0^\infty \int_0^\infty t^\beta \partial_t^\beta P_t^\lambda(f)(x) t^\beta \partial_t^\beta P_t^\lambda(g)(x) \frac{dx\,dt}{t} \\
&\quad= e^{i2\pi\beta} \int_0^\infty \int_0^\infty (tx)^{2\beta} e^{-2xt} h_\lambda(f)(x) h_\lambda(g)(x) \frac{dx\,dt}{t} \\
&\quad= e^{i2\pi\beta} \int_0^\infty h_\lambda(f)(x) h_\lambda(g)(x) \int_0^\infty (tx)^{2\beta} e^{-2xt} \frac{dt\,dx}{t} \\
&\quad= e^{i2\pi\beta} \frac{\Gamma(2\beta)}{2^{-2\beta}} \int_0^\infty h_\lambda(f)(x) h_\lambda(g)(x)\,dx \\
&\quad= e^{i2\pi\beta} \frac{\Gamma(2\beta)}{2^{-2\beta}} \int_0^\infty f(x) g(x)\,dx.
\end{aligned}$$

□

By using now Lemma 3.2, the arguments developed in Section 2.2 allow us to show that (3.12) holds, for every $f \in L^p((0,\infty), \mathbb{B})$.

Thus, the proof of Theorem 1.2 is completed.

4. Proof of Theorem 1.3

The Riesz transform R_λ associated with the Bessel operator Δ_λ is the principal value integral operator defined, for every $f \in L^p(0,\infty)$, by

$$R_\lambda(f)(x) = \lim_{\varepsilon \to 0^+} \int_{0, |x-y| > \varepsilon}^\infty R_\lambda(x,y) f(y)\,dy, \quad \text{a.e. } x \in (0,\infty),$$

where

$$R_\lambda(x,y) = \int_0^\infty D_\lambda P_t^\lambda(x,y)\,dt, \quad x, y \in (0,\infty), x \neq y,$$

and $D_\lambda = x^\lambda \frac{d}{dx} x^{-\lambda}$. Main properties of Riesz transform R_λ can be encountered in [2]. We denote by R_λ^* the "adjoint" operator of R_λ defined, for every $f \in L^p(0,\infty)$, by

$$R_\lambda^*(f)(x) = \lim_{\varepsilon \to 0^+} \int_{0, |x-y| > \varepsilon}^\infty R_\lambda(y,x) f(y)\,dy, \quad \text{a.e. } x \in (0,\infty).$$

Riesz transforms R_λ and R_λ^* are bounded from $L^p(0,\infty)$ into itself. Moreover, since $\mathbb{B}$ is a UMD Banach space, by defining R_λ and R_λ^* on $L^p(0,\infty) \otimes \mathbb{B}$ in the natural way, R_λ and R_λ^* can be extended to $L^p((0,\infty), \mathbb{B})$ as bounded operators on $L^p((0,\infty), \mathbb{B})$ into itself (see [7, Theorem 2.1]).

We define, for every $f \in L^p(0,\infty)$, the function $\mathcal{Q}_t^\lambda(f)$ by

$$\mathcal{Q}_t^\lambda(f)(x) = \int_0^\infty \mathcal{Q}_t^\lambda(x,y)f(y)\,dy, \quad t,x \in (0,\infty),$$

where

$$\mathcal{Q}_t^\lambda(x,y) = \frac{2\lambda(xy)^\lambda}{\pi}\int_0^\pi \frac{(x-y\cos\theta)(\sin\theta)^{2\lambda-1}}{(x^2+y^2+t^2-2xy\cos\theta)^{\lambda+1}}\,d\theta, \quad t,x,y \in (0,\infty).$$

The following Cauchy–Riemann equations hold:

$$D_\lambda P_t^\lambda(f) = \partial_t \mathcal{Q}_t^\lambda(f), \qquad D_\lambda^* \mathcal{Q}_t^\lambda(f) = \partial_t P_t^\lambda(f), \quad t>0.$$

These relations motivate that $\mathcal{Q}_t^\lambda(f)$ is called Δ_λ-conjugated to the Poisson integral $P_t^\lambda(f)$.

The adjoint Δ_λ-conjugated $\mathbb{Q}_t^\lambda(f)$ of $f \in L^p(0,\infty)$ is defined by

$$\mathbb{Q}_t^\lambda(f)(x) = \int_0^\infty \mathcal{Q}_t^\lambda(y,x)f(y)\,dy, \quad t,x \in (0,\infty).$$

We have that

$$D_\lambda^* P_t^{\lambda+1}(f) = \partial_t \mathbb{Q}_t^\lambda(f), \qquad D_\lambda \mathbb{Q}_t^\lambda(f) = \partial_t P_t^{\lambda+1}(f), \quad t>0.$$

By using the Hankel transform (see [39, (16.5)]) we can see that, for every $f \in \mathcal{S}_\lambda(0,\infty)$,

$$P_t^\lambda(R_\lambda^* f) = \mathbb{Q}_t^\lambda(f), \quad t>0.$$

Then, for every $f \in \mathcal{S}_\lambda(0,\infty)$,

$$\partial_t P_t^\lambda(R_\lambda^* f) = D_\lambda^* P_t^{\lambda+1}(f), \quad t>0. \tag{4.1}$$

Equality (4.1) also holds for every $f \in \mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$. Then

$$\mathcal{G}_{P,\mathbb{B}}^\lambda(f) = G_{P,\mathbb{B}}^{\lambda,1}(R_\lambda^* f), \quad f \in \mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}. \tag{4.2}$$

Since R_λ^* can be extended to $L^p((0,\infty),\mathbb{B})$ boundedly from $L^p((0,\infty),\mathbb{B})$ into itself, Theorem 1.2 implies that the operator $\mathcal{G}_{P,\mathbb{B}}^\lambda$ can be extended from $\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$ as a bounded operator from $L^p((0,\infty),\mathbb{B})$ into $L^p((0,\infty),\gamma(H,\mathbb{B}))$. We denote this extension by $\widetilde{\mathcal{G}}_{P,\mathbb{B}}^\lambda$.

We define

$$\mathcal{G}_{P,\mathbb{B}}^\lambda(t,x,y) = tD_\lambda^* P_t^{\lambda+1}(x,y), \quad t,x,y \in (0,\infty).$$

We have that, for each $t,x,y \in (0,\infty)$,

$$\begin{aligned}
&\mathcal{G}_{P,\mathbb{B}}^\lambda(t,x,y)\\
&\quad = -\frac{2(\lambda+1)}{\pi}t^2y^{\lambda+1}x^{-\lambda}\partial_x\Big(x^{2\lambda+1}\int_0^\pi \frac{(\sin\theta)^{2\lambda+1}}{[(x-y)^2+t^2+2xy(1-\cos\theta)]^{\lambda+2}}\,d\theta\Big)\\
&\quad = -\frac{2(\lambda+1)(2\lambda+1)}{\pi}t^2x^\lambda y^{\lambda+1}\int_0^\pi \frac{(\sin\theta)^{2\lambda+1}}{[(x-y)^2+t^2+2xy(1-\cos\theta)]^{\lambda+2}}\,d\theta\\
&\qquad + \frac{4(\lambda+1)(\lambda+2)}{\pi}t^2(xy)^{\lambda+1}\int_0^\pi \frac{[(x-y)+y(1-\cos\theta)](\sin\theta)^{2\lambda+1}}{[(x-y)^2+t^2+2xy(1-\cos\theta)]^{\lambda+3}}\,d\theta.
\end{aligned}$$

Then

$$\begin{aligned}
&\big|\mathcal{G}^{\lambda}_{P,\mathbb{B}}(t,x,y)\big|\\
&\quad\le C\sqrt{t}\Big\{x^{\lambda}y^{\lambda+1}\Big(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}\Big)\frac{(\sin\theta)^{2\lambda+1}}{[(x-y)^2+t^2+2xy(1-\cos\theta)]^{\lambda+5/4}}\,d\theta\\
&\qquad+(xy)^{\lambda+1}\Big(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}\Big)\frac{(\sin\theta)^{2\lambda+1}}{[(x-y)^2+t^2+2xy(1-\cos\theta)]^{\lambda+7/4}}\,d\theta\Big\}\\
&\quad=\mathcal{G}^{\lambda,1,1}_{P,\mathbb{B}}(t,x,y)+\mathcal{G}^{\lambda,1,2}_{P,\mathbb{B}}(t,x,y)+\mathcal{G}^{\lambda,2,1}_{P,\mathbb{B}}(t,x,y)+\mathcal{G}^{\lambda,2,2}_{P,\mathbb{B}}(t,x,y),\quad t,x,y>0.
\end{aligned}$$

Let $\varepsilon>0$. Since, for every $x,y\in(0,\infty)$ and $\theta\in(0,\pi/2)$,

$$\Big(\int_{\varepsilon}^{\infty}\frac{dt}{[(x-y)^2+t^2+2xy(1-\cos\theta)]^{2\lambda+5/2}}\Big)^{1/2}\le\frac{C}{(|x-y|+\varepsilon+\sqrt{xy}\theta)^{2\lambda+2}},$$

and

$$\Big(\int_{\varepsilon}^{\infty}\frac{dt}{[(x-y)^2+t^2+2xy(1-\cos\theta)]^{2\lambda+7/2}}\Big)^{1/2}\le\frac{C}{(|x-y|+\varepsilon+\sqrt{xy}\theta)^{2\lambda+3}},$$

we obtain

$$\begin{aligned}
&\big\|\mathcal{G}^{\lambda,1,1}_{P,\mathbb{B}}(\cdot,x,y)\big\|_{L^2((\varepsilon,\infty),dt/t)}+\big\|\mathcal{G}^{\lambda,2,1}_{P,\mathbb{B}}(\cdot,x,y)\big\|_{L^2((\varepsilon,\infty),dt/t)}\\
&\quad\le C\Big(x^{\lambda}y^{\lambda+1}\int_0^{\pi/2}\frac{\theta^{2\lambda+1}}{(|x-y|+\varepsilon+\sqrt{xy}\theta)^{2\lambda+2}}\,d\theta\\
&\qquad+(xy)^{\lambda+1}\int_0^{\pi/2}\frac{\theta^{2\lambda+1}}{(|x-y|+\varepsilon+\sqrt{xy}\theta)^{2\lambda+3}}\,d\theta\Big)\\
&\quad\le C\Big(\frac{y}{(|x-y|+\varepsilon)^2}+\frac{xy}{(|x-y|+\varepsilon)^3}\Big)\\
&\quad\le C\begin{cases}1/(x+\varepsilon), & 0<y<x/2,\\ y/\varepsilon^2+y^2/\varepsilon^3, & x/2<y<2x,\\ 1/(y+\varepsilon), & y>2x>0.\end{cases}
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\big\|\mathcal{G}^{\lambda,1,2}_{P,\mathbb{B}}(\cdot,x,y)\big\|_{L^2((\varepsilon,\infty),dt/t)}+\big\|\mathcal{G}^{\lambda,2,2}_{P,\mathbb{B}}(\cdot,x,y)\big\|_{L^2((\varepsilon,\infty),dt/t)}\\
&\quad\le C\Big(x^{\lambda}y^{\lambda+1}\int_{\pi/2}^{\pi}\frac{(\sin\theta)^{2\lambda+1}}{(x+y+\varepsilon)^{2\lambda+2}}\,d\theta+(xy)^{\lambda+1}\int_{\pi/2}^{\pi}\frac{(\sin\theta)^{2\lambda+1}}{(x+y+\varepsilon)^{2\lambda+3}}\,d\theta\Big)\\
&\quad\le\frac{C}{x+y+\varepsilon},\quad x,y\in(0,\infty).
\end{aligned}$$

Hence, for every $x\in(0,\infty)$, $\|\mathcal{G}^{\lambda}_{P,\mathbb{B}}(\cdot,x,y)\|_{L^2((\varepsilon,\infty),dt/t)}\in L^{p'}(0,\infty)$.

By proceeding now as in Section 2.1, we conclude that

$$\mathcal{G}^{\lambda}_{P,\mathbb{B}}(f)=\widetilde{\mathcal{G}}^{\lambda}_{P,\mathbb{B}}(f),\quad f\in L^p\big((0,\infty),\mathbb{B}\big),$$

and the proof of Theorem 1.3 is completed.

5. Proof of Theorem 1.4

5.1. **Proof of (i) ⇒ (ii) and (i) ⇒ (iii).** In Theorems 1.2 and 1.3 it was proved that if $\mathbb{B}$ is a UMD Banach space, then (1.6), (1.7), and (1.8) are satisfied, for every $1 < p < \infty$.

5.2. **Proof of (ii) ⇒ (i).** Let $1 < p < \infty$. Suppose that (1.6) and (1.7) hold. Let $f \in \mathcal{S}_\lambda(0,\infty) \otimes \mathbb{B}$. Since R_λ^* is bounded from $L^p(0,\infty)$ into itself (see [2, Theorem 4.2]), $R_\lambda^* f \in L^p(0,\infty) \otimes \mathbb{B}$. According to (4.2), we obtain

$$\begin{aligned}\|R_\lambda^* f\|_{L^p((0,\infty),\mathbb{B})} &\leq C\big\|G_{P,\mathbb{B}}^{\lambda,1}(R_\lambda^* f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))} = C\big\|\mathcal{G}_{P,\mathbb{B}}^{\lambda}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}\\ &\leq C\|f\|_{L^p((0,\infty),\mathbb{B})}.\end{aligned}$$

Since $\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$ is dense in $L^p((0,\infty),\mathbb{B})$, R_λ^* can be extended to $L^p((0,\infty),\mathbb{B})$ as a bounded operator from $L^p((0,\infty),\mathbb{B})$ into itself. By using [7, Theorem 2.1], we deduce that $\mathbb{B}$ is UMD.

5.3. **Proof of (iii) ⇒ (i).** Assume now that (1.8) holds. In order to show that $\mathbb{B}$ is UMD, we prove first a characterization of UMD Banach spaces involving L^p-boundedness properties of the imaginary powers $\Delta_\lambda^{i\omega}$, $\omega \in \mathbb{R}\setminus\{0\}$, of the Bessel operator Δ_λ.

Let $\omega \in \mathbb{R}\setminus\{0\}$. The $i\omega$-power $\Delta_\lambda^{i\omega}$ of Δ_λ is the Hankel multiplier defined by

$$\Delta_\lambda^{i\omega} f = h_\lambda\big(y^{2i\omega} h_\lambda(f)\big), \quad f \in L^2(0,\infty). \tag{5.1}$$

Since h_λ is an isometry in $L^2(0,\infty)$, the operator $\Delta_\lambda^{i\omega}$ is bounded from $L^2(0,\infty)$ into itself. Moreover,

$$y^{2i\omega} = y^2 \int_0^\infty e^{-y^2 u} \frac{u^{-i\omega}}{\Gamma(1-i\omega)}\, du, \quad y \in (0,\infty),$$

and hence $\Delta_\lambda^{i\omega}$ is a Hankel multiplier of Laplace transform type. This type of Hankel multiplier was studied in [3] and [9]. Proceeding as in [3, Theorem 1.2], for every $f \in C_c^\infty(0,\infty)$, we have

$$\Delta_\lambda^{i\omega} f(x) = \lim_{\varepsilon\to 0^+}\Big(\alpha(\varepsilon) f(x) - \int_{0,|x-y|>\varepsilon}^\infty K_\omega^\lambda(x,y) f(y)\, dy\Big), \quad \text{a.e. } x \in (0,\infty), \tag{5.2}$$

where

$$K_\omega^\lambda(x,y) = \int_0^\infty \frac{t^{-i\omega}}{\Gamma(1-i\omega)} \partial_t W_t^\lambda(x,y)\, dt, \quad x,y \in (0,\infty), x \neq y,$$

and $W_t^\lambda(x,y)$ is the Bessel heat kernel

$$W_t^\lambda(x,y) = \frac{1}{\sqrt{2t}}\Big(\frac{xy}{2t}\Big)^{1/2} I_{\lambda-1/2}\Big(\frac{xy}{2t}\Big) e^{-(x^2+y^2)/4t}, \quad t,x,y \in (0,\infty).$$

Here α denotes a bounded function on $(0,\infty)$ and I_ν is the modified Bessel function of the first kind and order ν. By [9, Theorem 1.2], $\Delta_\lambda^{i\omega} f$ can be extended to $L^p(0,\infty)$ as a bounded operator from $L^p(0,\infty)$ into itself. Moreover, as in [3, Theorem 1.4], we can see that this extension, that we will continue denoting by $\Delta_\lambda^{i\omega}$, is given by the limit in (5.2) for every $f \in L^p(0,\infty)$. The operator $\Delta_\lambda^{i\omega}$ is defined on $L^p(0,\infty) \otimes \mathbb{B}$ in the usual way.

The following result is a Bessel version from [18].

Proposition 5.1 ([18, Theorem, p. 402])**.** *Let X be a Banach space and let $\lambda > 0$. Then X is UMD if and only if, for some (equivalently, for every) $1 < q < \infty$, the operator $\Delta_\lambda^{i\omega}$, $\omega \in \mathbb{R} \setminus \{0\}$, can be extended from $L^q(0,\infty) \otimes X$ to $L^q((0,\infty), X)$ as a bounded operator from $L^q((0,\infty), X)$ into itself.*

Proof. According to [18, Theorem, p. 402], X is UMD if and only if, for every $\omega \in \mathbb{R} \setminus \{0\}$ and for some (equivalently, for every) $1 < q < \infty$, the $i\omega$-power $(-\frac{d^2}{dx^2})^{i\omega}$ of the operator $-\frac{d^2}{dx^2}$ can be extended from $L^q(\mathbb{R}) \otimes X$ to $L^q(\mathbb{R}, X)$ as a bounded operator from $L^q(\mathbb{R}, X)$ into itself.

We recall that (see [6, Appendix] for a proof) for every $f \in L^q(\mathbb{R})$, $1 < q < \infty$, and $\omega \in \mathbb{R} \setminus \{0\}$,

$$\Big(-\frac{d^2}{dx^2}\Big)^{i\omega} f(x) = \lim_{\varepsilon \to 0^+} \Big(\alpha(\varepsilon) f(x) - \int_{|x-y|>\varepsilon} K_\omega(x,y) f(y)\, dy\Big), \quad \text{a.e. } x \in \mathbb{R},$$

where

$$K_\omega(x,y) = -\int_0^\infty \frac{t^{-i\omega}}{\Gamma(1-i\omega)} \partial_t \mathbb{W}_t(x-y)\, dt, \quad x, y \in \mathbb{R}, x \neq y,$$

and $\mathbb{W}_t(z)$ denotes the classical heat kernel (1.2). Here α represents the same function that appears in (5.2). The operator $(-\frac{d^2}{dx^2})^{i\omega}$, $\omega \in \mathbb{R} \setminus \{0\}$, is defined on $L^q(\mathbb{R}) \otimes X$, $1 < q < \infty$, in the natural way.

Let $\omega \in \mathbb{R} \setminus \{0\}$. We are going to obtain some estimates for the kernels $K_\omega^\lambda(x,y)$ and $K_\omega(x,y)$, $x, y \in (0,\infty)$, that will allow us to get our characterization of the UMD spaces by using imaginary powers of Bessel operators.

Note first that, for every $x, y \in (0,\infty)$,

$$\big|K_\omega(x,-y)\big| \leq C \int_0^\infty \big|\partial_t \mathbb{W}_t(x+y)\big|\, dt \leq C \int_0^\infty \frac{e^{-c(x+y)^2/t}}{t^{3/2}}\, dt \leq \frac{C}{x+y}. \tag{5.3}$$

In a similar way we obtain, for every $x \in (0,\infty)$,

$$\big|K_\omega(x,y)\big| \leq C \begin{cases} 1/x, & 0 < y < x/2, \\ 1/y, & y > 2x. \end{cases} \tag{5.4}$$

Let I_ν, $\nu > -1$, be the modified Bessel function of the first kind and order ν, which is given by

$$I_\nu(z) = \sum_{k=0}^\infty \frac{z^{\nu+2k}}{2^{\nu+2k}\Gamma(k+1)\Gamma(k+\nu+1)}, \quad z \in (0,\infty).$$

The main properties of I_ν can be found in [33, Section 5.7]. According to [33, pp. 108 and 123], if $\nu > -1$, then we have

$$I_\nu(z) \sim \frac{z^\nu}{2^\nu \Gamma(\nu+1)}, \quad \text{as } z \to 0^+, \tag{5.5}$$

and

$$\sqrt{z}I_\nu(z) = \frac{e^z}{\sqrt{2\pi}}\Big(\sum_{r=0}^{n}\frac{(-1)^r[\nu,r]}{(2z)^r} + \mathcal{O}\Big(\frac{1}{z^{n+1}}\Big)\Big), \tag{5.6}$$

where $[\nu,0]=1$ and

$$[\nu,r] = \frac{(4\nu^2-1)(4\nu^2-3^2)\cdots(4\nu^2-(2r-1)^2)}{2^{2r}\Gamma(r+1)}, \quad r=1,2,\dots$$

Since (see [33, p. 110])

$$\frac{d}{dz}\big(z^{-\nu}I_\nu(z)\big) = z^{-\nu}I_{\nu+1}(z), \quad z\in(0,\infty), \nu>-1, \tag{5.7}$$

it follows that, for every $t,x,y\in(0,\infty)$,

$$\begin{aligned}
&\partial_t\big[W_t^\lambda(x,y)-\mathbb{W}_t(x-y)\big] \\
&\quad= \partial_t\Big[\mathbb{W}_t(x-y)\Big\{\sqrt{2\pi}\Big(\frac{xy}{2t}\Big)^{\nu+1/2}\Big(\frac{xy}{2t}\Big)^{-\nu}I_\nu\Big(\frac{xy}{2t}\Big)e^{-xy/2t}-1\Big\}\Big] \\
&\quad= \partial_t\mathbb{W}_t(x-y)\Big\{\sqrt{2\pi}\Big(\frac{xy}{2t}\Big)^{1/2}I_\nu\Big(\frac{xy}{2t}\Big)e^{-xy/2t}-1\Big\} \\
&\qquad - \sqrt{2\pi}\mathbb{W}_t(x-y)\Big\{(\nu+1/2)\Big(\frac{xy}{2t}\Big)^{\nu-1/2}\frac{xy}{2t^2}\Big(\frac{xy}{2t}\Big)^{-\nu}I_\nu\Big(\frac{xy}{2t}\Big) \\
&\qquad + \Big(\frac{xy}{2t}\Big)^{\nu+1/2}\frac{xy}{2t^2}\Big(\frac{xy}{2t}\Big)^{-\nu}I_{\nu+1}\Big(\frac{xy}{2t}\Big) - \frac{xy}{2t^2}\Big(\frac{xy}{2t}\Big)^{1/2}I_\nu\Big(\frac{xy}{2t}\Big)\Big\}e^{-xy/2t} \\
&\quad= \partial_t\mathbb{W}_t(x-y)\Big\{\sqrt{2\pi}\Big(\frac{xy}{2t}\Big)^{1/2}I_\nu\Big(\frac{xy}{2t}\Big)e^{-xy/2t}-1\Big\} \\
&\qquad - \sqrt{2\pi}\mathbb{W}_t(x-y)\frac{xy}{2t^2}e^{-xy/2t} \\
&\qquad \times\Big\{(\nu+1/2)\frac{2t}{xy}\Big(\frac{xy}{2t}\Big)^{1/2}I_\nu\Big(\frac{xy}{2t}\Big) \\
&\qquad + \Big(\frac{xy}{2t}\Big)^{1/2}I_{\nu+1}\Big(\frac{xy}{2t}\Big) - \Big(\frac{xy}{2t}\Big)^{1/2}I_\nu\Big(\frac{xy}{2t}\Big)\Big\},
\end{aligned}$$

with $\nu=\lambda-1/2$.

From (5.5), we deduce that

$$\begin{gathered}\big|\partial_t\big[W_t^\lambda(x,y)-\mathbb{W}_t(x-y)\big]\big| \le C\frac{e^{-c(x-y)^2/t}}{t^{3/2}}, \\ \text{for every } t,x,y\in(0,\infty) \text{ and } xy\le 2t,\end{gathered} \tag{5.8}$$

and by using (5.6), that

$$\begin{gathered}\big|\partial_t\big[W_t^\lambda(x,y)-\mathbb{W}_t(x-y)\big]\big| \le C\frac{e^{-c(x-y)^2/t}}{t^{1/2}xy}, \\ \text{for every } t,x,y\in(0,\infty) \text{ and } xy\ge 2t.\end{gathered} \tag{5.9}$$

Combining (5.8) and (5.9), we obtain

$$\begin{aligned}\big|K_\omega(x,y)-K_\omega^\lambda(x,y)\big| &\le C\int_0^\infty \big|\partial_t\big[W_t^\lambda(x,y)-\mathbb{W}_t(x-y)\big]\big|\,dt\\ &\le C\Big(\int_0^{xy/2}\frac{e^{-c(x-y)^2/t}}{t^{1/2}xy}\,dt+\int_{xy/2}^\infty \frac{e^{-c(x^2+y^2)/t}}{t^{3/2}}\,dt\Big)\\ &\le \frac{C}{(xy)^{1/2}}\le\frac{C}{x},\quad x/2<y<2x, x\in(0,\infty). \end{aligned}\tag{5.10}$$

Moreover, (5.8) and (5.9) imply that, for each $x\in(0,\infty)$,

$$\begin{aligned}\big|K_\omega^\lambda(x,y)\big| &\le C\int_0^\infty \frac{e^{-c(x-y)^2/t}}{t^{3/2}}\,dt\le\frac{C}{|x-y|}\\ &\le C\begin{cases}1/x, & 0<y<x/2,\\ 1/y, & y>2x.\end{cases}\end{aligned}\tag{5.11}$$

Suppose that X is UMD and $1<q<\infty$. Let $f\in L^q(0,\infty)\otimes X$. We define the function $\tilde f$ by

$$\tilde f(x)=\begin{cases}0, & x\le 0,\\ f(x), & x>0.\end{cases}$$

Thus, $\tilde f\in L^q(\mathbb{R})\otimes X$. We have that

$$\Big(-\frac{d^2}{dx^2}\Big)^{i\omega}\tilde f(x)=\lim_{\varepsilon\to0^+}\Big(f(x)\alpha(\varepsilon)-\int_{0,|x-y|>\varepsilon}^\infty K_\omega(x,y)f(y)\,dy\Big),\quad \text{a.e. } x\in(0,\infty),$$

and

$$\Delta_\lambda^{i\omega}f(x)=\lim_{\varepsilon\to0^+}\Big(f(x)\alpha(\varepsilon)-\int_{0,|x-y|>\varepsilon}^\infty K_\omega^\lambda(x,y)f(y)\,dy\Big),\quad \text{a.e. } x\in(0,\infty).$$

Then, (5.4), (5.10), and (5.11) lead to

$$\begin{aligned}&\Big\|\Big(-\frac{d^2}{dx^2}\Big)^{i\omega}\tilde f(x)-\Delta_\lambda^{i\omega}f(x)\Big\|_X\\ &\quad\le\varlimsup_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty\big|K_\omega(x,y)-K_\omega^\lambda(x,y)\big|\big\|f(y)\big\|_X\,dy\\ &\quad\le C\big[H_0\big(\|f\|_X\big)(x)+H_\infty\big(\|f\|_X\big)(x)\big],\quad \text{a.e. } x\in(0,\infty).\end{aligned}$$

Hence, according to [20, p. 244, (9.9.1) and (9.9.2)], there exists $C>0$ such that

$$\Big\|\Big(-\frac{d^2}{dx^2}\Big)^{i\omega}\tilde f-\Delta_\lambda^{i\omega}f\Big\|_{L^q((0,\infty),X)}\le C\|f\|_{L^q((0,\infty),X)},\quad f\in L^q(0,\infty)\otimes X.$$

Moreover, by [18, Theorem, p. 402], we also have

$$\Big\|\Big(-\frac{d^2}{dx^2}\Big)^{i\omega}\tilde f\Big\|_{L^q((0,\infty),X)}\le C\|f\|_{L^q((0,\infty),X)},\quad f\in L^q(0,\infty)\otimes X.$$

We conclude that

$$\|\Delta_\lambda^{i\omega}f\|_{L^q((0,\infty),X)}\le C\|f\|_{L^q((0,\infty),X)},\quad f\in L^q(0,\infty)\otimes X.$$

Suppose now that $\Delta_\lambda^{i\omega}$ can be extended from $L^q(0,\infty)\otimes X$ to $L^q((0,\infty),X)$ as a bounded operator from $L^q((0,\infty),X)$ into itself. According to [18, Theorem, p. 402], in order to see that X is UMD, it is sufficient to see that, for a certain $C>0$,

$$\Big\|\Big(-\frac{d^2}{dx^2}\Big)^{i\omega}f\Big\|_{L^q(\mathbb{R},X)}\leq C\|f\|_{L^q(\mathbb{R},X)},\quad f\in L^q(\mathbb{R})\otimes X.$$

Let $f\in L^p(\mathbb{R})\otimes X$. By defining

$$f_+(x)=f(x),\qquad\text{and}\qquad f_-(x)=f(-x),\quad x\in(0,\infty),$$

we have that

$$\begin{aligned}\Big(-\frac{d^2}{dx^2}\Big)^{i\omega}&f(x)\\ &=\lim_{\varepsilon\to 0^+}\Big(f_+(x)\alpha(\varepsilon)-\int_{0,|x-y|>\varepsilon}^{\infty}K_\omega(x,y)f_+(y)\,dy-\int_{-\infty}^0K_\omega(x,y)f(y)\,dy\Big)\\ &=\lim_{\varepsilon\to 0^+}\Big(f_+(x)\alpha(\varepsilon)-\int_{0,|x-y|>\varepsilon}^{\infty}K_\omega(x,y)f_+(y)\,dy\Big)\\ &\quad-\int_0^\infty K_\omega(x,-y)f_-(y)\,dy\quad\text{a.e. } x\in(0,\infty),\end{aligned}$$

and

$$\begin{aligned}\Big(-\frac{d^2}{dx^2}\Big)^{i\omega}&f(x)\\ &=\lim_{\varepsilon\to 0^+}\Big(f(x)\alpha(\varepsilon)-\int_{-\infty,|x-y|>\varepsilon}^{0}K_\omega(x,y)f(y)\,dy-\int_0^\infty K_\omega(x,y)f(y)\,dy\Big)\\ &=\lim_{\varepsilon\to 0^+}\Big(f_-(-x)\alpha(\varepsilon)-\int_{0,|x+y|>\varepsilon}^{\infty}K_\omega(x,-y)f(-y)\,dy-\int_0^\infty K_\omega(x,y)f(y)\,dy\Big)\\ &=\lim_{\varepsilon\to 0^+}\Big(f_-(-x)\alpha(\varepsilon)-\int_{0,|x+y|>\varepsilon}^{\infty}K_\omega(x,-y)f_-(y)\,dy\Big)\\ &\quad-\int_0^\infty K_\omega(x,y)f_+(y)\,dy,\quad\text{a.e. } x\in(-\infty,0).\end{aligned}$$

We consider the operators

$$T_{\omega,1}(g)(x)=\lim_{\varepsilon\to 0^+}\Big(g(x)\alpha(\varepsilon)-\int_{0,|x-y|>\varepsilon}^{\infty}K_\omega(x,y)g(y)\,dy\Big),\quad x\in(0,\infty),$$

and

$$T_{\omega,2}(g)(x)=\int_0^\infty K_\omega(x,-y)g(y)\,dy,\quad x\in(0,\infty),$$

for every $g\in L^q(0,\infty)\otimes X$.

We can write

$$\begin{aligned}\Big\|\Big(-\frac{d^2}{dx^2}\Big)^{i\omega}f\Big\|_{L^q(\mathbb{R},X)}^q&=\big\|T_{\omega,1}(f_+)\big\|_{L^q((0,\infty),X)}^q+\big\|T_{\omega,2}(f_-)\big\|_{L^q((0,\infty),X)}^q\\ &\quad+\big\|T_{\omega,1}(f_-)\big\|_{L^q((0,\infty),X)}^q+\big\|T_{\omega,2}(f_+)\big\|_{L^q((0,\infty),X)}^q.\end{aligned}\tag{5.12}$$

According to (5.3), we get, for every $g \in L^q(0,\infty) \otimes X$,

$$\begin{aligned}\big\|T_{\omega,2}(g)(x)\big\|_X &\leq C \int_0^\infty \frac{\|g(y)\|_X}{x+y}\,dy \\ &\leq C\big[H_0\big(\|g\|_X\big)(x) + H_\infty\big(\|g\|_X\big)(x)\big], \quad x>0.\end{aligned}$$

Also, by combining (5.4), (5.10), and (5.11), we obtain, for each $g \in L^q(0,\infty)\otimes X$,

$$\big\|T_{\omega,1}(g)(x) - \Delta_\lambda^{i\omega}(g)(x)\big\|_X \leq C\big[H_0\big(\|g\|_X\big)(x) + H_\infty\big(\|g\|_X\big)(x)\big], \quad x \in (0,\infty).$$

Then, by [20, p. 244, (9.9.1) and (9.9.2)] it follows that, for every $g \in L^q(0,\infty)\otimes X$,

$$\big\|T_{\omega,2}(g)\big\|_{L^q((0,\infty),X)} + \big\|T_{\omega,1}(g) - \Delta_\lambda^{i\omega}(g)\big\|_{L^q((0,\infty),X)} \leq C\|g\|_{L^q((0,\infty),X)}. \tag{5.13}$$

Since $\Delta_\lambda^{i\omega}$ can be extended from $L^q(0,\infty)\otimes X$ to $L^q((0,\infty),X)$ as a bounded operator from $L^q((0,\infty),X)$ into itself, (5.12) and (5.13) imply that

$$\Big\|\Big(-\frac{d^2}{dx^2}\Big)^{i\omega} f\Big\|_{L^q(\mathbb{R},X)} \leq C\big(\|f_+\|_{L^q((0,\infty),X)} + \|f_-\|_{L^q((0,\infty),X)}\big) \leq C\|f\|_{L^q(\mathbb{R},X)},$$

for every $f \in L^q(\mathbb{R})\otimes X$. $\square$

Let $\beta > 0$ and $f \in \mathcal{S}_\lambda(0,\infty)$. According to Theorem 1.2, there exists a set $\Omega \subset (0,\infty)$, such that $|(0,\infty)\setminus\Omega| = 0$ and for every $x \in \Omega$, the functions $G_{P,\mathbb{C}}^{\lambda,\beta}(\Delta_\lambda^{i\omega} f)(\cdot,x)$ and $G_{P,\mathbb{C}}^{\lambda,\beta+1}(f)(\cdot,x)$ are in H. Let $x\in\Omega$. We denote by A_1 and A_2 the linear bounded operators from H into $\mathbb{C}$ defined by

$$A_1(h) = \int_0^\infty G_{P,\mathbb{C}}^{\lambda,\beta}(\Delta_\lambda^{i\omega} f)(t,x)h(t)\frac{dt}{t}, \quad h \in H,$$

and

$$A_2(h) = \int_0^\infty G_{P,\mathbb{C}}^{\lambda,\beta+1}(f)(t,x)h(t)\frac{dt}{t}, \quad h \in H.$$

We also define, for every $h \in H$,

$$T_{\omega,\beta}(h)(t) = \frac{1}{t^\beta}\int_0^t (t-s)^{\beta-1}h(t-s)\phi_\omega(s)\,ds, \quad t\in(0,\infty),$$

where $\phi_\omega(s) = s^{-2i\omega}/\Gamma(1-2i\omega)$, $s \in (0,\infty)$. Thus, $T_{\omega,\beta}$ is a linear bounded operator from H into itself. Indeed, Jensen's inequality leads to

$$\begin{aligned}\big\|T_{\omega,\beta}(h)\big\|_H &\leq \Big(\int_0^\infty \frac{1}{t^{2\beta+1}}\Big(\int_0^t \big|h(t-s)(t-s)^{\beta-1}\phi_\omega(s)\big|\,ds\Big)^2 dt\Big)^{1/2} \\ &\leq C\Big(\int_0^\infty \frac{1}{t}\Big(\int_0^t \big|h(u)\big|\frac{u^{\beta-1}\,du}{t^\beta}\Big)^2 dt\Big)^{1/2} \\ &\leq C\Big(\int_0^\infty \frac{1}{t^{\beta+1}}\int_0^t \big|h(u)\big|^2 u^{\beta-1}\,du\,dt\Big)^{1/2} \leq C\|h\|_H, \quad h\in H.\end{aligned}$$

We now show that

$$A_1(h) = -A_2(T_{\omega,\beta}h), \quad h \in H. \tag{5.14}$$

Indeed, let $h \in H$. Since $T_{\omega,\beta}h \in H$, we can write

$$A_2(T_{\omega,\beta}h) = \int_0^\infty G_{P,\mathbb{C}}^{\lambda,\beta+1}(f)(t,x)(T_{\omega,\beta}h)(t)\frac{dt}{t} = \int_0^\infty t^{\beta+1}\partial_t^{\beta+1}P_t^\lambda(f)(x)(T_{\omega,\beta}h)(t)\frac{dt}{t}.$$

By Lemma 3.1 we have that

$$\partial_t^{\beta+1}P_t^\lambda(f)(x) = e^{i\pi(\beta+1)}h_\lambda\big(y^{\beta+1}e^{-yt}h_\lambda(f)(y)\big)(x), \quad t,x \in (0,\infty). \tag{5.15}$$

Interchanging the order of integration twice, we get

$$\begin{aligned}
&A_2(T_{\omega,\beta}h)\\
&= e^{i\pi(\beta+1)}\int_0^\infty t^\beta h_\lambda\big(y^{\beta+1}e^{-yt}h_\lambda(f)(y)\big)(x)(T_{\omega,\beta}h)(t)\,dt\\
&= e^{i\pi(\beta+1)}h_\lambda\Big(h_\lambda(f)(y)y^{\beta+1}\int_0^\infty e^{-yt}t^\beta(T_{\omega,\beta}h)(t)\,dt\Big)(x)\\
&= e^{i\pi(\beta+1)}h_\lambda\Big(h_\lambda(f)(y)y^{\beta+1}\int_0^\infty e^{-yt}\int_0^t (t-s)^{\beta-1}h(t-s)\phi_\omega(s)\,ds\,dt\Big)(x)\\
&= e^{i\pi(\beta+1)}h_\lambda\Big(y^{\beta+2i\omega}h_\lambda(f)(y)\int_0^\infty e^{-yu}u^{\beta-1}h(u)\,du\Big)(x)\\
&= e^{i\pi(\beta+1)}\int_0^\infty h_\lambda\big[y^{\beta+2i\omega}h_\lambda(f)(y)e^{-yu}u^\beta\big](x)h(u)\frac{du}{u}\\
&= -\int_0^\infty h_\lambda\big[u^\beta e^{i\pi\beta}y^{\beta+2i\omega}e^{-yu}h_\lambda(f)(y)\big](x)h(u)\frac{du}{u}\\
&= -\int_0^\infty u^\beta\partial_u^\beta P_u^\lambda\big[h_\lambda\big(y^{2i\omega}h_\lambda(f)(y)\big)\big](x)h(u)\frac{du}{u}\\
&= -A_1(h), \quad h \in H,
\end{aligned}$$

and (5.14) is established. Note that the interchanges in the order of integration are justified because the function $\sqrt{z}J_{\lambda-1/2}(z)$ is bounded on $(0,\infty)$ and $h_\lambda(f) \in \mathcal{S}_\lambda(0,\infty)$.

From (5.14) we deduce that, for every $f \in \mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$,

$$G_{P,\mathbb{B}}^{\lambda,\beta}(\Delta_\lambda^{i\omega}f)(\cdot,x) = -G_{P,\mathbb{B}}^{\lambda,\beta+1}(f)(\cdot,x)\circ T_{\omega,\beta}, \quad \text{a.e. } x \in (0,\infty),$$

as elements of $L(H,\mathbb{B})$, the space of linear bounded operators from H into $\mathbb{B}$.

Let $f \in \mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$. Since $\Delta_\lambda^{i\omega}f \in L^p(0,\infty)\otimes\mathbb{B}$, (1.8) implies that

$$G_{P,\mathbb{B}}^{\lambda,\beta}(\Delta_\lambda^{i\omega}f)(\cdot,x) = -G_{P,\mathbb{B}}^{\lambda,\beta+1}(f)(\cdot,x)\circ T_{\omega,\beta}, \quad \text{a.e. } x \in (0,\infty), \tag{5.16}$$

as elements of $\gamma(H,\mathbb{B})$. Moreover, according to the ideal property for γ-radonifying operators [50, Theorem 6.2], we get

$$\big\|G_{P,\mathbb{B}}^{\lambda,\beta+1}(f)(\cdot,x)\circ T_{\omega,\beta}\big\|_{\gamma(H,\mathbb{B})} \le \|T_{\omega,\beta}\|_{L(H,H)}\big\|G_{P,\mathbb{B}}^{\lambda,\beta+1}(f)(\cdot,x)\big\|_{\gamma(H,\mathbb{B})},$$

for a.e. $x \in (0,\infty)$.

Then, (1.8) and (5.16) lead to

$$\begin{aligned}\big\|\Delta_\lambda^{i\omega}(f)\big\|_{L^p((0,\infty),\mathbb{B})} &\leq C\big\|G_{P,\mathbb{B}}^{\lambda,\beta}(\Delta_\lambda^{i\omega}f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}\\ &= C\big\|G_{P,\mathbb{B}}^{\lambda,\beta+1}(f)\circ T_{\omega,\beta}\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}\\ &\leq C\big\|G_{P,\mathbb{B}}^{\lambda,\beta+1}(f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}\\ &\leq C\|f\|_{L^p((0,\infty),\mathbb{B})}.\end{aligned}$$

Hence, $\Delta_\lambda^{i\omega}$ can be extended from $L^p(0,\infty)\otimes\mathbb{B}$ to $L^p((0,\infty),\mathbb{B})$ as a bounded operator from $L^p((0,\infty),\mathbb{B})$ into itself. By Proposition 5.1 we conclude that $\mathbb{B}$ is UMD, and the proof of Theorem 1.4 is complete.

6. Proof of Theorem 1.5

The Bessel operator Δ_λ is positive in $L^2(0,\infty)$. Then the square root $\sqrt{\Delta_\lambda}$ of Δ_λ is defined by

$$\sqrt{\Delta_\lambda}f = h_\lambda\big(yh_\lambda(f)\big),\quad f\in D(\sqrt{\Delta_\lambda}),$$

where, since h_λ is an isometry in $L^2(0,\infty)$, the domain $D(\sqrt{\Delta_\lambda})$ of $\sqrt{\Delta_\lambda}$ is the following set:

$$D(\sqrt{\Delta_\lambda}) = \big\{f\in L^2(0,\infty) : yh_\lambda(f)\in L^2(0,\infty)\big\}.$$

The Poisson semigroup $\{P_t^\lambda\}_{t>0}$ is the one generated by the operator $-\sqrt{\Delta_\lambda}$. We define $M(y)=m(y^2)$, $y\in(0,\infty)$. It is clear that the $\sqrt{\Delta_\lambda}$-multiplier associated with M coincides with the Δ_λ-multiplier defined by m. Since the function M satisfies the conditions specified in [37, Theorem 1], from the proof of [37, Theorem 1] we deduce that, for every $n\in\mathbb{N}$, and $f\in\mathcal{S}_\lambda(0,\infty)$,

$$\begin{aligned}&t^{n+1}\partial_t^{n+1}P_t^\lambda\big(M(\sqrt{\Delta_\lambda})f\big)(x)\\ &\quad=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{M}_n(t,u)t\partial_t P_{t/2}^\lambda(\Delta_\lambda^{iu/2}f)(x)\,du,\quad t,x>0,\end{aligned}\tag{6.1}$$

where

$$\mathcal{M}_n(t,u)=\int_0^\infty y^{-iu-1}M_n(t,y)\,dy,\quad u\in\mathbb{R}\text{ and }t\in(0,\infty),$$

and

$$M_n(t,y)=(ty)^n e^{-ty/2}M(y),\quad t,y\in(0,\infty).$$

We also have that, for every $n\in\mathbb{N}$ and $f\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$,

$$t^{n+1}\partial_t^{n+1}P_t^\lambda\big(M(\sqrt{\Delta_\lambda})f\big)(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{M}_n(t,u)t\partial_t P_{t/2}^\lambda(\Delta_\lambda^{iu/2}f)(x)\,du,\quad t,x>0.$$

Moreover, according to [37, Theorem 1], $M(\sqrt{\Delta_\lambda})f\in L^p(0,\infty)\otimes\mathbb{B}$, $f\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$.

Let $n\in\mathbb{N}$. We define, for every $u\in\mathbb{R}$, the operator

$$L_{n,u}(h)(t)=\mathcal{M}_n(t,u)h(t),\quad t\in(0,\infty).$$

Since

$$\sup_{\substack{u\in\mathbb{R}\\ t\in(0,\infty)}} \big|\mathcal{M}_n(t,u)\big| \le C\|m\|_{L^\infty(0,\infty)},$$

the family of operators $\{L_{n,u}\}_{u\in\mathbb{R}}$ is bounded in $L(H,H)$.

Let $f\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$. Since h_λ is an isometry in $L^2(0,\infty)$, (5.1) and (5.15) allow us to write

$$t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x) = -\frac{1}{2}h_\lambda\big(tye^{-ty/2}y^{iu}h_\lambda(f)(y)\big)(x),$$
$$\text{for every } t,x\in(0,\infty) \text{ and } u\in\mathbb{R}.$$

Then, Minkowski's inequality leads to

$$\begin{aligned}
&\Big(\int_0^\infty \big\|t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x)\big\|^2_{\mathbb{B}}\frac{dt}{t}\Big)^{1/2}\\
&\quad\le C\int_0^\infty \big\|h_\lambda(f)(y)\big\|_{\mathbb{B}}\Big(\int_0^\infty |tye^{-ty/2}|^2\frac{dt}{t}\Big)^{1/2}dy\\
&\quad\le C\int_0^\infty \big\|h_\lambda(f)(y)\big\|_{\mathbb{B}}\,dy<\infty,\quad x\in(0,\infty)\text{ and } u\in\mathbb{R},
\end{aligned}$$

because $h_\lambda(f)\in\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$ and the function $\sqrt{z}J_\nu(z)$ is bounded on $(0,\infty)$ when $\nu>-1/2$. We conclude that

$$t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x)\in\gamma(H,\mathbb{B}),\quad u\in\mathbb{R}\text{ and } x\in(0,\infty).$$

According to [37, p. 642], we get

$$\int_{\mathbb{R}}\big|\mathcal{M}_n(t,u)\big|\big\|t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x)\big\|_{\mathbb{B}}\,du\in L^p\big((0,\infty),L^2\big((0,\infty),dt/t\big)\big),$$

and we infer that

$$\int_0^\infty\Big(\int_{\mathbb{R}}\big|\mathcal{M}_n(t,u)\big|\big\|t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x)\big\|_{\mathbb{B}}\,du\Big)^2\frac{dt}{t}<\infty,\quad \text{a.e. } x\in(0,\infty).$$

If $h\in H$, then we have that

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}\mathcal{M}_n(t,u)t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x)h(t)\frac{du\,dt}{t}\\
&\quad=\int_{\mathbb{R}}\int_0^\infty\mathcal{M}_n(t,u)t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x)h(t)\frac{dt\,du}{t},\quad \text{a.e. } x\in(0,\infty).
\end{aligned}$$

Hence, if $\{h_j\}_{j=1}^k$ is an orthonormal system in H, we can write

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_{j=1}^k\gamma_j\int_0^\infty\int_{\mathbb{R}}\mathcal{M}_n(t,u)t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x)h_j(t)\frac{du\,dt}{t}\Big\|^2_{\mathbb{B}}\Big)^{1/2}\\
&\quad=\Big(\mathbb{E}\Big\|\int_{\mathbb{R}}\sum_{j=1}^k\gamma_j\int_0^\infty\mathcal{M}_n(t,u)t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2}f)(x)h_j(t)\frac{dt\,du}{t}\Big\|^2_{\mathbb{B}}\Big)^{1/2}
\end{aligned}$$

$$\leq \int_{\mathbb{R}} \Big(\mathbb{E}\Big\|\sum_{j=1}^{k} \gamma_j \int_0^\infty \mathcal{M}_n(t,u) t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x) h_j(t) \frac{dt}{t}\Big\|_{\mathbb{B}}^2\Big)^{1/2} du$$
$$\leq \int_{\mathbb{R}} \big\|\mathcal{M}_n(t,u) t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x)\big\|_{\gamma(H,\mathbb{B})} du, \quad \text{a.e. } x \in (0,\infty).$$

Here $\{\gamma_j\}_{j=1}^\infty$ is a sequence of independent Gaussian variables.

We conclude that, for a.e. $x \in (0,\infty)$,

$$\Big\|\int_{\mathbb{R}} \mathcal{M}_n(t,u) t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x)\, du\Big\|_{\gamma(H,\mathbb{B})}$$
$$\leq C \int_{\mathbb{R}} \big\|\mathcal{M}_n(t,u) t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x)\big\|_{\gamma(H,\mathbb{B})} du. \tag{6.2}$$

For every $u \in \mathbb{R}$ we have that

$$\mathcal{M}_n(t,u) t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x) = t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x) \circ L_{n,u}, \quad \text{a.e. } x \in (0,\infty),$$

in the sense of equality in $L(H,\mathbb{B})$. According to [50, Theorem 6.2], we get

$$\big\|\mathcal{M}_n(t,u) t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x)\big\|_{\gamma(H,\mathbb{B})}$$
$$\leq \|L_{n,u}\|_{L(H,H)} \big\|t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x)\big\|_{\gamma(H,\mathbb{B})}$$
$$\leq C \sup_{t>0}\big|\mathcal{M}_n(t,u)\big| \big\|t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x)\big\|_{\gamma(H,\mathbb{B})}, \quad \text{a.e. } x \in (0,\infty). \tag{6.3}$$

Putting together (6.1), (6.2), and (6.3) and by taking into account Theorem 1.2 and Proposition 5.1, we obtain

$$\big\|m(\Delta_\lambda) f\big\|_{L^p((0,\infty),\mathbb{B})}$$
$$= \big\|M(\sqrt{\Delta_\lambda}) f\big\|_{L^p((0,\infty),\mathbb{B})}$$
$$\leq C \big\|G^{\lambda,n+1}_{P,\mathbb{B}}\big(M(\sqrt{\Delta_\lambda}) f\big)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}$$
$$\leq C \Big\|\int_{\mathbb{R}} \mathcal{M}_n(t,u) t\partial_t P^\lambda_{t/2}(\Delta_\lambda^{iu/2} f)(x)\, du\Big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))}$$
$$\leq C \int_{\mathbb{R}} \sup_{t>0}\big|\mathcal{M}_n(t,u)\big| \big\|G^{\lambda,1}_{P,\mathbb{B}}(\Delta_\lambda^{iu/2} f)\big\|_{L^p((0,\infty),\gamma(H,\mathbb{B}))} du$$
$$\leq C \int_{\mathbb{R}} \sup_{t>0}\big|\mathcal{M}_n(t,u)\big| \|\Delta_\lambda^{iu/2} f\|_{L^p((0,\infty),\mathbb{B})} du$$
$$\leq C\Big(\int_{\mathbb{R}} \sup_{t>0}\big|\mathcal{M}_n(t,u)\big| \|\Delta_\lambda^{iu/2}\|_{L^p((0,\infty),\mathbb{B})\to L^p((0,\infty),\mathbb{B})} du\Big) \|f\|_{L^p((0,\infty),\mathbb{B})}.$$

Hence, $m(\Delta_\lambda)$ can be extended from $\mathcal{S}_\lambda(0,\infty)\otimes\mathbb{B}$ to $L^p((0,\infty),\mathbb{B})$ as a bounded operator from $L^p((0,\infty),\mathbb{B})$ into itself.

7. Proof of Theorem 1.6

In order to apply Theorem 1.5, it is necessary to know nice estimations for the norm

$$\|\Delta_\lambda^{i\omega}\|_{L^p((0,\infty),\mathbb{B})\to L^p((0,\infty),\mathbb{B})}, \quad \omega \in \mathbb{R}\setminus\{0\}.$$

For this purpose we use a Banach-valued version of [40, Theorem 4.3] concerning to local Calderón–Zygmund operators. By taking in mind [43], the same proof of [40, Theorem 4.3] also works to show the following result.

Proposition 7.1. *Let X be a Banach space. Assume that $K:(0,\infty)\times(0,\infty)\setminus\{(x,x):x\in(0,\infty)\}\to\mathbb{R}$ is a differentiable function satisfying that, for certain $M>0$,*

$$\big|K(x,y)\big| \le \frac{M}{|x-y|}, \quad x,y\in(0,\infty), x\neq y,$$

and

$$\big|\partial_x K(x,y)\big| + \big|\partial_y K(x,y)\big| \le \frac{M}{|x-y|^2}, \quad 0<\frac{x}{2}<y<2x, x\neq y.$$

Suppose that T is a bounded operator from $L^q((0,\infty),X)$ into itself, for some $1<q<\infty$, such that for every $f\in S_\lambda(0,\infty)\otimes X$,

$$(Tf)(x) = \int_0^\infty K(x,y)f(y)\,dy, \quad a.e.\ x\notin \operatorname{supp}(f).$$

Then,

(i) *for every $1<p<\infty$, T can be extended to $L^p((0,\infty),X)$ as a bounded operator T_p from $L^p((0,\infty),X)$ into itself and, for certain $C>0$,*

$$\|T_p\|_{L^p((0,\infty),X)\to L^p((0,\infty),X)} \le C\big(M + \|T\|_{L^q((0,\infty),X)\to L^q((0,\infty),X)}\big); \tag{7.1}$$

(ii) *T can be extended to $L^1((0,\infty),X)$ as a bounded operator T_1 from $L^1((0,\infty),X)$ into $L^{1,\infty}((0,\infty),X)$ and, for certain $C>0$,*

$$\|T_1\|_{L^1((0,\infty),X)\to L^{1,\infty}((0,\infty),X)} \le C\big(M + \|T\|_{L^q((0,\infty),X)\to L^q((0,\infty),X)}\big). \tag{7.2}$$

The constant C in (7.1) and (7.2) does not depend on T.

The next result cannot be deduced from [46, Theorem 2.5.1] when $0<\lambda<1$ and $p>1$ because the semigroup $\{P_t^\lambda\}_{t>0}$ is not contractive for $0<\lambda<1$.

Proposition 7.2. *Let X be a UMD Banach space, $\lambda>0$ and $1<p<\infty$. Then there exists $C>0$ such that*

$$\|\Delta_\lambda^{i\omega}\|_{L^p((0,\infty),X)\to L^p((0,\infty),X)} \le Ce^{\pi|\omega|}, \quad \omega\in\mathbb{R}.$$

Moreover, if $\lambda\ge 1$ for every $\omega\in\mathbb{R}\setminus\{0\}$, $\Delta_\lambda^{i\omega}$ can be extended to $L^1((0,\infty),X)$ as a bounded operator from $L^1((0,\infty),X)$ into $L^{1,\infty}((0,\infty),X)$, and

$$\|\Delta_\lambda^{i\omega}\|_{L^1((0,\infty),X)\to L^{1,\infty}((0,\infty),X)} \le Ce^{\pi|\omega|},$$

where $C>0$ does not depend on ω.

Proof. Let $\omega \in \mathbb{R} \setminus \{0\}$. According to Proposition 5.1, the operator $\Delta_\lambda^{i\omega}$ can be extended to $L^p((0,\infty),X)$ as a bounded operator from $L^p((0,\infty),X)$ into itself. Moreover, by (5.16), for every $f \in \mathcal{S}_\lambda(0,\infty)\otimes X$, we have

$$G_{P,X}^{\lambda,1}(\Delta_\lambda^{i\omega} f)(\cdot,x) = -G_{P,X}^{\lambda,2}(f)(\cdot,x)\circ T_\omega, \quad \text{a.e. } x \in (0,\infty),$$

as elements of $\gamma(H,X)$, where

$$T_\omega(h)(t) = \frac{1}{t}\int_0^t h(t-s)\frac{s^{-2i\omega}}{\Gamma(1-2i\omega)}\,ds, \quad h \in H.$$

As in the proof of Theorem 1.4, we can see that

$$\|T_\omega\|_{L(H,H)} \le \frac{1}{|\Gamma(1-2i\omega)|} \le e^{\pi|\omega|},$$

and, for every $f \in \mathcal{S}_\lambda(0,\infty)\otimes X$,

$$\begin{aligned}\|\Delta_\lambda^{i\omega} f\|_{L^p((0,\infty),X)} &\le C\big\|G_{P,X}^{\lambda,1}(\Delta_\lambda^{i\omega} f)\big\|_{L^p((0,\infty),\gamma(H,X))}\\ &\le Ce^{\pi|\omega|}\big\|G_{P,X}^{\lambda,2}(f)\big\|_{L^p((0,\infty),\gamma(H,X))} \le Ce^{\pi|\omega|}\|f\|_{L^p((0,\infty),X)},\end{aligned}$$

that is,

$$\|\Delta_\lambda^{i\omega}\|_{L^p((0,\infty),X)\to L^p((0,\infty),X)} \le Ce^{\pi|\omega|}, \tag{7.3}$$

where $C>0$ does not depend on ω.

We are going to show that $\Delta_\lambda^{i\omega}$ is an X-valued local Calderón–Zygmund operator. According to (5.8) and (5.9), we have

$$\big|\partial_t W_t^\lambda(x,y)\big| \le C\frac{e^{-c(x-y)^2/t}}{t^{3/2}}, \quad t,x,y \in (0,\infty).$$

Then

$$\begin{aligned}\big|K_\omega^\lambda(x,y)\big| &\le C\int_0^\infty \frac{|t^{-i\omega}|}{|\Gamma(1-i\omega)|}\frac{e^{-c(x-y)^2/t}}{t^{3/2}}\,dt\\ &\le C\frac{e^{\pi|\omega|/2}}{|x-y|}, \quad x,y\in(0,\infty), x\ne y.\end{aligned} \tag{7.4}$$

From (7.4) we deduce that, for every $f \in \mathcal{S}_\lambda(0,\infty)\otimes X$,

$$\int_0^\infty \big|K_\omega^\lambda(x,y)\big|\big|f(y)\big|\,dy < \infty, \quad x \notin \operatorname{supp}(f).$$

Hence, for each $f \in \mathcal{S}_\lambda(0,\infty)\otimes X$, (5.2) implies that

$$\Delta_\lambda^{i\omega} f(x) = \int_0^\infty K_\omega^\lambda(x,y)f(y)\,dy, \quad \text{a.e. } x \notin \operatorname{supp}(f).$$

We can write

$$\begin{aligned}\partial_x\,\partial_t W_t^\lambda(x,y) &= \partial_x\,\partial_t\Big[\mathbb{W}_t(x-y)\sqrt{2\pi}\Big(\frac{xy}{2t}\Big)^{1/2} I_{\lambda-1/2}\Big(\frac{xy}{2t}\Big)e^{-xy/2t}\Big]\\ &= \partial_x\,\partial_t\big[\mathbb{W}_t(x-y)\big]\sqrt{2\pi}\Big(\frac{xy}{2t}\Big)^{1/2} I_{\lambda-1/2}\Big(\frac{xy}{2t}\Big)e^{-xy/2t}\end{aligned}$$

$$
\begin{aligned}
&+ \partial_x\big[\mathbb{W}_t(x-y)\big]\sqrt{2\pi}\partial_t\Big[\Big(\frac{xy}{2t}\Big)^{1/2} I_{\lambda-1/2}\Big(\frac{xy}{2t}\Big)e^{-xy/2t}\Big] \\
&+ \partial_t\big[\mathbb{W}_t(x-y)\big]\sqrt{2\pi}\partial_x\Big[\Big(\frac{xy}{2t}\Big)^{1/2} I_{\lambda-1/2}\Big(\frac{xy}{2t}\Big)e^{-xy/2t}\Big] \\
&+ \mathbb{W}_t(x-y)\sqrt{2\pi}\partial_x\,\partial_t\Big[\Big(\frac{xy}{2t}\Big)^{1/2} I_{\lambda-1/2}\Big(\frac{xy}{2t}\Big)e^{-xy/2t}\Big] \\
= &\sum_{j=1}^{4}\mathcal{E}_j(t,x,y), \quad t,x,y \in (0,\infty).
\end{aligned}
$$

Applying (5.6) and (5.7), we obtain the following:

(A)
$$
\begin{aligned}
&\partial_t\Big[\Big(\frac{xy}{2t}\Big)^{1/2} I_\nu\Big(\frac{xy}{2t}\Big)e^{-xy/2t}\Big] \\
&= -\frac{xy}{2t^2}\frac{d}{dz}\big[z^{\nu+1/2}z^{-\nu}I_\nu(z)e^{-z}\big]_{|z=xy/2t} \\
&= -\frac{xy}{2t^2}\big[(\nu+1/2)z^{\nu-1/2}z^{-\nu}I_\nu(z)e^{-z} + z^{\nu+1/2}z^{-\nu}I_{\nu+1}(z)e^{-z} \\
&\quad - z^{\nu+1/2}z^{-\nu}I_\nu(z)e^{-z}\big]_{|z=xy/2t} \\
&= -\frac{1}{\sqrt{2\pi}}\frac{xy}{2t^2}\Big[\frac{\nu+1/2}{z}\Big\{1+\mathcal{O}\Big(\frac{1}{z}\Big)\Big\} + 1 - \frac{[\nu+1,1]}{2z} + \mathcal{O}\Big(\frac{1}{z^2}\Big) - 1 \\
&\quad + \frac{[\nu,1]}{2z} + \mathcal{O}\Big(\frac{1}{z^2}\Big)\Big]_{|z=xy/2t} \\
&= \frac{xy}{t^2}\mathcal{O}\Big(\Big(\frac{t}{xy}\Big)^2\Big), \quad t,x,y\in(0,\infty);
\end{aligned}
$$

(B)
$$
\partial_x\Big[\Big(\frac{xy}{2t}\Big)^{1/2} I_\nu\Big(\frac{xy}{2t}\Big)e^{-xy/2t}\Big] = \frac{y}{t}\mathcal{O}\Big(\Big(\frac{t}{xy}\Big)^2\Big), \quad t,x,y\in(0,\infty);
$$

(C)
$$
\begin{aligned}
&\partial_x\,\partial_t\Big[\Big(\frac{xy}{2t}\Big)^{1/2} I_\nu\Big(\frac{xy}{2t}\Big)e^{-xy/2t}\Big] \\
&= \partial_x\Big[-\frac{xy}{2t^2}\frac{d}{dz}\big[z^{\nu+1/2}z^{-\nu}I_\nu(z)e^{-z}\big]_{|z=xy/2t}\Big] \\
&= -\frac{y}{2t^2}\frac{d}{dz}\big[z^{\nu+1/2}z^{-\nu}I_\nu(z)e^{-z}\big]_{|z=xy/2t} \\
&\quad - \frac{xy^2}{4t^3}\frac{d^2}{dz^2}\big[z^{\nu+1/2}z^{-\nu}I_\nu(z)e^{-z}\big]_{|z=xy/2t} \\
&= \frac{y}{t^2}\mathcal{O}\Big(\Big(\frac{t}{xy}\Big)^2\Big) - \frac{xy^2}{4t^3}\frac{d}{dz}\big[(\nu+1/2)z^{\nu-1/2}z^{-\nu}I_\nu(z)e^{-z} \\
&\quad + z^{\nu+1/2}z^{-\nu}I_{\nu+1}(z)e^{-z} - z^{\nu+1/2}z^{-\nu}I_\nu(z)e^{-z}\big]_{|z=xy/2t} \\
&= \frac{y}{t^2}\mathcal{O}\Big(\Big(\frac{t}{xy}\Big)^2\Big) - \frac{xy^2}{4t^3}\Big[\frac{\nu^2-1/4}{z^2}\sqrt{z}I_\nu(z)e^{-z} + \frac{2\nu+2}{z}\sqrt{z}I_{\nu+1}(z)e^{-z} \\
&\quad + \sqrt{z}I_{\nu+2}(z)e^{-z} - 2\sqrt{z}I_{\nu+1}(z)e^{-z} + \sqrt{z}I_\nu(z)e^{-z} \\
&\quad - \frac{2\nu+1}{z}\sqrt{z}I_\nu(z)e^{-z}\Big]_{|z=xy/2t}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{y}{t^2}\mathcal{O}\Big(\Big(\frac{t}{xy}\Big)^2\Big) - \frac{xy^2}{4\sqrt{2\pi}t^3}\Big[\frac{\nu^2-1/4}{z^2}\Big\{1+\mathcal{O}\Big(\frac{1}{z}\Big)\Big\}\\
&\quad + \frac{2\nu+2}{z}\Big\{1-\frac{[\nu+1,1]}{2z}+\mathcal{O}\Big(\frac{1}{z^2}\Big)\Big\} - \frac{2\nu+1}{z}\Big\{1-\frac{[\nu,1]}{2z}+\mathcal{O}\Big(\frac{1}{z^2}\Big)\Big\}\\
&\quad + \Big\{1-\frac{[\nu+2,1]}{2z}+\frac{[\nu+2,2]}{4z^2}+\mathcal{O}\Big(\frac{1}{z^3}\Big)\Big\}\\
&\quad - 2\Big\{1-\frac{[\nu+1,1]}{2z}+\frac{[\nu+1,2]}{4z^2}+\mathcal{O}\Big(\frac{1}{z^3}\Big)\Big\}\\
&\quad + \Big\{1-\frac{[\nu,1]}{2z}+\frac{[\nu,2]}{4z^2}+\mathcal{O}\Big(\frac{1}{z^3}\Big)\Big\}\Big]_{|z=xy/2t}\\
&= \frac{xy^2}{t^3}\mathcal{O}\Big(\Big(\frac{t}{xy}\Big)^3\Big), \quad t,x,y\in(0,\infty).
\end{aligned}
$$

Here $\nu=\lambda-1/2$. Then we deduce the following:

- $|\mathcal{E}_1(t,x,y)| \le C\frac{e^{-c(x-y)^2/t}}{t^2}$, $t,x,y\in(0,\infty)$,
- $|\mathcal{E}_2(t,x,y)| \le C\frac{e^{-c(x-y)^2/t}}{t}\frac{xy}{t^2}\frac{t^2}{(xy)^2}$, $t,x,y\in(0,\infty)$,
- $|\mathcal{E}_3(t,x,y)| \le C\frac{e^{-c(x-y)^2/t}}{t^{3/2}}\frac{y}{t}\frac{t^2}{(xy)^2}$, $t,x,y\in(0,\infty)$,

and

- $|\mathcal{E}_4(t,x,y)| \le C\frac{e^{-c(x-y)^2/t}}{t^{1/2}}\frac{xy^2}{t^3}\frac{t^3}{(xy)^3}$, $t,x,y\in(0,\infty)$.

We now estimate

$$\int_0^{xy/2}\big|\mathcal{E}_j(t,x,y)\big|\,dt, \quad j=1,2,3,4.$$

First, we have that, when $x,y\in(0,\infty), x\neq y$,

$$\int_0^{xy/2}\big|\mathcal{E}_1(t,x,y)\big|\,dt \le C\int_0^\infty \frac{e^{-c(x-y)^2/t}}{t^2}\,dt \le \frac{C}{|x-y|^2},$$

and also

$$\int_0^{xy/2}\big|\mathcal{E}_2(t,x,y)\big|\,dt \le C\int_0^{xy/2}\frac{e^{-c(x-y)^2/t}}{txy}\,dt \le C\int_0^\infty \frac{e^{-c(x-y)^2/t}}{t^2}\,dt \le \frac{C}{|x-y|^2}.$$

To study $\mathcal{E}_3$ and $\mathcal{E}_4$, we distinguish two cases:

$$
\begin{aligned}
&\int_0^{xy/2}\big(\big|\mathcal{E}_3(t,x,y)\big|+\big|\mathcal{E}_4(t,x,y)\big|\big)\,dt\\
&\quad\le C\int_0^{xy/2}\frac{e^{-c(x-y)^2/t}}{\sqrt{t}}\frac{dt}{x^2y}\\
&\quad\le \begin{cases} C\int_0^{xy/2}\frac{e^{-c(x-y)^2/t}}{\sqrt{t}}\frac{1}{x^2y}\big(\frac{xy}{t}\big)^{3/2}\,dt \le C\sqrt{\frac{y}{x}}\int_0^{xy/2}\frac{e^{-c(x-y)^2/t}}{t^2}\,dt,\\ C\int_0^{xy/2}\frac{e^{-c(x-y)^2/t}}{\sqrt{t}}\frac{1}{x^2y}\big(\frac{xy}{t}\big)^{2}\,dt \le Cy\int_0^{xy/2}\frac{e^{-c(x-y)^2/t}}{t^{5/2}}\,dt\end{cases}\\
&\quad\le \begin{cases} C\int_0^{\infty}\frac{e^{-c(x-y)^2/t}}{t^2}\,dt \le \frac{C}{|x-y|^2}, & y<2x, x\in(0,\infty),\\ Cy\int_0^{\infty}\frac{e^{-c(x-y)^2/t}}{t^{5/2}}\,dt \le C\frac{y}{|x-y|^3}\le\frac{C}{|x-y|^2}, & y\ge 2x, x>0.\end{cases}
\end{aligned}
$$

Hence, we conclude that

$$\int_0^{xy/2} \left|\partial_x \partial_t W_t^\lambda(x,y)\right| dt \leq \frac{C}{|x-y|^2}, \quad x,y \in (0,\infty), x \neq y. \tag{7.5}$$

According to (5.5) and by taking in mind the above calculations, we get

$$\left|\partial_x \partial_t W_t^\lambda(x,y)\right| \leq C\frac{e^{-c(x^2+y^2)/t}}{t^2}\left[\left(\frac{xy}{t}\right)^\lambda + \frac{y}{\sqrt{t}}\left(\frac{xy}{t}\right)^{\lambda-1}\right] \leq C\frac{x^{2\lambda-1}}{t^{\lambda+3/2}},$$

for every $t,x,y \in (0,\infty)$ such that $xy \leq 2t$ and $x/2 \leq y \leq 2x$. Then

$$\int_{xy/2}^\infty \left|\partial_x \partial_t W_t^\lambda(x,y)\right| dt \leq C\int_{xy/2}^\infty \frac{x^{2\lambda-1}}{t^{\lambda+3/2}}\, dt \leq \frac{C}{x^2}, \quad 0 < \frac{x}{2} \leq y \leq 2x. \tag{7.6}$$

From (7.5) and (7.6) we deduce that

$$\left|\partial_x K_\omega^\lambda(x,y)\right| \leq C\frac{e^{\pi|\omega|/2}}{|x-y|^2}, \quad 0 < \frac{x}{2} \leq y \leq 2x, x \neq y. \tag{7.7}$$

Since $K_\omega^\lambda(x,y) = K_\omega^\lambda(y,x)$, $x,y \in (0,\infty)$, we also have that

$$\left|\partial_y K_\omega^\lambda(x,y)\right| \leq C\frac{e^{\pi|\omega|/2}}{|x-y|^2}, \quad 0 < \frac{x}{2} \leq y \leq 2x, x \neq y. \tag{7.8}$$

By (7.4), (7.7), and (7.8), K_ω^λ is a local Calderón–Zygmund kernel.

By applying now Proposition 7.1, we obtain that the operator $\Delta_\lambda^{i\omega}$ can be extended to $L^1((0,\infty),X)$ as a bounded operator, that we continue denoting by Δ_λ^{iw}, from $L^1((0,\infty),X)$ into $L^{1,\infty}((0,\infty),X)$. Moreover, (7.3), (7.4), (7.7), and (7.8) lead to

$$\|\Delta_\lambda^{i\omega}\|_{L^1((0,\infty),X)\to L^{1,\infty}((0,\infty),X)} \leq Ce^{\pi|\omega|},$$

where $C > 0$ does not depend on ω. □

Proposition 7.3. *Let $\mathbb{H}$ be a Hilbert space and $\lambda > 0$. Then, $\|\Delta_\lambda^{i\omega}\|_{L^2((0,\infty),\mathbb{H})} = 1$, for every $\omega \in \mathbb{R} \setminus \{0\}$.*

Proof. We consider $f \in L^2(0,\infty) \otimes \mathbb{H}$, that is, $f = \sum_{j=1}^n a_j f_j$ where $a_j \in \mathbb{H}$ and $f_j \in L^2(0,\infty)$. By using the Plancherel equality for Hankel transforms on $L^2(0,\infty)$, we can write

$$\begin{aligned}
\int_0^\infty \left\|h_\lambda(f)(x)\right\|_{\mathbb{H}}^2 dx &= \int_0^\infty \left\langle h_\lambda(f)(x), h_\lambda(f)(x)\right\rangle_{\mathbb{H}} dx \\
&= \sum_{i,j=1}^n \langle a_i, a_j\rangle_{\mathbb{H}} \int_0^\infty h_\lambda(f_i)(x) h_\lambda(f_j)(x)\, dx \\
&= \sum_{i,j=1}^n \langle a_i, a_j\rangle_{\mathbb{H}} \int_0^\infty f_i(x) f_j(x)\, dx \\
&= \int_0^\infty \left\langle f(x), f(x)\right\rangle_{\mathbb{H}} dx = \int_0^\infty \left\|f(x)\right\|_{\mathbb{H}}^2 dx.
\end{aligned}$$

Hence, h_λ can be extended to $L^2((0,\infty),\mathbb{H})$ boundedly from $L^2((0,\infty),\mathbb{H})$ into itself. Since $|y^{2i\omega}|=1$, $y\in(0,\infty)$ and $\omega\in\mathbb{R}\setminus\{0\}$, by (5.1) we conclude that, for every $\omega\in\mathbb{R}\setminus\{0\}$, $\Delta_\lambda^{i\omega}$ is bounded from $L^2((0,\infty),\mathbb{H})$ into itself and

$$\|\Delta_\lambda^{i\omega}\|_{L^2((0,\infty),\mathbb{H})\to L^2((0,\infty),\mathbb{H})}=1. \qquad \square$$

Let $\omega\in\mathbb{R}\setminus\{0\}$ and assume that $\mathbb{B}=[\mathbb{H},X]_\theta$, where $\mathbb{H}$ is a Hilbert space and X is a UMD space, $0<\theta<\vartheta/\pi$. Then, by using the interpolation theorem for vector-valued Lebesgue spaces [1, Theorem 5.1.2] and Propositions 7.2 and 7.3, we deduce that $\Delta_\lambda^{i\omega}$ is a bounded operator from $L^p((0,\infty),\mathbb{B})$ into itself, with $p=2/(1+\theta)$ and

$$\begin{aligned}\|\Delta_\lambda^{i\omega}\|&_{L^p((0,\infty),\mathbb{B})\to L^p((0,\infty),\mathbb{B})}\\ &\le C\|\Delta_\lambda^{i\omega}\|^{1-\theta}_{L^2((0,\infty),\mathbb{H})\to L^2((0,\infty),\mathbb{H})}\|\Delta_\lambda^{i\omega}\|^{\theta}_{L^1((0,\infty),X)\to L^{1,\infty}((0,\infty),X)}\\ &\le Ce^{2\pi(1/p-1/2)|\omega|}.\end{aligned}$$

Here $C>0$ does not depend on ω.

Since $\Delta_\lambda^{i\omega}$ is self-adjoint, by using duality and that $[\mathbb{H},X]_\theta^*=[\mathbb{H}^*,X^*]_\theta$ (see [23, p. 1007]), we get

$$\|\Delta_\lambda^{i\omega}\|_{L^{p'}((0,\infty),\mathbb{B})\to L^{p'}((0,\infty),\mathbb{B})}\le Ce^{2\pi(1/p-1/2)|\omega|}.$$

Hence, another interpolation leads to

$$\|\Delta_\lambda^{i\omega}\|_{L^q((0,\infty),\mathbb{B})\to L^q((0,\infty),\mathbb{B})}\le Ce^{2\pi(1/p-1/2)|\omega|},\quad p\le q\le p'. \tag{7.9}$$

Since m is a bounded holomorphic function in $\sum_\vartheta$, the function $M(y)=m(y^2)$, $y\in(0,\infty)$, is bounded and holomorphic in $\sum_{\vartheta/2}$. The proof now can be finished by proceeding as in the proof of [37, Theorem 3] and by using (7.9).

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