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MULTIPLE HILBERT-TYPE INEQUALITIES INVOLVING SOME DIFFERENTIAL OPERATORS

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Abstract. In this article, we derive several multidimensional Hilbert-type inequalities, including certain differential operators. Further, we determine the conditions under which the constants appearing on the right-hand sides of the established inequalities are the best possible. As an application, some particular examples are also studied.

1. Introduction

The Hilbert inequality asserts that

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y}\,dx\,dy \le \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|g\|_q \tag{1.1}$$

holds for nonnegative functions $f \in L^p(\mathbb{R}_+)$, $g \in L^q(\mathbb{R}_+)$. Here, and throughout this paper, $\|\cdot\|_r$ stands for the usual norm in $L^r(\mathbb{R}_+)$; that is, $\|f\|_r = (\int_{\mathbb{R}_+}|f(x)|^r\,dx)^{1/r}$, $r > 1$. The parameters p and q appearing in (1.1) are mutually conjugate; that is, $\frac{1}{p}+\frac{1}{q}=1$, where $p>1$. In addition, the constant $\pi/\sin\frac{\pi}{p}$ is the best possible in the sense that it cannot be replaced with a smaller constant, so that (1.1) still holds.

The Hilbert inequality is one of the most interesting inequalities in mathematical analysis. Applications of this inequality in diverse fields of mathematics

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have certainly contributed to its importance. After its discovery at the beginning of the twentieth century, the Hilbert inequality was studied by numerous authors, who either re-proved it using various techniques, or improved and generalized it in many different directions. For a comprehensive inspection of the initial development of the Hilbert inequality, the reader is referred to the classical monograph [5].

Nowadays, more than a century after its discovery, this problem area is still of interest to numerous authors. The most important recent results regarding Hilbert-type inequalities are collected in the monograph [6].

In the last few years, considerable attention has been given to a class of Hilbert-type inequalities in which the functions and sequences are replaced by certain integral or discrete operators. As an example, the classical Hardy operator $f \mapsto \frac{1}{x}\int_0^x f(t)\,dt$ represents the arithmetic mean in the integral case. Such inequalities may easily be derived by virtue of general Hilbert-type inequalities (see [6, Chapters 1, 2]) and several well-known classical inequalities, such as the Hardy inequality (see [9]), the Knopp inequality, and so on. But the most interesting fact in connection with this topic is that the constants appearing in these inequalities remain the best possible (see, for example, [1] and references therein).

Recently, Adiyasuren et al. [2], derived several Hilbert-type inequalities involving some differential operators. Denote by $\mathcal{D}_+^n$, $n \geq 0$, a differential operator defined by $\mathcal{D}_+^n f(x) = f^{(n)}(x)$, where $f^{(n)}$ stands for the nth derivative of a function $f : \mathbb{R}_+ \to \mathbb{R}$. In addition, throughout this article, Λ_+^n denotes the set of nonnegative measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $f^{(n)}(x) > 0$, a.e. on $\mathbb{R}_+$, and $f^{(k)}(0) = 0$, $k = 0, 1, 2, \dots, n-1$. The authors proved in [2] that if $p, q > 1$ are conjugate parameters, $a_1, a_2 \in (n-1, s-1)$, $a_1 + a_2 = s - 2$, where n is a fixed nonnegative integer, and $K : \mathbb{R}_+^2 \to \mathbb{R}$ is a nonnegative homogeneous function of degree $-s$, then the inequalities

$$\int_{\mathbb{R}_+^2} K(x,y) f(x) g(y)\,dx\,dy \leq M \|x^{n-a_1-\frac{1}{p}} \mathcal{D}_+^n f\|_p \|y^{n-a_2-\frac{1}{q}} \mathcal{D}_+^n g\|_q \tag{1.2}$$

and

$$\Big[\int_{\mathbb{R}_+} y^{(p-1)(1+qa_2)} \Big(\int_{\mathbb{R}_+} K(x,y) f(x)\,dx\Big)^p dy\Big]^{\frac{1}{p}} \leq m \|x^{n-a_1-\frac{1}{p}} \mathcal{D}_+^n f\|_p \tag{1.3}$$

hold for all $f, g \in \Lambda_+^n$, such that $x^{n-a_1-\frac{1}{p}} \mathcal{D}_+^n f \in L^p(\mathbb{R}_+)$, $y^{n-a_2-\frac{1}{q}} \mathcal{D}_+^n g \in L^q(\mathbb{R}_+)$. In addition, the constants $M = k(a_2)\frac{\Gamma(a_1-n+1)\Gamma(a_2-n+1)}{\Gamma(a_1+1)\Gamma(a_2+1)}$ and $m = k(a_2)\frac{\Gamma(a_1-n+1)}{\Gamma(a_1+1)}$, where $k(a) = \int_{\mathbb{R}_+} K(1,t) t^a\,dt$ and $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\,dt$, $a > 0$, is the usual gamma function, are the best possible in the corresponding inequalities.

The main objective of this paper is to extend inequalities (1.2) and (1.3) to a multidimensional case. The multidimensional extensions of (1.2) and (1.3) will be given in the setting with nonconjugate parameters. Then, we determine the conditions under which the constants appearing in the established inequalities are the best possible. This leads us again to the case of conjugate parameters. As an application, we consider some particular inequalities with the best possible constants. We first need to cite some auxiliary results needed for deriving our results.

2. Preliminaries

The Hilbert-type inequalities can also be considered in the case of nonconjugate parameters. Let p_i be the real parameters satisfying

$$\sum_{i=1}^{n} \frac{1}{p_i} > 1, \quad p_i > 1, \tag{2.1}$$

and let p_i' be their respective conjugates; that is,

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1, 2, \dots, n. \tag{2.2}$$

Since $p_i > 1$, it follows that $p_i' > 1$, $i = 1, 2, \dots, n$. In addition, we define

$$\lambda_n = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{p_i'}. \tag{2.3}$$

Clearly, relations (2.1) and (2.2) imply that $0 < \lambda_n < 1$. Finally, let q_i be defined by

$$\frac{1}{q_i} = \lambda_n - \frac{1}{p_i'}, \quad i = 1, 2, \dots, n, \tag{2.4}$$

provided that $q_i > 0$, $i = 1, 2, \dots, n$. The above conditions (2.1)–(2.4) provide the n-tuple of nonconjugate exponents and were given by Bonsall [3] more than half a century ago. Note also that $\lambda_n = \sum_{i=1}^{n} 1/q_i$ and $1/q_i + 1 - \lambda_n = 1/p_i$, $i = 1, 2, \dots, n$. Of course, if $\lambda_n = 1$, then $\sum_{i=1}^{n} 1/p_i = 1$, which represents the setting with conjugate parameters.

In 2011, Perić and Vuković [8], provided a unified treatment of multidimensional Hilbert-type inequalities with a homogeneous kernel in the case of nonconjugate parameters. Before we state the corresponding result, we introduce some notation.

Recall that the function $K : \mathbb{R}_+^n \to \mathbb{R}$ is said to be *homogeneous* of degree $-s$, $s > 0$, if $K(t\mathbf{x}) = t^{-s} K(\mathbf{x})$ for all $t > 0$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$. Furthermore, if $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^{n} u_j^{a_j} \, \hat{d}^i\mathbf{u}, \quad i = 1, 2, \dots, n, \tag{2.5}$$

where $\hat{\mathbf{u}}^i = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$ and $\hat{d}^i\mathbf{u} = du_1 \cdots du_{i-1}\, du_{i+1} \cdots du_n$, and provided that the above integral converges. Further, in the remainder of the paper, $d\mathbf{u}$ is an abbreviation for $du_1, du_2, \dots, du_n$.

The following pair of multidimensional Hilbert-type inequalities, in a slightly altered notation, can be found in [8, p. 38, (3.4), (3.5)].

Theorem 2.1. *Let p_i, p_i', q_i, $i = 1, 2, \dots, n$, let λ_n be as in (2.1)–(2.4), and let A_{ij}, $i, j = 1, 2, \dots, n$, be real parameters such that $\sum_{i=1}^{n} A_{ij} = 0$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a nonnegative measurable homogeneous function of degree $-s$, $s > 0$, and*

$f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \ldots, n$, *are nonnegative measurable functions, then the following two inequalities hold and are equivalent:*

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mathbf{x} \leq \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n} \|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i} \tag{2.6}$$

and

$$\begin{aligned}\Big[\int_{\mathbb{R}_+} {x_n}^{(1-\lambda_n p_n')(n-1-s)-p_n'\alpha_n} \Big(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x}\Big)^{p_n'} dx_n\Big]^{1/p_n'} \\ \leq \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n-1} \|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i},\end{aligned} \tag{2.7}$$

where $\alpha_i = \sum_{j=1}^{n} A_{ij}$, $\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$, $x_i^{(n-1-s)/q_i+\alpha_i} f_i \in L^{p_i}(\mathbb{R}_+)$, *and* $k_i(q_i\mathbf{A_i}) < \infty$, $i = 1, 2, \ldots, n$.

Inequalities related to (2.6) are usually called *Hilbert-type inequalities*, while the equivalent forms such as (2.7) are usually referred to as the *Hardy–Hilbert-type* inequalities.

The previous theorem will be the crucial step in proving our results. In addition, we need the well-known Hardy and dual Hardy inequalities.

In 1928, Hardy [4], proved an estimate for the integration operator (or the Hardy operator) $\mathcal{H}f(x) = \int_0^x f(t)\, dt$, from which the first weighted modification of the Hardy inequality followed, namely, the inequality

$$\|x^{-\frac{r}{p}} \mathcal{H}f\|_p \leq \frac{p}{r-1} \|x^{1-\frac{r}{p}} f\|_p, \tag{2.8}$$

valid with $p > 1$, $r > 1$, and $x^{1-\frac{r}{p}} f \in L^p(\mathbb{R}_+)$, where the constant $\frac{p}{r-1}$ is the best possible (for more details, see [5, Theorem 330] and [7]).

The dual Hardy inequality, accompanied with the dual integration operator or the dual Hardy operator $\mathcal{H}^*f(x) = \int_x^\infty f(t)\, dt$, asserts that

$$\|x^{-\frac{r}{p}} \mathcal{H}^*f\|_p \leq \frac{p}{1-r} \|x^{1-\frac{r}{p}} f\|_p \tag{2.9}$$

holds for $p > 1$ and $r < 1$, provided that $x^{1-\frac{r}{p}} f \in L^p(\mathbb{R}_+)$.

3. Multidimensional inequalities with nonconjugate exponents

Now we give the multidimensional extension of inequalities (1.2) and (1.3) in the case of nonconjugate parameters.

It should be noted here that the constants appearing in our extended inequalities are also expressed in terms of the gamma function. Therefore, we first give the definition of rising and falling factorial powers. The rising factorial power $x^{\overline{n}}$, where n is a nonnegative integer, also known as a *Pochhammer symbol*, is defined by $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$, while the falling factorial power $x^{\underline{n}}$ is given by $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$. The rising and falling factorial powers may be expressed in terms of the usual gamma function; that is,

$$x^{\overline{n}} = \frac{\Gamma(x+n)}{\Gamma(x)} \qquad \text{and} \qquad x^{\underline{n}} = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}.$$

Our first result is a consequence of Theorem 2.1 and the weighted Hardy inequality (2.8).

Theorem 3.1. *Suppose that p_i, p_i', q_i, $i = 1, 2, \ldots, n$, and λ_n are as in (2.1)–(2.4), and that A_{ij}, $i, j = 1, 2, \ldots, n$, are real parameters satisfying $\sum_{i=1}^{n} A_{ij} = 0$. Further, let $\alpha_i = \sum_{j=1}^{n} A_{ij}$, and let $s > 0$ be a real parameter such that $\frac{s-n}{q_i} + \lambda_n - \alpha_i > m_i$, $m_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, n$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a nonnegative measurable homogeneous function of degree $-s$, and $f_i \in \Lambda_+^{m_i}$, $i = 1, 2, \ldots, n$, then*

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mathbf{x} \le C_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}) \prod_{i=1}^{n} \|x_i^{(n-1-s)/q_i + \alpha_i + m_i} \mathcal{D}_+^{m_i} f_i\|_{p_i} \tag{3.1}$$

and

$$\begin{aligned} &\left[\int_{\mathbb{R}_+} x_n{}^{(1-\lambda_n p_n')(n-1-s) - p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'} \\ &\qquad \le C_{n-1}^s(\mathbf{p}, \mathbf{q}, \mathbf{A}) \prod_{i=1}^{n-1} \|x_i^{(n-1-s)/q_i + \alpha_i + m_i} \mathcal{D}_+^{m_i} f_i\|_{p_i}, \end{aligned} \tag{3.2}$$

where

$$C_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}) = \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n} \frac{\Gamma(\frac{s-n}{q_i} + \lambda_n - \alpha_i - m_i)}{\Gamma(\frac{s-n}{q_i} + \lambda_n - \alpha_i)},$$

$$C_{n-1}^s(\mathbf{p}, \mathbf{q}, \mathbf{A}) = \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n-1} \frac{\Gamma(\frac{s-n}{q_i} + \lambda_n - \alpha_i - m_i)}{\Gamma(\frac{s-n}{q_i} + \lambda_n - \alpha_i)},$$

$\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$, $x_i^{(n-1-s)/q_i + \alpha_i + m_i} \mathcal{D}_+^{m_i} f_i \in L^{p_i}(\mathbb{R}_+)$, and $k_i(q_i \mathbf{A_i}) < \infty$, $i = 1, 2, \ldots, n$.

Proof. First suppose that $m_i \in \mathbb{N}$, $i = 1, 2, \ldots, n$. In order to prove (3.1) we will rewrite the right-hand side of inequality (2.6) in a form that is more suitable for the application of the Hardy inequality. Namely, since

$$\mathcal{H}(\mathcal{D}_+ f)(x) = \int_0^x f'(t)\, dt = f(x) - f(0) = f(x),$$

we have that

$$\begin{aligned} &\prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n} \|x_i^{(n-1-s)/q_i + \alpha_i} f_i\|_{p_i} \\ &\quad = \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n} \left\|x_i^{(n-1-s)/q_i + \alpha_i} \mathcal{H}(\mathcal{D}_+ f_i)\right\|_{p_i}. \end{aligned} \tag{3.3}$$

Now, due to the weighted Hardy inequality (2.8), it follows that

$$\left\|x_i^{(n-1-s)/q_i+\alpha_i}\mathcal{H}(\mathcal{D}_+f_i)\right\|_{p_i} \le \frac{1}{\frac{s-n}{q_i}+\lambda_n-\alpha_i-1}\|x_i^{(n-1-s)/q_i+\alpha_i+1}\mathcal{D}_+f_i\|_{p_i},$$

$i=1,2,\dots,n$. Moreover, applying the Hardy inequality to the right-hand side of the above inequality m_i-1 times, yields the relation

$$\begin{aligned}&\left\|x_i^{(n-1-s)/q_i+\alpha_i}\mathcal{H}(\mathcal{D}_+f_i)\right\|_{p_i}\\ &\quad\le \frac{1}{(\frac{s-n}{q_i}+\lambda_n-\alpha_i-1)^{\underline{m_i}}}\cdot\|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathcal{D}_+^{m_i}f_i\|_{p_i}.\end{aligned} \tag{3.4}$$

Finally, taking into account that $(\frac{s-n}{q_i}+\lambda_n-\alpha_i)^{\underline{m_i}} = \frac{\Gamma(\frac{s-n}{q_i}+\lambda_n-\alpha_i)}{\Gamma(\frac{s-n}{q_i}+\lambda_n-\alpha_i-m_i)}$, the inequality (3.1) holds due to (2.6), (3.3), and (3.4). It remains to consider the case when $m_i=0$ for some $i\in\{1,2,\dots,n\}$. In that case, the relation (3.4) reduces to a trivial equality, and so (3.1) holds.

In the same way, the inequality (3.2) holds by virtue of (2.7) and (3.4). The proof is completed. □

The Theorem 3.1 is a multidimensional extension of the inequalities (1.2) and (1.3), which will be discussed in the next section. Moreover, this result may be regarded as an extension of Theorem 2.1. Namely, if $m_i=0$, $i=1,2,\dots,n$, then Theorem 3.1 trivially reduces to Theorem 2.1.

The previous theorem holds when the corresponding parameters fulfill the set of conditions $\frac{s-n}{q_i}+\lambda_n-\alpha_i>m_i$, $i=1,2,\dots,n$. If $\frac{s-n}{q_i}+\lambda_n-\alpha_i<1$, $i=1,2,\dots,n$, we can also derive a pair of inequalities which are in some way dual to inequalities (3.1) and (3.2). Namely, this result relies on the dual Hardy inequality (2.9).

In order to state the corresponding result, we define a differential operator $\mathcal{D}_\pm^n$ by $\mathcal{D}_\pm^n f(x)=(-1)^n f^{(n)}(x)$, where n is a nonnegative integer. Moreover, the following theorem holds for all nonnegative functions $f:\mathbb{R}_+\to\mathbb{R}$ such that the nth derivative $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $\mathcal{D}_\pm^n f(x)>0$, a.e. on $\mathbb{R}_+$, and $\lim_{x\to\infty} f^{(k)}(x)=0$ for $k=0,1,2,\dots,n-1$. This set of functions will be denoted by $\Lambda_\pm^n$.

Theorem 3.2. *Suppose that p_i, p_i', q_i, $i=1,2,\dots,n$, and λ_n are as in (2.1)–(2.4), and let A_{ij}, $i,j=1,2,\dots,n$, be real parameters satisfying $\sum_{i=1}^n A_{ij}=0$. Further, let $\alpha_i=\sum_{j=1}^n A_{ij}$, and let $s>0$ be real parameter such that $\frac{s-n}{q_i}+\lambda_n-\alpha_i<1$, $i=1,2,\dots,n$. If $K:\mathbb{R}_+^n\to\mathbb{R}$ is a nonnegative measurable homogeneous function of degree $-s$ and $f_i\in\Lambda_\pm^{m_i}$, $m_i\in\mathbb{N}\cup\{0\}$, $i=1,2,\dots,n$, then*

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x})\prod_{i=1}^n f_i(x_i)\,d\mathbf{x} \le E_n^s(\mathbf{p},\mathbf{q},\mathbf{A})\prod_{i=1}^n \|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathcal{D}_\pm^{m_i}f_i\|_{p_i} \tag{3.5}$$

and

$$\begin{aligned}\Big[\int_{\mathbb{R}_+} x_n{}^{(1-\lambda_n p_n')(n-1-s)-p_n'\alpha_n}\Big(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)\,\hat{d}^n\mathbf{x}\Big)^{p_n'}dx_n\Big]^{1/p_n'}\\ \leq E_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A})\prod_{i=1}^{n-1}\|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathcal{D}_{\pm}^{m_i} f_i\|_{p_i},\end{aligned}\tag{3.6}$$

where

$$E_n^s(\mathbf{p},\mathbf{q},\mathbf{A})=\prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^{n}\frac{\Gamma(\frac{n-s}{q_i}-\lambda_n+\alpha_i+1)}{\Gamma(\frac{n-s}{q_i}-\lambda_n+\alpha_i+m_i+1)},$$
$$E_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A})=\prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^{n-1}\frac{\Gamma(\frac{n-s}{q_i}-\lambda_n+\alpha_i+1)}{\Gamma(\frac{n-s}{q_i}-\lambda_n+\alpha_i+m_i+1)},$$

$\mathbf{A_i}=(A_{i1},A_{i2},\dots,A_{in})$, $x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathcal{D}_{+}^{m_i} f_i\in L^{p_i}(\mathbb{R}_+)$, *and* $k_i(q_i\mathbf{A_i})<\infty$, $i=1,2,\dots,n$.

Proof. The proof is similar to the proof of the previous theorem, except that we use the dual Hardy inequality (2.9) this time. In this regard, the right-hand side of (2.6) can be rewritten as

$$\begin{aligned}\prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^{n}\|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}\\ =\prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^{n}\big\|x_i^{(n-1-s)/q_i+\alpha_i}\mathcal{H}^*(\mathcal{D}_{\pm} f_i)\big\|_{p_i},\end{aligned}\tag{3.7}$$

since

$$\mathcal{H}^*(\mathcal{D}_{\pm} f)(x)=-\int_x^\infty f'(t)\,dt=f(x).$$

Now, by applying the dual Hardy inequality to the expressions on the right-hand side of (3.7) m_i times (when $m_i\in\mathbb{N}$), it follows that

$$\begin{aligned}&\big\|x_i^{(n-1-s)/q_i+\alpha_i}\mathcal{H}^*(\mathcal{D}_{\pm} f_i)\big\|_{p_i}\\ &\quad\leq\frac{1}{(\frac{n-s}{q_i}-\lambda_n+\alpha_i+1)^{\overline{m_i}}}\cdot\|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathcal{D}_{\pm}^{m_i} f_i\|_{p_i},\end{aligned}\tag{3.8}$$

$i=1,2,\dots,n$. Further, since $(\frac{n-s}{q_i}-\lambda_n+\alpha_i+1)^{\overline{m_i}}=\frac{\Gamma(\frac{n-s}{q_i}-\lambda_n+\alpha_i+m_i+1)}{\Gamma(\frac{n-s}{q_i}-\lambda_n+\alpha_i+1)}$, the inequality (3.5) holds due to (2.6), (3.7), and (3.8). In the same way, inequality (3.6) holds by virtue of (2.7) and (3.8). The trivial case when $m_i=0$ for some $i\in\{1,2,\dots,n\}$ is treated in the same way as in Theorem 3.1. □

It should be noted here that Theorem 3.2 may also be regarded as an extension of Theorem 2.1 since in the case when $m_i=0$, $i=1,2,\dots,n$, it reduces trivially to Theorem 2.1.

Our next step is to determine conditions under which the constants $C_n^s(\mathbf{p},\mathbf{q},\mathbf{A})$, $C_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A})$, $E_n^s(\mathbf{p},\mathbf{q},\mathbf{A})$, and $E_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A})$ appearing in Theorems 3.1 and 3.2

are the best possible. This happens in the case of conjugate parameters. Namely, the problem of the best constants in Hilbert-type inequalities with nonconjugate parameters seems to be a quite difficult issue and remains open.

4. Inequalities with conjugate parameters: The best possible constants

In order to obtain the best possible constants in inequalities (3.1), (3.2), (3.5), and (3.6), in this section we deal with their conjugate forms. Namely, if $p_i > 1$, $i = 1, 2, \ldots, n$, is the set of conjugate parameters, then inequalities (3.1) and (3.2) become, respectively,

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i)\, d\mathbf{x} \le \overline{C}_n^s(\mathbf{p},\mathbf{A}) \prod_{i=1}^n \|x_i^{(n-1-s)/p_i+\alpha_i+m_i} \mathcal{D}_+^{m_i} f_i\|_{p_i} \tag{4.1}$$

and

$$\begin{aligned} &\Big[\int_{\mathbb{R}_+} x_n{}^{(1-p_n')(n-1-s)-p_n'\alpha_n} \Big(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x}\Big)^{p_n'} dx_n\Big]^{1/p_n'} \\ &\quad \le \overline{C}_{n-1}^s(\mathbf{p},\mathbf{A}) \prod_{i=1}^{n-1} \|x_i^{(n-1-s)/p_i+\alpha_i+m_i} \mathcal{D}_+^{m_i} f_i\|_{p_i}, \end{aligned} \tag{4.2}$$

where

$$\overline{C}_n^s(\mathbf{p},\mathbf{A}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^n \frac{\Gamma(\frac{s-n}{p_i} - \alpha_i - m_i + 1)}{\Gamma(\frac{s-n}{p_i} - \alpha_i + 1)},$$

$$\overline{C}_{n-1}^s(\mathbf{p},\mathbf{A}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^{n-1} \frac{\Gamma(\frac{s-n}{p_i} - \alpha_i - m_i + 1)}{\Gamma(\frac{s-n}{p_i} - \alpha_i + 1)}.$$

In the same way, the conjugate forms of inequalities (3.5) and (3.6) read

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i)\, d\mathbf{x} \le \overline{E}_n^s(\mathbf{p},\mathbf{A}) \prod_{i=1}^n \|x_i^{(n-1-s)/p_i+\alpha_i+m_i} \mathcal{D}_\pm^{m_i} f_i\|_{p_i} \tag{4.3}$$

and

$$\begin{aligned} &\Big[\int_{\mathbb{R}_+} x_n{}^{(1-p_n')(n-1-s)-p_n'\alpha_n} \Big(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x}\Big)^{p_n'} dx_n\Big]^{1/p_n'} \\ &\quad \le \overline{E}_{n-1}^s(\mathbf{p},\mathbf{A}) \prod_{i=1}^{n-1} \|x_i^{(n-1-s)/p_i+\alpha_i+m_i} \mathcal{D}_\pm^{m_i} f_i\|_{p_i}, \end{aligned} \tag{4.4}$$

with the constants

$$\overline{E}_n^s(\mathbf{p},\mathbf{A}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^n \frac{\Gamma(\frac{n-s}{p_i} + \alpha_i)}{\Gamma(\frac{n-s}{p_i} + \alpha_i + m_i)},$$

$$\overline{E}_{n-1}^s(\mathbf{p},\mathbf{A}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^{n-1} \frac{\Gamma(\frac{n-s}{p_i} + \alpha_i)}{\Gamma(\frac{n-s}{p_i} + \alpha_i + m_i)}.$$

Now, our goal is to determine the conditions under which the inequalities (4.1), (4.2), (4.3), and (4.4) include the best possible constants on their right-hand sides. To do this, we establish some more specific conditions about the convergence of the integral $k_1(\mathbf{a})$, $\mathbf{a} = (a_1, a_2, \dots, a_n)$, defined by (2.5). More precisely, we assume that

$$k_1(\mathbf{a}) < \infty \quad \text{for } a_2, \dots, a_n > -1, \qquad \sum_{i=2}^{n} a_i < s - n + 1, \quad n \in \mathbb{N}, n \geq 2. \tag{4.5}$$

By reasoning similar to that in some recent papers (see [8], [10]), the best possible constants can be obtained if their parts related to a homogeneous kernel contain no exponents. In this regard, assume that

$$k_1(p_1\mathbf{A_1}) = k_2(p_2\mathbf{A_2}) = \cdots = k_n(p_n\mathbf{A_n}). \tag{4.6}$$

Our goal is to find suitable conditions under which (4.6) holds. To do this, we first express $k_2(p_2\mathbf{A_2})$ in terms of $k_1(\cdot)$, in accordance with definition (2.5). Hence, passing to new variables $t_2, t_3, \dots, t_n$, where $u_1 = 1/t_2, u_3 = t_3/t_2, u_4 = t_4/t_2, \dots, u_n = t_n/t_2$, yields the Jacobian of the transformation

$$\left|\frac{\partial(u_1, u_3, \dots, u_n)}{\partial(t_2, t_3, \dots, t_n)}\right| = t_2^{-n},$$

and so we have

$$\begin{aligned} k_2(p_2\mathbf{A_2}) &= \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{t}}^1) t_2^{s-n-p_2(\alpha_2 - A_{22})} \prod_{j=3}^{n} t_j^{p_2 A_{2j}} \, \hat{d}^1\mathbf{t} \\ &= k_1\big(p_1 A_{11}, s - n - p_2(\alpha_2 - A_{22}), p_2 A_{23}, \dots, p_2 A_{2n}\big). \end{aligned}$$

According to (4.6), it follows that $p_1 A_{12} = s - n - p_2(\alpha_2 - A_{22})$, $p_1 A_{13} = p_2 A_{23}, \dots, p_1 A_{1n} = p_2 A_{2n}$. In the same way we can express $k_i(p_i\mathbf{A_i})$, $i = 3, \dots, n$, in terms of $k_1(\cdot)$. Therefore the relation (4.6) is fulfilled if

$$p_j A_{ji} = s - n - p_i(\alpha_i - A_{ii}), \quad i, j = 1, 2, \dots, n, i \neq j. \tag{4.7}$$

The above set of relations also implies that $p_i A_{ik} = p_j A_{jk}$, when $k \neq i, j$. Therefore, we use the abbreviations $\widetilde{A}_1 = p_n A_{n1}$ and $\widetilde{A}_i = p_1 A_{1i}$, $i \neq 1$. Since $\sum_{i=1}^n A_{ij} = 0$, one easily obtains that $p_j A_{jj} = \widetilde{A}_j(1 - p_j)$. In addition, $\sum_{i=1}^n \widetilde{A}_i = s - n$ (see also [10]).

Now, if the set of conditions (4.7) is fulfilled, then, with the above abbreviations, inequalities (4.1) and (4.2) become, respectively,

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) \, d\mathbf{x} \leq L_n^s(\mathbf{p}, \mathbf{A}) \prod_{i=1}^{n} \|x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_+^{m_i} f_i\|_{p_i} \tag{4.8}$$

and

$$\begin{aligned}\Big[\int_{\mathbb{R}_+} {x_n}^{(p_n'-1)(1+p_n\widetilde{A}_n)}\Big(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)\,\hat{d}^n\mathbf{x}\Big)^{p_n'}\,dx_n\Big]^{1/p_n'}&\\ \leq L_{n-1}^s(\mathbf{p},\mathbf{A})\prod_{i=1}^{n-1}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathcal{D}_+^{m_i}f_i\|_{p_i},&\end{aligned}\tag{4.9}$$

where

$$L_n^s(\mathbf{p},\widetilde{\mathbf{A}})=k_1(\widetilde{\mathbf{A}})\prod_{i=1}^{n}\frac{\Gamma(\widetilde{A}_i-m_i+1)}{\Gamma(\widetilde{A}_i+1)},$$

$$L_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})=k_1(\widetilde{\mathbf{A}})\prod_{i=1}^{n-1}\frac{\Gamma(\widetilde{A}_i-m_i+1)}{\Gamma(\widetilde{A}_i+1)}.$$

In the same regard, the inequalities (4.3) and (4.4) read, respectively,

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n} f_i(x_i)\,d\mathbf{x}\leq M_n^s(\mathbf{p},\widetilde{\mathbf{A}})\prod_{i=1}^{n}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathcal{D}_\pm^{m_i}f_i\|_{p_i}\tag{4.10}$$

and

$$\begin{aligned}\Big[\int_{\mathbb{R}_+} {x_n}^{(p_n'-1)(1+p_n\widetilde{A}_n)}\Big(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)\,\hat{d}^n\mathbf{x}\Big)^{p_n'}\,dx_n\Big]^{1/p_n'}&\\ \leq M_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})\prod_{i=1}^{n-1}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathcal{D}_\pm^{m_i}f_i\|_{p_i},&\end{aligned}\tag{4.11}$$

with the corresponding constants

$$M_n^s(\mathbf{p},\widetilde{\mathbf{A}})=k_1(\widetilde{\mathbf{A}})\prod_{i=1}^{n}\frac{\Gamma(-\widetilde{A}_i)}{\Gamma(-\widetilde{A}_i+m_i)},$$

$$M_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})=k_1(\widetilde{\mathbf{A}})\prod_{i=1}^{n-1}\frac{\Gamma(-\widetilde{A}_i)}{\Gamma(-\widetilde{A}_i+m_i)}.$$

Remark 4.1. Note that inequalities (4.8) and (4.9) are multidimensional extensions of inequalities (1.2) and (1.3) since for $n=2$ and $m_1=m_2=m$ they reduce to (1.2) and (1.3), respectively. In that case, conditions (4.5) and (4.7) are equivalent to $\widetilde{A}_1,\widetilde{A}_2\in(m-1,s-1)$, $\widetilde{A}_1+\widetilde{A}_2=s-2$, as stated in the Introduction. In the same setting the inequalities (4.10) and (4.11) become the corresponding relations derived in [2].

Now, we are ready to state and prove our main results in this section. More precisely, we show that the constants $L_n^s(\mathbf{p},\widetilde{\mathbf{A}})$, $L_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})$, $M_n^s(\mathbf{p},\widetilde{\mathbf{A}})$, and $M_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})$ appearing on the right-hand sides of the above inequalities are the best possible.

Theorem 4.2. *Let $m_i \in \mathbb{N} \cup \{0\}$, $\widetilde{A}_i > m_i - 1$, $i = 1, 2, \dots, n$, and let the parameters $\widetilde{A}_i$, $i = 2, \dots, n$, fulfill conditions as in (4.5). Then, the constants $L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ and $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ are the best possible in inequalities (4.8) and (4.9) respectively.*

Proof. Suppose that the constant $L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is not the best possible in (4.8). Then, there exists a positive constant C_n, smaller than $L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$, such that inequality (4.8) is still valid if we replace $L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ by C_n. Now, consider the functions

$$\widetilde{f}_i(x_i) = \begin{cases} 0, & 0 < x_i < 1, \\ \frac{\Gamma(1+\widetilde{A}_i - \frac{\varepsilon}{p_i} - m_i)}{\Gamma(1+\widetilde{A}_i - \frac{\varepsilon}{p_i})} x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}}, & x_i \geq 1, \end{cases} \qquad i = 1, \dots, n,$$

where $\varepsilon > 0$ is a sufficiently small number. Since the m_ith derivative of the function $x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}}$ is equal to $\frac{\Gamma(1+\widetilde{A}_i - \frac{\varepsilon}{p_i})}{\Gamma(1+\widetilde{A}_i - \frac{\varepsilon}{p_i} - m_i)} x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i} - m_i}$, it follows that

$$\mathcal{D}_+^{m_i} \widetilde{f}_i(x_i) = \begin{cases} 0, & 0 < x_i < 1, \\ x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i} - m_i}, & x_i > 1, \end{cases} \qquad i = 1, \dots, n,$$

so in this setting the right-hand side of (4.8) reduces to

$$\begin{aligned} C_n \prod_{i=1}^n &\| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_+^{m_i} \widetilde{f}_i \|_{p_i} \\ &= C_n \prod_{i=1}^n \Big[\int_{\mathbb{R}_+} x_i^{p_i(m_i - \widetilde{A}_i) - 1} \big(\mathcal{D}_+^{m_i} \widetilde{f}_i(x_i) \big)^{p_i} \, dx_i \Big]^{\frac{1}{p_i}} \\ &= \frac{C_n}{\varepsilon}. \end{aligned} \tag{4.12}$$

On the other hand, the left-hand side of (4.8) can be rewritten as

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n \widetilde{f}_i(x_i) \, d\mathbf{x} = I \cdot \prod_{i=1}^n \frac{\Gamma(1 + \widetilde{A}_i - \frac{\varepsilon}{p_i} - m_i)}{\Gamma(1 + \widetilde{A}_i - \frac{\varepsilon}{p_i})},$$

where $I = \int_{[1,\infty)^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} \, d\mathbf{x}$. Now, since

$$I = \int_1^\infty x_1^{-1-\varepsilon} \Big[\int_{[1/x_1, \infty)^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/p_i} \, \hat{d}^1 \mathbf{u} \Big] \, dx_1,$$

we have the following estimate:

$$\begin{aligned} I \geq & \int_1^\infty x_1^{-1-\varepsilon}\Big[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i-\varepsilon/p_i}\,\hat{d}^1\mathbf{u}\Big]\,dx_1 \\ & - \int_1^\infty x_1^{-1-\varepsilon}\Big[\sum_{i=2}^n\int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{u}\Big]\,dx_1 \\ \geq & \frac{1}{\varepsilon}\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i-\varepsilon/p_i}\,\hat{d}^1\mathbf{u} \\ & - \int_1^\infty x_1^{-1}\Big[\sum_{i=2}^n\int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{u}\Big]\,dx_1, \end{aligned} \tag{4.13}$$

where $\mathbb{D}_i = \{(u_2,u_3,\dots,u_n); 0<u_i\leq 1/x_1, u_j>0, j\neq i\}$.

Without loss of generality, it suffices to find the appropriate estimate for the integral $\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{u}$. In fact, setting $\alpha>0$ such that $\widetilde{A}_2+1>\varepsilon/p_2+\alpha$, since $-u_2^\alpha\log u_2\to 0$ $(u_2\to 0^+)$, there exists $M\geq 0$ such that $-u_2^\alpha\log u_2\leq M$ $(u_2\in(0,1])$. On the other hand, it follows easily that the parameters $a_2=\widetilde{A}_2-(\varepsilon/p_2+\alpha)$ and $a_i=\widetilde{A}_i-\varepsilon/p_i, i=3,\dots,n$ satisfy conditions (4.5). Then, by virtue of the Fubini theorem, we have

$$\begin{aligned} 0 &\leq \int_1^\infty x_1^{-1}\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{u}\,dx_1 \\ &= \int_1^\infty x_1^{-1}\Big[\int_{\mathbb{R}_+^{n-2}}\int_0^{1/x_1} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{u}\Big]\,dx_1 \\ &= \int_{\mathbb{R}_+^{n-2}}\int_0^1 K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\Big(\int_1^{1/u_2} x_1^{-1}\,dx_1\Big)\,\hat{d}^1\mathbf{u} \\ &= \int_{\mathbb{R}_+^{n-2}}\int_0^1 K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}(-\log u_2)\,\hat{d}^1\mathbf{u} \\ &\leq M\int_{\mathbb{R}_+^{n-2}}\int_0^1 K(\hat{\mathbf{u}}^1)u_2^{\widetilde{A}_2-(\varepsilon/p_2+\alpha)}\prod_{j=3}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{u} \\ &\leq M\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)u_2^{\widetilde{A}_2-(\varepsilon/p_2+\alpha)}\prod_{j=3}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\,\hat{d}^1\mathbf{u} \\ &= M\cdot k_1\big(\widetilde{A}_2-(\varepsilon/p_2+\alpha),\widetilde{A}_3-\varepsilon/p_3,\dots,\widetilde{A}_n-\varepsilon/p_n\big)<\infty. \end{aligned}$$

Hence, taking into account (4.13), we obtain

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^n \widetilde{f}_i(x_i)\,d\mathbf{x} \geq \Big(\frac{1}{\varepsilon}k_1(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p})-O(1)\Big)\prod_{i=1}^n\frac{\Gamma(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i)}{\Gamma(1+\widetilde{A}_i-\frac{\varepsilon}{p_i})},$$

where $\mathbf{1}/\mathbf{p} = (1/p_1, \dots, 1/p_n)$. Moreover, the relation (4.12) implies that

$$C_n \geq \big(k_1(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}) - \varepsilon O(1)\big) \prod_{i=1}^{n} \frac{\Gamma(1 + \widetilde{A}_i - \frac{\varepsilon}{p_i} - n)}{\Gamma(1 + \widetilde{A}_i - \frac{\varepsilon}{p_i})}.$$

Obviously, letting $\varepsilon \to 0^+$, it follows that $C_n \geq L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$, which contradicts our assumption that $0 < C_n < L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$. Hence, $L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is the best possible in (4.8).

It remains to show that $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is the best possible constant in (4.9). Assume that there exists a positive constant C_{n-1}, smaller than $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$, such that inequality (4.9) holds when $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is replaced by C_{n-1}. Then, utilizing the well-known Hölder inequality and inequality (3.4), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mathbf{x} \\
&\quad = \int_{\mathbb{R}_+} \Big[x_n^{\frac{1+p_n \widetilde{A}_n}{p_n}} \int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n \mathbf{x} \Big] \cdot \big[x_n^{-\frac{1+p_n \widetilde{A}_n}{p_n}} f_n(x_n) \big]\, dx_n \\
&\quad \leq \Big[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n \widetilde{A}_n)} \Big(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n \mathbf{x} \Big)^{p_n'} dx_n \Big]^{1/p_n'} \\
&\qquad \times \Big[\int_{\mathbb{R}_+} x_n^{-1-p_n \widetilde{A}_n} f_n^{p_n}(x_n)\, dx_n \Big]^{1/p_n} \\
&\quad \leq C_{n-1} \frac{\Gamma(\widetilde{A}_n - m_n + 1)}{\Gamma(\widetilde{A}_n + 1)} \prod_{i=1}^{n} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_+^{m_i} f_i \|_{p_i}.
\end{aligned} \tag{4.14}$$

Finally, taking into account our assumption $0 < C_{n-1} < L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$, we have

$$\begin{aligned}
0 &< C_{n-1} \frac{\Gamma(\widetilde{A}_n - m_n + 1)}{\Gamma(\widetilde{A}_n + 1)} \\
&< L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}) \frac{\Gamma(\widetilde{A}_n - m_n + 1)}{\Gamma(\widetilde{A}_n + 1)} = L_n^s(\mathbf{p}, \widetilde{\mathbf{A}}).
\end{aligned}$$

Therefore, relation (4.14) contradicts with the fact that $L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is the best possible constant in inequality (4.8). Thus, the assumption that $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is not the best possible is false. □

Theorem 4.3. *Let $m_i \in \mathbb{N} \cup \{0\}$, $\widetilde{A}_i < 0$, $i = 1, 2, \dots, n$, and let the parameters $\widetilde{A}_i$, $i = 2, \dots, n$, fulfill conditions as in (4.5). Then, the constants $M_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ and $M_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ are the best possible in (4.10) and (4.11), respectively.*

Proof. We follow the same procedure as in the proof of Theorem 4.2; that is, we suppose that the inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mathbf{x} \leq C_n^* \prod_{i=1}^{n} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_\pm^{m_i} f_i \|_{p_i} \tag{4.15}$$

holds with a positive constant C_n^*, smaller than $M_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$. Now, we consider this inequality with the functions

$$\widetilde{f}_i^*(x_i) = \begin{cases} 0, & 0 < x_i < 1, \\ \frac{\Gamma(-\widetilde{A}_i + \frac{\varepsilon}{p_i})}{\Gamma(-\widetilde{A}_i + m_i + \frac{\varepsilon}{p_i})} x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}}, & x_i \geq 1, \end{cases} \qquad i = 1, \ldots, n,$$

where ε is a sufficiently small number. Then, similarly as in the proof of Theorem 4.2, we have the following lower bound for the left-hand side of (4.15):

$$\begin{aligned} &\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} \widetilde{f}_i^*(x_i)\, d\mathbf{x} \\ &\geq \Big(\frac{1}{\varepsilon} k_1(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}) - O(1)\Big) \prod_{i=1}^{n} \frac{\Gamma(-\widetilde{A}_i + \frac{\varepsilon}{p_i})}{\Gamma(-\widetilde{A}_i + m_i + \frac{\varepsilon}{p_i})}. \end{aligned} \tag{4.16}$$

On the other hand, since the m_ith derivative of the function $x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}}$ is equal to $(-1)^{m_i} \frac{\Gamma(-\widetilde{A}_i + m_i + \frac{\varepsilon}{p_i})}{\Gamma(-\widetilde{A}_i + \frac{\varepsilon}{p_i})} x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i} - m_i}$, it follows that

$$\mathcal{D}_{\pm}^{m_i} \widetilde{f}_i^*(x_i) = \begin{cases} 0, & 0 < x_i < 1, \\ x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i} - m_i}, & x_i > 1, \end{cases} \qquad i = 1, \ldots, n,$$

so the right-hand side of (4.15) reduces to

$$C_n^* \prod_{i=1}^{n} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_{\pm}^{m_i} \widetilde{f}_i^* \|_{p_i} = \frac{C_n^*}{\varepsilon}. \tag{4.17}$$

Consequently, comparing (4.15), (4.16), and (4.17), it follows that

$$C_n^* \geq \big(k_1(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}) - \varepsilon O(1)\big) \prod_{i=1}^{n} \frac{\Gamma(-\widetilde{A}_i + \frac{\varepsilon}{p_i})}{\Gamma(-\widetilde{A}_i + m_i + \frac{\varepsilon}{p_i})}.$$

Therefore, as $\varepsilon \to 0$, it follows that $M_n^s(\mathbf{p}, \widetilde{\mathbf{A}}) \leq C_n^*$, which contradicts with our assumption. This means that the constant $M_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is the best possible in (4.10).

To conclude the proof, we suppose that, contrary to our claim, there exists a constant $0 < C_{n-1}^* < M_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ such that the inequality

$$\begin{aligned} &\Big[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n \widetilde{A}_n)} \Big(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n \mathbf{x}\Big)^{p_n'} dx_n\Big]^{1/p_n'} \\ &\leq C_{n-1}^* \prod_{i=1}^{n-1} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_{\pm}^{m_i} f_i \|_{p_i} \end{aligned}$$

holds. Then, utilizing the Hölder inequality and inequality (3.8), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, d\mathbf{x} \\
&= \int_{\mathbb{R}_+} \Big[x_n^{\frac{1+p_n\widetilde{A}_n}{p_n}} \int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x} \Big] \cdot \big[x_n^{-\frac{1+p_n\widetilde{A}_n}{p_n}} f_n(x_n) \big]\, dx_n \\
&\le \Big[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \Big(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x} \Big)^{p_n'} dx_n \Big]^{1/p_n'} \\
&\quad \times \Big[\int_{\mathbb{R}_+} x_n^{-1-p_n\widetilde{A}_n} f_n^{p_n}(x_n)\, dx_n \Big]^{1/p_n} \\
&\le C_{n-1}^{*} \frac{\Gamma(-\widetilde{A}_n)}{\Gamma(-\widetilde{A}_n + m_n)} \prod_{i=1}^{n} \|x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_{\pm}^{m_i} f_i\|_{p_i}.
\end{aligned}$$

Now, according to our assumption, it follows that

$$\begin{aligned}
0 &< C_{n-1}^{*} \frac{\Gamma(-\widetilde{A}_n)}{\Gamma(-\widetilde{A}_n + m_n)} \\
&< M_{n-1}^{s}(\mathbf{p}, \widetilde{\mathbf{A}}) \frac{\Gamma(-\widetilde{A}_n)}{\Gamma(-\widetilde{A}_n + m_n)} = M_n^s(\mathbf{p}, \widetilde{\mathbf{A}}),
\end{aligned}$$

which means that $M_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is not the best constant in (4.10). This is a clear contradiction of our assumption, and the proof is completed. □

5. Applications and concluding remarks

In order to conclude the paper, we consider the inequalities (4.8), (4.9), (4.10), and (4.11) in some particular settings. The resulting inequalities will include the best possible constants on their right-hand sides.

The standard example of a homogeneous kernel with the negative degree of homogeneity is the function $K_1 : \mathbb{R}_+^n \to \mathbb{R}$, defined by $K_1(\mathbf{x}) = (\sum_{i=1}^n x_i)^{-s}$, $s > 0$. Clearly, K_1 is a homogeneous function of degree $-s$, and the constant $k_1(\widetilde{\mathbf{A}})$, appearing in (4.8), (4.9), (4.10), and (4.11), can be expressed in terms of the usual gamma function Γ. Namely, utilizing the formula

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{a_i-1}}{(1+\sum_{i=1}^{n-1} u_i)^{\sum_{i=1}^n a_i}}\, \hat{d}^n\mathbf{u} = \frac{\prod_{i=1}^n \Gamma(a_i)}{\Gamma(\sum_{i=1}^n a_i)},$$

which holds for $a_i > 0$, $i = 1, 2, \ldots, n$, it follows that

$$k_1(\widetilde{\mathbf{A}}) = \frac{1}{\Gamma(s)} \prod_{i=1}^{n} \Gamma(1+\widetilde{A}_i), \quad i = 1, 2, \ldots, n,$$

provided that $\widetilde{A}_i > -1$, $i = 1, 2, \dots, n$, and $\sum_{i=1}^n \widetilde{A}_i = s - n$. With this kernel, inequalities (4.8), (4.9), (4.10), and (4.11) reduce, respectively, to

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \frac{1}{(\sum_{i=1}^n x_i)^s} \prod_{i=1}^n f_i(x_i)\, d\mathbf{x} \\
&\quad \le \frac{1}{\Gamma(s)} \prod_{i=1}^n \Gamma(\widetilde{A}_i - m_i + 1) \prod_{i=1}^n \|x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_+^{m_i} f_i\|_{p_i},
\end{aligned}$$

$$\begin{aligned}
&\Big[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \Big(\int_{\mathbb{R}_+^{n-1}} \frac{1}{(\sum_{i=1}^n x_i)^s} \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x}\Big)^{p_n'} dx_n\Big]^{1/p_n'} \\
&\quad \le \frac{\Gamma(1+\widetilde{A}_n)}{\Gamma(s)} \prod_{i=1}^{n-1} \Gamma(\widetilde{A}_i - m_i + 1) \prod_{i=1}^{n-1} \|x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_+^{m_i} f_i\|_{p_i},
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \frac{1}{(\sum_{i=1}^n x_i)^s} \prod_{i=1}^n f_i(x_i)\, d\mathbf{x} \\
&\quad \le \frac{1}{\Gamma(s)} \prod_{i=1}^n \frac{\mathrm{B}(1+\widetilde{A}_i, -\widetilde{A}_i)}{\Gamma(-\widetilde{A}_i + m_i)} \prod_{i=1}^n \|x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_\pm^{m_i} f_i\|_{p_i},
\end{aligned}$$

and

$$\begin{aligned}
&\Big[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \Big(\int_{\mathbb{R}_+^{n-1}} \frac{1}{(\sum_{i=1}^n x_i)^s} \prod_{i=1}^{n-1} f_i(x_i)\, \hat{d}^n\mathbf{x}\Big)^{p_n'} dx_n\Big]^{1/p_n'} \\
&\quad \le \frac{\Gamma(1+\widetilde{A}_n)}{\Gamma(s)} \prod_{i=1}^{n-1} \frac{\mathrm{B}(1+\widetilde{A}_i, -\widetilde{A}_i)}{\Gamma(-\widetilde{A}_i + m_i)} \prod_{i=1}^{n-1} \|x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_\pm^{m_i} f_i\|_{p_i}.
\end{aligned}$$

It should be noted here that the last two inequalities include the constants with the usual beta function since $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$.

Another interesting example of a homogeneous kernel with degree $-s$ is the function $K_2(\mathbf{x}) = 1/\max\{x_1^s, \dots, x_n^s\}$, $s > 0$. One can easily check the integral formula

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} x_i^{\widetilde{A}_i}}{\max\{1, x_1^s, \dots, x_{n-1}^s\}}\, \hat{d}^n\mathbf{u} = \frac{s}{\prod_{i=1}^n (1+\widetilde{A}_i)},$$

provided that $\widetilde{A}_i > -1$ and $\sum_{i=1}^n \widetilde{A}_i = s - n$. Based on this formula, inequalities (4.8), (4.9), (4.10), and (4.11) reduce to

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \frac{1}{\max\{x_1^s, \dots, x_n^s\}} \prod_{i=1}^n f_i(x_i)\, d\mathbf{x} \\
&\quad \le s \prod_{i=1}^n \frac{\Gamma(\widetilde{A}_i - m_i + 1)}{\Gamma(\widetilde{A}_i + 2)} \prod_{i=1}^n \|x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathcal{D}_+^{m_i} f_i\|_{p_i},
\end{aligned}$$

$$\Big[\int_{\mathbb{R}_+} x_n{}^{(p_n'-1)(1+p_n\widetilde{A}_n)}\Big(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\max\{x_1^s,\ldots,x_n^s\}}\prod_{i=1}^{n-1} f_i(x_i)\,\hat{d}^n\mathbf{x}\Big)^{p_n'}dx_n\Big]^{1/p_n'}$$
$$\le \frac{s}{(1+\widetilde{A}_n)}\prod_{i=1}^{n-1}\frac{\Gamma(\widetilde{A}_i-m_i+1)}{\Gamma(\widetilde{A}_i+2)}\prod_{i=1}^{n-1}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathcal{D}_+^{m_i}f_i\|_{p_i},$$
$$\int_{\mathbb{R}_+^n}\frac{1}{\max\{x_1^s,\ldots,x_n^s\}}\prod_{i=1}^{n} f_i(x_i)\,d\mathbf{x}$$
$$\le s\prod_{i=1}^{n}\frac{\Gamma(-\widetilde{A}_i)}{(1+\widetilde{A}_i)\Gamma(-\widetilde{A}_i+m_i)}\prod_{i=1}^{n}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathcal{D}_\pm^{m_i}f_i\|_{p_i},$$

and

$$\Big[\int_{\mathbb{R}_+} x_n{}^{(p_n'-1)(1+p_n\widetilde{A}_n)}\Big(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\max\{x_1^s,\ldots,x_n^s\}}\prod_{i=1}^{n-1} f_i(x_i)\,\hat{d}^n\mathbf{x}\Big)^{p_n'}dx_n\Big]^{1/p_n'}$$
$$\le \frac{s}{(1+\widetilde{A}_n)}\prod_{i=1}^{n-1}\frac{\Gamma(-\widetilde{A}_i)}{(1+\widetilde{A}_i)\Gamma(-\widetilde{A}_i+m_i)}\prod_{i=1}^{n-1}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathcal{D}_\pm^{m_i}f_i\|_{p_i},$$

where the constants appearing on the right-hand sides are the best possible.

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