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SCHATTEN-CLASS GENERALIZED VOLTERRA COMPANION INTEGRAL OPERATORS

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ABSTRACT. We study the Schatten-class membership of generalized Volterra companion integral operators on the standard Fock spaces $\mathcal{F}_\alpha^2$. The Schatten $\mathcal{S}_p(\mathcal{F}_\alpha^2)$ membership of the operators are characterized in terms of $L^{p/2}$-integrability of certain generalized Berezin-type integral transforms on the complex plane. We also give a more simplified and easy-to-apply description in terms of L^p-integrability of the symbols inducing the operators against super-exponentially decreasing weights. Asymptotic estimates for the $\mathcal{S}_p(\mathcal{F}_\alpha^2)$ norms of the operators have also been provided.

1. INTRODUCTION AND MAIN RESULTS

For functions f and g, we consider the Volterra-type integral operator V_g and its companion I_g defined by

$$V_g f(z) = \int_0^z f(w)g'(w)\,dw \qquad \text{and} \qquad I_g f(z) = \int_0^z f'(w)g(w)\,dw.$$

Performing integration by parts in any one of the above integrals gives the relation

$$V_g f + I_g f = M_g f - f(0)g(0),$$

where $M_g f = gf$ is the multiplication operator induced by g. These integral operators have been studied extensively on various spaces of holomorphic functions with the aim to explore the connection between their behaviors with the function-theoretic properties of the symbols g, especially after the works of Pommerenke

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[15] and subsequently by Aleman, Cima, and Siskakis [1]–[3], [16]. Later, in 2008, S. Li and S. Stević took the study further by introducing the following operators induced by pairs of holomorphic symbols (g, ψ),

$$I_{(g,\psi)}f(z) = \int_0^z f'\big(\psi(w)\big)g(w)\,dw, \qquad C_{(g,\psi)}f(z) = \int_0^{\psi(z)} f'(w)g(w)\,dw, \quad (1.1)$$

$$V_g^{\psi}f(z) = \int_0^z f\big(\psi(w)\big)g'(w)\,dw, \qquad C_g^{\psi}f(z) = \int_0^{\psi(z)} f(w)g'(w)\,dw, \quad (1.2)$$

and studied their operator-theoretic properties on some spaces of analytic functions on the unit disk (see, e.g., [8]–[10]). Since then, these classes of generalized integral operators have constituted an active area of research. In particular, there has been a growing interest in studying the operators V_g^{ψ} and C_g^{ψ}, partly because some of their properties are related to the notion of Carleson measures and properties of Toeplitz operators, which are readily available for several known spaces. In contrast, the operators $I_{(g,\psi)}$ and $C_{(g,\psi)}$ have drawn little attention, even if they have found applications in the study of linear isometries of spaces of holomorphic functions. An interesting example in this arena could be the following: if D^p denotes the space of all analytic functions f in the unit disk for which its derivative f' belongs to the Hardy space H^p, then, for $p \neq 2$, any surjective isometry U of D^p under the norm $\|f\|_{D^p} = |f(0)| + \|f'\|_{H^p}$ is of the form

$$Uf = \lambda f(0) + \lambda I_{(g,\psi)}f$$

for some unimodular λ in $\mathbb{C}$, a nonconstant inner function ψ, and a function g in H^p (see [5]).

The bounded and compact Schatten-class properties of the operators in (1.2) acting on the classical Fock spaces were studied in [12] and [11]. Recently, this study was pursued further and the bounded and compact properties of the operators in (1.1) were addressed in [13]. In this article, we continue those lines of research and address the question of Schatten-class membership for this class of operators. It turns out that such maps belong to the Schatten $\mathcal{S}_p(\mathcal{F}_\alpha^2) =: \mathcal{S}_p$ class if and only if certain Berezin-type integral transforms are $L^{p/2}$-integrable on the complex plane $\mathbb{C}$. After that, an easier and simple-to-apply description is given. As will be seen later, an immediate consequence of our main results shows that the operators in (1.2) belong to the $\mathcal{S}_p$ class whenever the class of operators in (1.1) do, while the converse in general fails.

We note that the operators in (1.1) are called the *generalized Volterra companion operators* because the particular choice $\psi(z) = z$ reduces both $I_{(g,\psi)}$ and $C_{(g,\psi)}$ to the Volterra companion operator I_g. Some call them the *generalized composition operators* because the choices $g = \psi'$ and $g = 1$ reduce the operators $I_{(g,\psi)}$ and $C_{(g,\psi)}$, respectively, to the composition operator C_ψ up to certain constants.

The classical Fock space $\mathcal{F}_\alpha^2$ consists of all entire functions f for which

$$\|f\|^2 = \frac{\alpha}{\pi}\int_{\mathbb{C}} \big|f(z)\big|^2 e^{-\alpha|z|^2}\,dm(z) < \infty, \quad (1.3)$$

where dm denotes the usual Lebesgue area measure on $\mathbb{C}$ and α is a positive parameter. The space $\mathcal{F}_\alpha^2$ is a reproducing kernel Hilbert space with kernel function $K_w(z) = e^{\alpha\langle z,w\rangle}$ and normalized kernel function $k_w(z) = e^{\alpha\langle z,w\rangle - \alpha|w|^2/2}$. Because of the reproducing property of the kernel and Parseval identity, it holds that

$$\begin{aligned} K_w(z) &= \sum_{n=1}^{\infty} \langle K_w, e_n\rangle e_n(z) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)} \qquad \text{and} \\ \|K_w\|^2 &= \sum_{n=1}^{\infty} \big|e_n(w)\big|^2 \end{aligned} \tag{1.4}$$

for any orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{F}_\alpha^2$. These series representations of K_w and its norm will be used several times in our subsequent considerations. An immediate consequence of (1.4) is that

$$\frac{\partial}{\partial \overline{w}} K_w(z) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n'(w)} \qquad \text{and} \qquad \Big\|\frac{\partial}{\partial \overline{w}} K_w\Big\|^2 = \sum_{n=1}^{\infty} \big|e_n'(w)\big|^2. \tag{1.5}$$

We set $Q_g(z) = |g(z)|e^{-\frac{\alpha}{2}|z|^2}(1+|z|)^{-1}$. Then our first result is expressed in terms of generalized Berezin-type integral transforms

$$\begin{aligned} B_{(|g|,\psi)}(w) &= \int_{\mathbb{C}} \big|\big(|w|+1\big)k_w\big(\psi(z)\big)Q_g(z)\big|^2\, dm(z) \qquad \text{and} \\ B_{(|g(\psi)|,\psi)}(w) &= \int_{\mathbb{C}} \big|\big(|w|+1\big)k_w\big(\psi(z)\big)\psi'(z)Q_{g(\psi)}(z)\big|^2\, dm(z). \end{aligned}$$

Having fixed the notions, we may now state our first main result.

Theorem 1.1. *Let $0 < p < \infty$ and (g,ψ) be a pair of entire functions on $\mathbb{C}$. Then the operator*

(i) $I_{(g,\psi)} : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$ *belongs to the Schatten $\mathcal{S}_p$ class if and only if $B_{(|g|,\psi)}$ belongs to $L^{p/2}(\mathbb{C}, dm)$. In this case, we also have the asymptotic norm estimate*

$$\|I_{(g,\psi)}\|_{\mathcal{S}_p} \simeq \Big(\int_{\mathbb{C}} B_{(|g|,\psi)}^{p/2}(z)\, dm(z)\Big)^{1/p}. \tag{1.6}$$

(ii) $C_{(g,\psi)} : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$ *belongs to the Schatten $\mathcal{S}_p$ class if and only if $B_{(|g(\psi)|,\psi)}$ belongs to $L^{p/2}(\mathbb{C}, dm)$. Furthermore, we have*

$$\|C_{(g,\psi)}\|_{\mathcal{S}_p} \simeq \Big(\int_{\mathbb{C}} B_{(|g(\psi)|,\psi)}^{p/2}(z)\, dm(z)\Big)^{1/p}.$$

Note that notation $U(z) \lesssim V(z)$ (or, equivalently, $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \le CV(z)$ holds for all z in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

We note that Theorem 1.1 is formulated in terms of a condition that involves a double integral. In what follows we give a simplified and easy-to-apply description in terms of L^p-integrability of the symbol g against a super-exponentially decreasing weight.

Theorem 1.2. *Let $0 < p < \infty$ and (g, ψ) be a pair of entire functions on $\mathbb{C}$. Then*

(i) $I_{(g,\psi)} : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$ *belongs to the Schatten $\mathcal{S}_p$ class if and only if*

$$\int_{\mathbb{C}} \big|g(z)\big|^p e^{\frac{p\alpha}{2}(|\psi(z)|^2 - |z|^2)}\, dm(z) < \infty. \tag{1.7}$$

(ii) $C_{(g,\psi)} : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$ *belongs to the Schatten $\mathcal{S}_p$ class if and only if*

$$\int_{\mathbb{C}} \big|g\big(\psi(z)\big)\big|^p e^{\frac{p\alpha}{2}(|\psi(z)|^2 - |z|^2)}\, dm(z) < \infty.$$

As mentioned earlier, setting $\psi(z) = z$ reduces the operators in (1.1) to I_g. By Corollary 3.1 of [13], I_g belongs to $\mathcal{S}_p$ if and only if g is the zero function. This fails to hold in general for the operators $I_{(g,\psi)}$ and $C_{(g,\psi)}$. One such example could be seen by scaling ψ as $\psi_0(z) = \frac{1}{2}z$. In this case, for $p = 2$, condition (1.7) holds if and only if

$$\int_{\mathbb{C}} \big|g(z)\big|^2 e^{-\frac{3\alpha}{4}|z|^2}\, dm(z) < \infty.$$

Then $I_{(g_0,\psi_0)}$ belongs to $\mathcal{S}_2$ if we set, for instance, $g_0(z) = z$ since

$$\int_{\mathbb{C}} |z|^2 e^{-\frac{3\alpha}{4}|z|^2}\, dm(z) \simeq \int_0^\infty r^3 e^{-\frac{3\alpha}{4}r^2}\, dr = \frac{2}{9}\alpha^{-2}\Gamma(2) < \infty.$$

Seemingly, for this particular choice (g_0, ψ_0), the operator $C_{(g_0,\psi_0)}$ also belongs to the Schatten class $\mathcal{S}_2$. This example, in addition, verifies that the operators $I_{(g,\psi)}$ and $C_{(g,\psi)}$ have a much richer operator-theoretic structure than the operator I_g.

Our main results, coupled with a similar result from [12] for the class of operators in (1.2), give the following sufficient conditions for the Schatten-class membership of V_g^ψ and C_g^ψ .

Corollary 1.3. *Let $0 < p < \infty$ and (g, ψ) be a pair of entire functions on $\mathbb{C}$. Then if the operator*

(i) $I_{(g,\psi)} : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$ *belongs to $\mathcal{S}_p$, so does the map* $V_g^\psi : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$.

(ii) $C_{(g,\psi)} : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$ *belongs to $\mathcal{S}_p$, so does the map* $C_g^\psi : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$.

The corollary shows that the conditions for Schatten-class membership of the operators $I_{(g,\psi)}$ and $C_{(g,\psi)}$ are respectively stronger than the corresponding conditions for V_g^ψ and C_g^ψ. But the converses of the statements both in (i) and (ii) in general fail. To see this, we may in particular set $\psi(z) = z$ and observe that the class of operators in (1.2) reduces to the operator V_g. By Corollary 4 of [12], any compact V_g belongs to $\mathcal{S}_p$ for all g whenever $p > 2$, while its $\mathcal{S}_p$ membership for $p \le 2$ holds if and only if g is a constant function. On the other hand, by Corollary 3.1 of [13], I_g belongs to $\mathcal{S}_p$ if and only if g is the zero function. A similar observation was recorded in [13], contrasting the boundedness and compactness conditions for the two classes of maps in (1.1) and (1.2).

2. Preliminaries

Before embarking on the proof of our first main result, we give a key lemma, which provides a link to how the Berezin-type integral transforms in the condition of the theorem come into play.

Lemma 2.1. *Let (g, ψ) be a pair of entire functions on $\mathbb{C}$. Then for any function f in $\mathcal{F}_\alpha^2$, the following estimates hold:*

$$\|I_{(g,\psi)}f\|^2 \lesssim \int_{\mathbb{C}} \big|f'(w)\big|^2 \frac{e^{-\alpha|w|^2}}{(1+|w|)^2} B_{(|g|,\psi)}(w)\, dm(w), \tag{2.1}$$

$$\|C_{(g,\psi)}f\|^2 \lesssim \int_{\mathbb{C}} \big|f'(w)\big|^2 \frac{e^{-\alpha|w|^2}}{(1+|w|)^2} B_{(|g(\psi)|,\psi)}(w)\, dm(w). \tag{2.2}$$

Proof. The proof of the lemma is implicitly contained in the proof of Theorem 3.1 in [13]. We explicitly reproduce it here for the sake of completeness. Since $|f'|^2$ is subharmonic for each holomorphic function f, by Lemma 1 of [6], we have the local estimate

$$\big|f'(z)\big|^2 e^{-\alpha|z|^2} \lesssim \int_{D(z,1)} \big|f'(w)\big|^2 e^{-\alpha|w|^2}\, dm(w). \tag{2.3}$$

On the other hand, a recent result of Constantin [4, Proposition 1.4] ensures that, for each entire function f, the Littlewood–Paley-type estimate

$$\int_{\mathbb{C}} \big|f(z)\big|^p e^{-\frac{\alpha p}{2}|z|^2}\, dm(z) \simeq \big|f(0)\big|^p + \int_{\mathbb{C}} \big|f'(z)\big|^p \big(1+|z|\big)^{-p} e^{-\frac{\alpha p}{2}|z|^2}\, dm(z) \tag{2.4}$$

holds for all $0 < p < \infty$. Applying this for $p = 2$ and (2.3), we obtain

$$\begin{aligned} \|I_{(g,\psi)}f\|^2 \lesssim \int_{\mathbb{C}} & e^{\alpha(|\psi(z)|^2 - |z|^2)} \frac{|g(z)|^2}{(1+|z|)^2} \\ & \times \int_{\mathbb{C}} \chi_{D(\psi(z),1)}(w) \big|f'(w)\big|^2 e^{-\alpha|w|^2}\, dm(w)\, dm(z), \end{aligned}$$

where $\chi_{D(\psi(z),1)}$ refers to the characteristic function on the set $D(\psi(z), 1)$. Since $\chi_{D(\psi(z),1)}(w) = \chi_{D(w,1)}(\psi(z))$, for each point w and z in $\mathbb{C}$, by Fubini's theorem it follows that the right-hand side of the above inequality is equal to

$$\begin{aligned} &\int_{\mathbb{C}} \big|f'(w)\big|^2 e^{-\alpha|w|^2} \int_{D(w,1)} e^{\alpha|\xi|^2}\, d\mu_{(g,\psi)}(\xi)\, dm(w) \\ &\simeq \int_{\mathbb{C}} \big|f'(w)\big|^2 \frac{e^{-\alpha|w|^2}}{(1+|w|)^2} \int_{D(w,1)} \big(1+|\xi|\big)^2 e^{\alpha|\xi|^2}\, d\mu_{(g,\psi)}(\xi)\, dm(w), \end{aligned} \tag{2.5}$$

where we set $\xi = \psi(z)$,

$$d\mu_{(g,\psi)}(E) = \int_{\psi^{-1}(E)} \frac{|g(z)|^2}{(1+|z|)^2} e^{-\alpha|z|^2}\, dm(z)$$

for every Borel subset E of $\mathbb{C}$, and use the fact that $1 + |w| \simeq 1 + |\xi|$ whenever ξ belongs to the disk $D(w, 1)$. To arrive at the desired conclusion, it suffices to show that

$$\int_{D(w,1)} \big(1+|\xi|\big)^2 e^{\alpha|\xi|^2}\, d\mu_{(g,\psi)}(\xi) \lesssim B_{(|g|,\psi)}(w).$$

But this estimate easily holds because

$$\begin{aligned}\int_{D(w,1)} \big(1+|\xi|\big)^2 e^{\alpha|\xi|^2}\, d\mu_{(g,\psi)}(\xi) &\simeq \big(1+|w|\big)^2 \int_{D(w,1)} e^{\alpha|\xi|^2}\, d\mu_{(g,\psi)}(\xi)\\ &\lesssim B_{(|g|,\psi)}(w),\end{aligned}$$

where in the last relationship we have used a simple fact that if $\xi \in D(w,1)$, then

$$\big|k_w(\xi)\big|^2 = |e^{-\frac{\alpha}{2}|w|^2+\alpha\overline{w}\xi}|^2 = e^{\alpha(|\xi|^2-|\xi-w|^2)} \gtrsim e^{\alpha|\xi|^2}, \tag{2.6}$$

and integrating (2.6) against the measure $\mu_{(g,\psi)}$ we have that

$$\int_{D(w,1)} e^{\alpha|\xi|^2}\, d\mu_{(g,\psi)}(\xi) \lesssim \int_{\mathbb{C}} \big|k_w(\xi)\big|^2\, d\mu_{(g,\psi)}(\xi) = \frac{B_{(|g|,\psi)}(w)}{(1+|w|)^2}.$$

The proof of the estimate in (2.2) is very similar to the proof of (2.1). Thus we omit it. ☐

Lemma 2.2. *Let (g,ψ) be a pair of entire functions on $\mathbb{C}$. Then:*

(i) *If $0<p\le 2$, we have the estimate*

$$\int_{\mathbb{C}} \big|k_w\big(\psi(\zeta)\big)\big|^2 \frac{|g(\zeta)|^2 e^{-\alpha|\zeta|^2}}{(1+|\zeta|)^2}\, dm(\zeta) \lesssim \Big(\int_{\mathbb{C}} \big|k_w\big(\psi(\zeta)\big)\big|^p \frac{|g(\zeta)|^p e^{-\frac{\alpha p}{2}|\zeta|^2}}{(1+|\zeta|)^p}\, dm(\zeta)\Big)^{\frac{2}{p}}.$$

(ii) *If $p>2$, we have the reverse estimate*

$$\int_{\mathbb{C}} \big|k_w\big(\psi(\zeta)\big)\big|^p \frac{|g(\zeta)|^p e^{-\frac{\alpha p}{2}|\zeta|^2}}{(1+|\zeta|)^p}\, dm(\zeta) \lesssim \Big(\int_{\mathbb{C}} \big|k_w\big(\psi(\zeta)\big)\big|^2 \frac{|g(\zeta)|^2 e^{-\alpha|\zeta|^2}}{(1+|\zeta|)^2}\, dm(\zeta)\Big)^{\frac{p}{2}}.$$

Proof. Using the fact that $\mathcal{F}_\alpha^p \subset \mathcal{F}_\alpha^2$ for $0<p\le 2$ (see [7, Theorem 7.2]) and the Littlewood–Paley estimate for Fock spaces, we have

$$\begin{aligned}&\Big(\int_{\mathbb{C}} \big|k_w\big(\psi(\zeta)\big)\big|^2 \frac{|g(\zeta)|^2 e^{-\alpha|\zeta|^2}}{(1+|\zeta|)^2}\, dm(\zeta)\Big)^{\frac{1}{2}}\\ &\quad\simeq \Big(\int_{\mathbb{C}} \Big|\int_0^z k_w\big(\psi(\zeta)\big) g(\zeta)\, dm(\zeta)\Big|^2 e^{-\alpha|z|^2}\, dm(z)\Big)^{\frac{1}{2}}\\ &\quad\lesssim \Big(\int_{\mathbb{C}} \Big|\int_0^z k_w\big(\psi(\zeta)\big) g(\zeta)\, dm(\zeta)\Big|^p e^{-\frac{\alpha p}{2}|z|^2}\, dm(z)\Big)^{\frac{1}{p}}\\ &\quad\simeq \Big(\int_{\mathbb{C}} \big|k_w\big(\psi(\zeta)\big)\big|^p \frac{|g(\zeta)|^p e^{-\frac{\alpha p}{2}|\zeta|^2}}{(1+|\zeta|)^p}\, dm(\zeta)\Big)^{\frac{1}{p}},\end{aligned}$$

from which the assertion in (i) follows.

The proof of part (ii) is similar to the preceding proof. This time we only have to use the inclusion $\mathcal{F}_\alpha^2 \subset \mathcal{F}_\alpha^p$ for $p>2$, which can be read, for instance, in Theorem 2.10 of [19]. ☐

Lemma 2.3. *Let (g,ψ) be a pair of entire functions on $\mathbb{C}$, and let $I_{(g,\psi)}$ be a compact operator on $\mathcal{F}_\alpha^2$. Then $\psi(z)=az+b$ for some a and b in $\mathbb{C}$, and $|a|<1$.*

Proof. Let $\mathcal{F}_\alpha^\infty$ denote the space of all entire functions f for which

$$\sup_{z\in\mathbb{C}}\big|f(z)\big|e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

Since $\mathcal{F}_\alpha^2 \subset \mathcal{F}_\alpha^\infty$, it follows that $I_{(g,\psi)} : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^\infty$ is also compact. Then Theorem 3.1 of [13] ensures that

$$\begin{aligned} \sup_{z\in\mathbb{C}} \frac{|g(z)\psi(z)|}{1+|z|} e^{\frac{\alpha}{2}(|\psi(z)|^2-|z|^2)} < \infty \qquad \text{and} \\ \lim_{|\psi(z)|\to\infty} \frac{|g(z)\psi(z)|}{1+|z|} e^{\frac{\alpha}{2}(|\psi(z)|^2-|z|^2)} = 0. \end{aligned} \tag{2.7}$$

Observe that the first part of (2.7) implies that

$$M_\infty\big(g\psi, |z|\big) \lesssim \frac{1+|z|}{e^{\frac{\alpha}{2}(|\psi(z)|^2-|z|^2)}}, \tag{2.8}$$

where $M_\infty(g\psi, |z|)$ is the integral mean (maximum modulus) of the function $g\psi$. Now (2.8), along with the fact that $M_\infty(g\psi, |z|)$ is a nondecreasing function of $|z|$, gives

$$\limsup_{|z|\to\infty}\big(\big|\psi(z)\big| - |z|\big) \le 0; \tag{2.9}$$

otherwise, there would be a sequence (z_j) such that $|z_j| \to \infty$ as $j \to \infty$ and

$$\limsup_{j\to\infty}\big(\big|\psi(z_j)\big| - |z_j|\big) > 0.$$

This, along with the fact that ψ is an entire function, implies that

$$M_\infty\big(g\psi, |z_j|\big) \lesssim \frac{1+|z_j|}{e^{\frac{\alpha}{2}(|\psi(z_j)|^2-|z_j|^2)}}$$

is bounded, which gives a contradiction whenever $g\psi$ is unbounded. The case for bounded $g\psi$ follows easily.

From relation (2.9), we deduce that ψ has the linear form $\psi(z) = az + b$ for some a and b in $\mathbb{C}$ and $|a| \le 1$, and $b = 0$ whenever $|a| = 1$. From the second part of (2.7), we easily see that $|a| < 1$. □

3. Proof of Theorem 1.1

We first prove the necessity of the condition following a classical approach, as, for example, in [14] and [17]. Since $I_{(g,\psi)} : \mathcal{F}_\alpha^2 \to \mathcal{F}_\alpha^2$ is compact, it admits a Schmidt decomposition, and there exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{F}_\alpha^2$ and a sequence of nonnegative numbers $(\lambda_{(n,g,\psi)})_{n\in\mathbb{N}}$ with $\lambda_{(n,g,\psi)} \to 0$ as $n \to \infty$ such that, for all f in $\mathcal{F}_\alpha^2$,

$$I_{(g,\psi)}f = \sum_{n=1}^\infty \lambda_{(n,g,\psi)}\langle f, e_n\rangle e_n. \tag{3.1}$$

The operator $I_{(g,\psi)}$ with such a decomposition belongs to the $\mathcal{S}_p$ class if and only if

$$\|I_{(g,\psi)}\|_{\mathcal{S}_p}^p = \sum_{n=1}^\infty |\lambda_{(n,g,\psi)}|^p < \infty. \tag{3.2}$$

Applying (3.1), in particular, to the kernel function, we obtain the relation

$$\|I_{(g,\psi)}K_z\|^2 = \sum_{n=1}^{\infty} |\lambda_{(n,g,\psi)}|^2 \big|e_n(z)\big|^2,$$

from which we have

$$\int_{\mathbb{C}} \|I_{(g,\psi)}k_z\|^p \, dm(z) = \int_{\mathbb{C}} \Big(\sum_{n=1}^{\infty} |\lambda_{(n,g,\psi)}|^2 \big|e_n(z)\big|^2\Big)^{\frac{p}{2}} e^{-\frac{p\alpha}{2}|z|^2} \, dm(z). \tag{3.3}$$

We may now consider two different cases depending on the size of the exponent p and proceed first to show the case for $p \geq 2$. Applying Hölder's inequality to the sum shows that the left-hand side in (3.3) is bounded by

$$\begin{aligned}
&\int_{\mathbb{C}} \sum_{n=1}^{\infty} |\lambda_{(n,g,\psi)}|^p \big|e_n(z)\big|^2 \Big(\sum_{n=1}^{\infty} \big|e_n(z)\big|^2\Big)^{\frac{p-2}{2}} e^{-\frac{p\alpha}{2}|z|^2} \, dm(z) \\
&\quad = \sum_{n=1}^{\infty} |\lambda_{(n,g,\psi)}|^p \int_{\mathbb{C}} \big|e_n(z)\big|^2 e^{-\alpha|z|^2} \, dm(z) \\
&\quad \simeq \sum_{n=1}^{\infty} |\lambda_{(n,g,\psi)}|^p = \|I_{(g,\psi)}\|_{\mathcal{S}_p}^p,
\end{aligned} \tag{3.4}$$

where the last equality follows by (3.2).

We may now assume that $0 < p < 2$. Since $I_{(g,\psi)}$ is assumed to be in $\mathcal{S}_p$, the positive operator $I^*_{(g,\psi)} I_{(g,\psi)}$ also belongs to $\mathcal{S}_{p/2}$ (see [18]). In addition, there exists a sequence (f_n) of orthonormal basis in $\mathcal{F}^2_\alpha$ for which we have the Schmidt decomposition

$$I^*_{(g,\psi)} I_{(g,\psi)} f = \sum_{n=1}^{\infty} \beta_n \langle f, f_n \rangle_E f_n, \tag{3.5}$$

where the sequence (β_n) comprises the singular values of $I^*_{(g,\psi)} I_{(g,\psi)}$ and $\langle \cdot, \cdot \rangle_E$ is an inner product in $\mathcal{F}^2_\alpha$ defined by

$$\langle f, h \rangle_E = f(0)\overline{h(0)} + \int_{\mathbb{C}} f'(z)\overline{h'(z)} \frac{e^{-\alpha|z|^2}}{(| + |z|)^2} \, dm(z). \tag{3.6}$$

Observe that, because of (2.4), the inner product in (3.6) gives a norm on $\mathcal{F}^2_\alpha$ equivalent to the classical norm. Now using (2.4) and since $0 < p < 2$, it follows that

$$\begin{aligned}
\int_{\mathbb{C}} \|I_{(g,\psi)}k_z\|^p \, dm(z) &\simeq \int_{\mathbb{C}} \Big(\int_{\mathbb{C}} \big|wk_w\big(\psi(\zeta)\big)\big|^2 \frac{|g(\zeta)|^2 e^{-\alpha|\zeta|^2}}{(1 + |\zeta|)^2} \, dm(\zeta)\Big)^{\frac{p}{2}} \, dm(w) \\
&\lesssim \int_{\mathbb{C}} \int_{\mathbb{C}} \big|wk_w\big(\psi(\zeta)\big)\big|^p \frac{|g(\zeta)|^p e^{-\frac{\alpha p}{2}|\zeta|^2}}{(1 + |\zeta|)^p} \, dm(\zeta) \, dm(w),
\end{aligned} \tag{3.7}$$

where the second estimate follows by Lemma 2.2.

By completing the square in the inner product from the kernel function again and making a change of variables, we obtain

$$\begin{aligned}\int_{\mathbb{C}} \big|wk_w\big(\psi(\zeta)\big)\big|^p\, dm(w) &= e^{\frac{p\alpha}{2}|\psi(\zeta)|^2} \int_{\mathbb{C}} |w|^p e^{-\frac{\alpha p}{2}|\psi(\zeta)-w|^2}\, dm(w)\\ &\simeq \big|\psi(\zeta)\big|^p e^{\frac{p\alpha}{2}|\psi(\zeta)|^2}. \end{aligned} \tag{3.8}$$

Applying (1.3) and the techniques above, we also estimate

$$\Big\|\frac{\partial}{\partial \overline{w}} K_w\Big\|^2 \simeq \int_{\mathbb{C}} \big|zK_w(z)\big|^2 e^{-\alpha|z|^2}\, dm(z) \simeq |w|^2 e^{\alpha|w|^2}, \tag{3.9}$$

and from (1.4), (3.8), and (1.5) we find that the double integral in (3.7) is in turn bounded by a positive multiple of

$$\begin{aligned}&\int_{\mathbb{C}} \frac{|g(\zeta)\psi(\zeta)|^p e^{\frac{p\alpha}{2}|\psi(\zeta)|^2}}{(1+|\zeta|)^p e^{\frac{p\alpha}{2}|\zeta|^2}}\, dm(\zeta)\\ &\simeq \int_{\mathbb{C}} \frac{|g(\zeta)\psi(\zeta)|^p}{(1+|\zeta|)^p} e^{\frac{p\alpha}{2}(|\psi(\zeta)|^2-|\zeta|^2)} \frac{\|\frac{\partial}{\partial\overline{\psi(\zeta)}} K_{\psi(\zeta)}\|^2}{(|\psi(\zeta)|+1)^2 e^{\alpha|\psi(\zeta)|^2}}\, dm(\zeta)\\ &= \sum_{n=1}^{\infty} \int_{\mathbb{C}} \frac{|g(\zeta)\psi(\zeta)|^p}{(1+|\zeta|)^p} e^{\frac{p\alpha}{2}(|\psi(\zeta)|^2-|\zeta|^2)} \frac{|f_n'(\psi(\zeta))|^2}{(|\psi(\zeta)|+1)^2 e^{\alpha|\psi(\zeta)|^2}}\, dm(\zeta). \end{aligned} \tag{3.10}$$

Applying Hölder's inequality, it follows that the above sum is bounded by

$$\begin{aligned}&\sum_{n=1}^{\infty} \Big(\int_{\mathbb{C}} \frac{|g(\zeta)|^2}{(1+|\zeta|)^2} |f_n'\big(\psi(\zeta)\big)|^2 e^{-\alpha|\zeta|^2}\, dm(\zeta)\Big)^{\frac{p}{2}}\\ &\qquad\times \Big(\int_{\mathbb{C}} \frac{|f_n'(\psi(\zeta))|^2}{(|\psi(\zeta)|+1)^2} e^{-\alpha|\psi(\zeta)|^2}\, dm(\zeta)\Big)^{\frac{2-p}{2}}. \end{aligned} \tag{3.11}$$

Again, since $I_{(g,\psi)}$ belongs to the Schatten $\mathcal{S}_p$ class, it is compact, and by Lemma 2.3 ψ has the linear form $\psi(z) = az + b$ for some a and b in $\mathbb{C}$ and $|a| < 1$. This together with (2.4) and substitution yield

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{C}} \frac{|f_n'(\psi(\zeta))|^2}{(|\psi(\zeta)|+1)^2} e^{-\alpha|\psi(\zeta)|^2}\, dm(\zeta) < \infty.$$

Making use of this, (3.5), and (3.6), we observe that the quantity in (3.11) is bounded up, to a positive multiple, by

$$\begin{aligned}&\sum_{n=1}^{\infty} \Big(\int_{\mathbb{C}} \frac{|g(\zeta)|^2}{(1+|\zeta|)^2} \big|f_n'\big(\psi(\zeta)\big)\big|^2 e^{-\alpha|\zeta|^2}\, dm(\zeta)\Big)^{\frac{p}{2}}\\ &\lesssim \sum_{n=1}^{\infty} \langle I^*_{(g,\psi)} I_{(g,\psi)} f_n, f_n\rangle_E^{\frac{p}{2}}\\ &= \sum_{n=1}^{\infty} \beta_n^{\frac{p}{2}} = \|I^*_{(g,\psi)} I_{(g,\psi)}\|^{p/2}_{\mathcal{S}_{p/2}} = \|I_{(g,\psi)}\|^p_{\mathcal{S}_p}. \end{aligned} \tag{3.12}$$

From the series of estimates in (3.7)–(3.12), together with (3.3) and (3.4), we deduce that

$$\int_{\mathbb{C}}\Big(\int_{\mathbb{C}}\big|wk_w\big(\psi(\zeta)\big)\big|^2\frac{|g(\zeta)|^2e^{-\alpha|\zeta|^2}}{(1+|\zeta|^2)}\,dm(\zeta)\Big)^{\frac{p}{2}}\,dm(w)\lesssim\|I_{(g,\psi)}\|_{\mathcal{S}_p}^p.$$

From this and (3.7), we conclude the estimate

$$\int_{\mathbb{C}}\|I_{(g,\psi)}k_z\|^p\,dm(z)\lesssim\|I_{(g,\psi)}\|_{\mathcal{S}_p}^p. \tag{3.13}$$

We may first note that

$$\int_{\mathbb{C}}B_{(|g|,\psi)}^{p/2}(w)\,dm(w)=\int_{|w|<1}B_{(|g|,\psi)}^{p/2}(w)\,dm(w)+\int_{|w|\geq1}B_{(|g|,\psi)}^{p/2}(w)\,dm(w).$$

As for $|w|>1$, one has $B_{(|g|,\psi)}^{p/2}(w)\leq\|I_{(g,\psi)}k_w\|^p$, and the estimate in (3.13) implies

$$\int_{|w|\geq1}B_{(|g|,\psi)}^{p/2}(w)\,dm(w)\lesssim\int_{\mathbb{C}}\|I_{(g,\psi)}k_z\|^p\,dm(z). \tag{3.14}$$

On the other hand, since $I_{(g,\psi)}$ is in the Schatten $\mathcal{S}_p$ class, it is bounded with $\|I_{(g,\psi)}\|\lesssim\|I_{(g,\psi)}\|_{\mathcal{S}_p}$, where $\|I_{(g,\psi)}\|$ denotes the operator norm of the bounded operator $I_{(g,\psi)}$. Therefore, by Theorem 2.1 of [13], we have

$$\sup_{w\in\mathbb{C}}B_{(|g|,\psi)}^{p/2}(w)\lesssim\|I_{(g,\psi)}\|^p,$$

from which we have the remaining estimate

$$\int_{|w|<1}B_{(|g|,\psi)}^{p/2}(w)\,dm(w)\lesssim\|I_{(g,\psi)}\|^p m\big\{|w|<1\big\}\lesssim\|I_{(g,\psi)}\|_{\mathcal{S}_p}^p. \tag{3.15}$$

Taking into account the estimates in (3.13), (3.14), and (3.15), we get

$$\int_{\mathbb{C}}B_{(|g|,\psi)}^{p/2}(w)\,dm(w)\lesssim\|I_{(g,\psi)}\|_{\mathcal{S}_p}^p,$$

from which we also have one part of the estimate in (1.6).

We now turn to the proof of the sufficiency of the condition in part (i) of the main result. First observe that relation (3.2) implies

$$\|I_{(g,\psi)}\|_{\mathcal{S}_p}^p=\sum_{n=1}^{\infty}|\lambda_{(n,g,\psi)}|^p\|e_n\|^2\simeq\sum_{n=1}^{\infty}|\lambda_{(n,g,\psi)}|^p\int_{\mathbb{C}}\big|e_n(z)\big|^2\|K_z\|^{-2}\,dm(z),$$

from which for $p<2$, Hölder's inequality applied with exponent $2/p$, and subsequently invoking relations (3.3) give

$$\begin{aligned}\|I_{(g,\psi)}\|_{\mathcal{S}_p}^p&\leq\int_{\mathbb{C}}\Big(\sum_{n=1}^{\infty}|\lambda_{(n,g,\psi)}|^2\big|e_n(z)\big|^2\Big)^{\frac{p}{2}}\Big(\sum_{n=1}^{\infty}\big|e_n(z)\big|^2\Big)^{\frac{2-p}{2}}\|K_z\|^{-2}\,dm(z)\\&=\int_{\mathbb{C}}\Big(\sum_{n=1}^{\infty}|\lambda_{(n,g,\psi)}|^2\big|e_n(z)\big|^2\Big)^{\frac{p}{2}}\|K_z\|^{-p}\,dm(z)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{C}} \|I_{(g,\psi)} k_z\|^p \, dm(z) \\
&\le \int_{\mathbb{C}} B^{\frac{p}{2}}_{(|g|,\psi)}(z)\, dm(z). \qquad (3.16)
\end{aligned}
$$

It remains to prove the assertion for $p \ge 2$. We first note that condition (i) in Theorem 1.1 along with Theorem 3.1 of [13] ensure that $I_{(g,\psi)}$ is a compact operator. We also recall that a compact map $I_{(g,\psi)}$ belongs to $\mathcal{S}_p$ if and only if the sequence $\|I_{(g,\psi)} e_n\|$, $n \in \mathbb{N}$ belongs to ℓ^p for any orthonormal set $\{e_n\}$ of $\mathcal{F}^2_\alpha$ (see [18, Theorem 1.33]). This fact together with Lemma 2.1 imply

$$
\begin{aligned}
\sum_{n=1}^{\infty} \|I_{(g,\psi)} e_n\|^p &\simeq \sum_{n=1}^{\infty} \Big(\int_{\mathbb{C}} \big| e_n'\big(\psi(z)\big)\big|^2 \frac{|g(z)|^2 e^{-\alpha|z|^2}}{(1+|z|)^2}\, dm(z)\Big)^{\frac{p}{2}} \\
&\lesssim \sum_{n=1}^{\infty} \Big(\int_{\mathbb{C}} \big| e_n'(w)\big|^2 \frac{e^{-\alpha|w|^2}}{(1+|w|)^2} B_{(|g|,\psi)}(w)\, dm(w)\Big)^{\frac{p}{2}}. \qquad (3.17)
\end{aligned}
$$

Applying Hölder's inequality again and subsequently taking into account (2.4), (1.5), and (3.9), we see that the right-hand side above is bounded by

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \Big(\int_{\mathbb{C}} \big| e_n'(w)\big|^2 \frac{e^{-\alpha|w|^2}}{(1+|w|)^2}\, dm(w)\Big)^{(p-2)/2} \int_{\mathbb{C}} \big| e_n'(w)\big|^2 \frac{e^{-\alpha|w|^2}}{(1+|w|)^2} B^{\frac{p}{2}}_{(|g|,\psi)}(w)\, dm(w) \\
&\simeq \int_{\mathbb{C}} \Big(\sum_{n=1}^{\infty} \big| e_n'(w)\big|^2 \frac{e^{-\alpha|w|^2}}{(1+|w|)^2}\Big) B^{\frac{p}{2}}_{(|g|,\psi)}(w)\, dm(w) \\
&\simeq \int_{\mathbb{C}} B^{\frac{p}{2}}_{(|g|,\psi)}(w)\, dm(w). \qquad (3.18)
\end{aligned}
$$

From (3.16), (3.17), and (3.18), we conclude our assertion and also establish the remaining estimate in (1.6).

The statement in part (ii) follows from a simple variant of the proof of part (i). This is because $(C_{(g,\psi)} f)'(z) = f'(\psi(z)) g(\psi(z)) \psi'(z)$, which shows that we only need to replace the quantity $g(z)$ by $g(\psi(z))\psi'(z)$ and proceed as in the proof of the preceding part. We omit the details and leave it to the interested reader.

4. Proof of Theorem 1.2

We first note that Theorem 1.1 simply means that $I_{(g,\psi)}$ is in the $\mathcal{S}_p$ class if and only if the function $w \to \|I_{(g,\psi)} k_w\|$ belongs to $L^p(\mathbb{C}, dm)$. Thus the sufficiency of the condition for the case $0 < p \le 2$ follows easily from Theorem 1.1 and the estimates in (3.8) and (3.7). On the other hand, we notice that the series of estimates from (3.10)–(3.12) and Lemma 2.3 give

$$
\begin{aligned}
&\int_{\mathbb{C}} \big|g(z)\big|^p e^{\frac{p\alpha}{2}(|\psi(z)|^2 - |z|^2)}\, dm(z) \\
&\quad \simeq \int_{\mathbb{C}} \Big(\frac{1+|\psi(z)|}{1+|z|}\Big)^p \big|g(z)\big|^p e^{\frac{p\alpha}{2}(|\psi(z)|^2 - |z|^2)}\, dm(z)
\end{aligned}
$$

$$\simeq \int_{\mathbb{C}}\int_{\mathbb{C}}\big|wk_w\big(\psi(\zeta)\big)\big|^p \frac{|g(\zeta)|^p e^{-\frac{\alpha p}{2}|\zeta|^2}}{(1+|\zeta|)^p}\,dm(\zeta)\,dm(w)$$
$$\lesssim \|I_{(g,\psi)}\|^p_{\mathcal{S}_p},$$

which verifies the necessity part of the case.

Next we prove the case for $p>2$. Taking into account the estimate in (3.8) and the case for $p>2$ of Lemma 2.2, we have

$$\int_{\mathbb{C}}\big|g(\zeta)\big|^p e^{\frac{p\alpha}{2}(|\psi(\zeta)|^2-|\zeta|^2)}\frac{|\psi(\zeta)|^p}{(1+|\zeta|)^p}\,dm(\zeta)$$
$$\simeq \int_{\mathbb{C}}\int_{\mathbb{C}}\big|k_w\big(\psi(\zeta)\big)\big|^p \frac{|g(\zeta)|^p e^{-\frac{p\alpha}{2}|\zeta|^2}}{(1+|\zeta|)^p}\,dm(\zeta)\,dm(w)$$
$$\lesssim \int_{\mathbb{C}}\|I_{(g,\psi)}k_w\|^p\,dm(w),$$

from which the necessity condition follows after an application of Lemma 2.3.

For sufficiency as done before, it is enough to prove that

$$\sum_{n=1}^{\infty}\|I_{(g,\psi)}e_n\|^p \le C<\infty$$

for any orthonormal set $\{e_n\}$ of $\mathcal{F}^2_\alpha$. From (3.17), we have

$$\sum_{n=1}^{\infty}\|I_{(g,\psi)}e_n\|^p \simeq \sum_{n=1}^{\infty}\Big(\int_{\mathbb{C}}\big|e_n'\big(\psi(z)\big)\big|^2\frac{|g(z)|^2e^{-\alpha|z|^2}}{(1+|z|)^2}\,dm(z)\Big)^{\frac{p}{2}}.$$

Applying Hölder's inequality we get

$$\begin{aligned} I_n &:= \Big(\int_{\mathbb{C}}\big|e_n'\big(\psi(z)\big)\big|^2\frac{|g(z)|^2e^{-\alpha|z|^2}}{(1+|z|)^2}\,dm(z)\Big)^{\frac{p}{2}}\\ &\le \Big(\int_{\mathbb{C}}\big|e_n'\big(\psi(z)\big)\big|^2\frac{|g(z)|^pe^{-\alpha\frac{p}{2}|z|^2}(1+|\psi(z)|)^p}{(1+|z|)^p(1+|\psi(z)|)^2}e^{\alpha(\frac{p}{2}-1)|\psi(z)|^2}\,dm(z)\Big)\\ &\quad\times\Big(\int_{\mathbb{C}}\big|e_n'\big(\psi(z)\big)\big|^2\frac{e^{-\alpha|\psi(z)|^2}}{(1+|\psi(z)|)^2}\,dm(z)\Big)^{\frac{p-2}{2}}. \end{aligned}$$

Making a change of variables again yields

$$\int_{\mathbb{C}}\big|e_n'(\psi(z)\big|^2\frac{e^{-\alpha|\psi(z)|^2}}{(1+|\psi(z)|)^2}\,dm(z)\lesssim \|e_n\|^2\lesssim 1,$$

which implies that

$$I_n\lesssim \int_{\mathbb{C}}\big|e_n'\big(\psi(z)\big)\big|^2\frac{|g(z)|^pe^{-\alpha\frac{p}{2}|z|^2}(1+|\psi(z)|)^p}{(1+|z|)^p(1+|\psi(z)|)^2}e^{\alpha(\frac{p}{2}-1)|\psi(z)|^2}\,dm(z).$$

From this and the estimate

$$\sum_{n=1}^{\infty}\big|e_n'\big(\psi(z)\big)\big|^2\simeq\big|\psi(z)\big|^2e^{\alpha|\psi(z)|^2},$$

we obtain

$$\sum_{n=1}^{\infty} \|I_{(g,\psi)} e_n\|^p \simeq \sum_{n=1}^{\infty} I_n \lesssim \int_{\mathbb{C}} \big|g(z)\big|^p e^{\alpha \frac{p}{2}(|\psi(z)|^2-|z|^2)} \frac{(1+|\psi(z)|)^p}{(1+|z|)^p}\, dm(z),$$

from which the result follows since, as done before, condition (1.7) implies that ψ is a linear map.

The statement in part (ii) of Theorem 1.2 follows from a simple variant of the proof of part (i) above. Thus we omit the details again.

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