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DUAL OF EXTREMAL ABSOLUTE NORMS ON $\mathbb{R}^2$ AND THE JAMES CONSTANT

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Abstract. The set of all absolute normalized norms on $\mathbb{R}^2$ (denoted by AN_2) has a convex structure with respect to the usual operation. In a previous article, N. Komuro, K.-S. Saito, and K.-I. Mitani calculated the James constants of $(\mathbb{R}^2, \|\cdot\|)$ when $\|\cdot\|$ is an extreme point of AN_2. In this article, we calculate the James constant of its dual space.

1. Introduction and preliminaries

For a Banach space X, let S_X be the unit sphere of X, that is, $S_X = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be *uniformly nonsquare* if there exists $\delta > 0$ such that $\|x - y\| \geq 2(1-\delta)$ and $x, y \in S_X$ imply $\|x + y\| \leq 2(1-\delta)$. The James constant $J(X)$ of X is defined by

$$J(X) = \sup\big\{\min\big\{\|x+y\|, \|x-y\|\big\} : x, y \in S_X\big\}$$

(Gao and Lau [3]). It has been recently studied by several authors (cf. [2], [3], and [12]). We collect some properties of the James constant:

(i) For any Banach space X, the James constant of X satisfies $\sqrt{2} \leq J(X) \leq 2$.
(ii) If X is a Hilbert space, then $J(X) = \sqrt{2}$. The converse is not true in general.
(iii) A Banach space X is uniformly nonsquare if and only if $J(X) < 2$.

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(iv) If $1 \leq p \leq \infty$ and $1/p + 1/q = 1$, then $J(L_p) = \max\{2^{1/p}, 2^{1/q}\}$, when $\dim L_p \geq 2$.

(v) $J(X) = \sup\{\varepsilon \in (0,2) : \delta_X(\varepsilon) \leq 1 - \varepsilon/2\}$, where δ_X is the modulus of convexity of X.

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(x_1, x_2)\| = \|(|x_1|, |x_2|)\|$ for all $(x_1, x_2) \in \mathbb{R}^2$, and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. Let AN_2 be the family of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ_2 be the set of all convex functions ψ on the interval $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$. Then it is known that AN_2 and Ψ_2 are in a one-to-one correspondence under the equation $\psi(t) = \|(1-t, t)\|_\psi$ for $t \in [0,1]$. The norm $\|\cdot\|_\psi$ associated with ψ is given by

$$\big\|(x_1, x_2)\big\|_\psi = \begin{cases} (|x_1| + |x_2|)\psi(\frac{|x_2|}{|x_1|+|x_2|}) & ((x_1, x_2) \neq (0,0)), \\ 0 & ((x_1, x_2) = (0,0)). \end{cases}$$

The sets AN_2 and Ψ_2 are convex, and the correspondence $\|\cdot\|_\psi \leftrightarrow \psi$ preserves the operation to take a convex combination. The extreme points $\mathrm{ext}(AN_2)$ of AN_2 were investigated by Grząślewicz [7]. It was shown that if $\|\cdot\| \in AN_2$, then $\|\cdot\|$ is an extreme point of AN_2 if and only if all extreme points of the unit ball of $(\mathbb{R}^2, \|\cdot\|)$ are contained in the unit sphere of $(\mathbb{R}^2, \|\cdot\|_\infty)$. In 2010, Komuro, Saito, and Mitani [9] showed this in terms of convex functions; that is, $\mathrm{ext}(\Psi_2) = \{\psi_{\alpha,\beta} : 0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1\}$, where

$$\psi_{\alpha,\beta}(t) = \begin{cases} 1-t & (0 \leq t \leq \alpha), \\ \frac{\alpha+\beta-1}{\beta-\alpha}t + \frac{\beta-2\alpha\beta}{\beta-\alpha} & (\alpha \leq t \leq \beta), \\ t & (\beta \leq t \leq 1). \end{cases}$$

The James constants of $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ were calculated by Komuro, Saito, and Mitani [10]. In this case, $\alpha \leq 1-\beta$ is essential. Indeed, for α, β with $\alpha > 1-\beta$, we have $\psi_{1-\beta,1-\alpha}(t) = \psi_{\alpha,\beta}(1-t)$. Then it follows that $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ is isometrically isomorphic to $(\mathbb{R}^2, \|\cdot\|_{\psi_{1-\beta,1-\alpha}})$. We define $\tilde{\psi}_{\alpha,\beta}(t) = \psi_{\alpha,\beta}(1-t)$. Then $(\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})$ is isometrically isomorphic to $(\mathbb{R}^2, \|\cdot\|_{\tilde{\psi}_{\alpha,\beta}})$. Since James constants are invariant under isometric isomorphism, we have $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = J((\mathbb{R}^2, \|\cdot\|_{\tilde{\psi}_{\alpha,\beta}}))$.

In 2011, Komuro, Saito, and Mitani [10] completely determined $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$. Let $0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1$ and $\alpha \leq 1-\beta$.

(i) If $\psi_{\alpha,\beta}(\frac{1}{2}) \leq \frac{1}{2(1-\alpha)}$, then

$$J\big((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})\big) = \frac{1}{\psi_{\alpha,\beta}(\frac{1}{2})}.$$

(ii) If $\frac{1}{2(1-\alpha)} \leq \psi_{\alpha,\beta}(\frac{1}{2}) \leq c(\alpha,\beta)$, then

$$J\big((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})\big) = 1 + \frac{1}{2\psi_{\alpha,\beta}(\frac{1}{2}) + \frac{2\beta-1}{\beta-\alpha}}.$$

(iii) If $\psi_{\alpha,\beta}(\frac{1}{2}) \geq c(\alpha,\beta)$, then

$$J\big((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})\big) = 2\psi_{\alpha,\beta}\Big(\frac{1}{2}\Big),$$

where

$$c(\alpha,\beta)=\frac{1}{4}\Big(1-\frac{2\beta-1}{\beta-\alpha}+\sqrt{\Big(1+\frac{2\beta-1}{\beta-\alpha}\Big)^2+4}\Big).$$

On the other hand, it is known that the equality $J(X^*)=J(X)$ does not hold in general. A counterexample can be found in Kato, Maligranda, and Takahashi [8]. Therefore it is natural to consider the behavior of $J((\mathbb{R}^2,\|\cdot\|_{\psi_{\alpha,\beta}})^*)$. We note, as was shown in [11], that the equality $J((\mathbb{R}^2,\|\cdot\|_{\psi_{1-\beta,\beta}})^*)=J((\mathbb{R}^2,\|\cdot\|_{\psi_{1-\beta,\beta}}))$ holds for each $\beta\in[1/2,1]$.

In this paper, we determine the value of $J((\mathbb{R}^2,\|\cdot\|_{\psi_{\alpha,\beta}})^*)$ for all $\psi_{\alpha,\beta}$ satisfying $0\le\alpha\le\frac{1}{2}\le\beta\le1$. As in the case of $(\mathbb{R}^2,\|\cdot\|_{\psi_{\alpha,\beta}})$, the case $\alpha<1-\beta$ is essential. Then we make use of the notion of dual functions. For $\psi\in\Psi_2$, we define

$$\psi^*(t)=\sup\Big\{\frac{(1-s)(1-t)+st}{\psi(s)}:0\le s\le1\Big\}$$

for t with $0\le t\le1$. Then we have $\psi^*\in\Psi_2$ and $(\mathbb{R}^2,\|\cdot\|_\psi)^*=(\mathbb{R}^2,\|\cdot\|_{\psi^*})$. In what follows, we denote $\|\cdot\|_{\psi^*_{\alpha,\beta}}$ by $\|\cdot\|^*_{\alpha,\beta}$ for short. Using this identification, the main theorem in this article is stated as follows.

Theorem 1.1. *Let $0\le\alpha\le\frac{1}{2}\le\beta\le1$ and $\alpha<1-\beta$.*

(i) *If $\psi^*_{\alpha,\beta}(\frac{1}{2})\ge\frac{3}{4}$, then $J((\mathbb{R}^2,\|\cdot\|_{\alpha,\beta})^*)=2\psi^*_{\alpha,\beta}(\frac{1}{2})$.*
(ii) *If $\frac{3}{4}\ge\psi^*_{\alpha,\beta}(\frac{1}{2})\ge\frac{1}{\sqrt{2}}$, then $J((\mathbb{R}^2,\|\cdot\|_{\alpha,\beta})^*)=\max\{2\psi^*_{\alpha,\beta}(\frac{1}{2}),A(\alpha,\beta)\}$.*
(iii) *If $\psi^*_{\alpha,\beta}(\frac{1}{2})\le\frac{1}{\sqrt{2}}$, then $J((\mathbb{R}^2,\|\cdot\|_{\alpha,\beta})^*)=A(\alpha,\beta)$,*

where

$$A(\alpha,\beta)=\frac{2(1-\alpha)((2\beta-1)^2\alpha+(1-2\alpha)\beta)}{(\beta-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))}.$$

Figure 1 represents the behavior of $J((\mathbb{R}^2,\|\cdot\|_{\alpha,\beta})^*)$ for any α, β with $0\le\alpha\le\frac{1}{2}\le\beta\le1$ and $\alpha<1-\beta$.

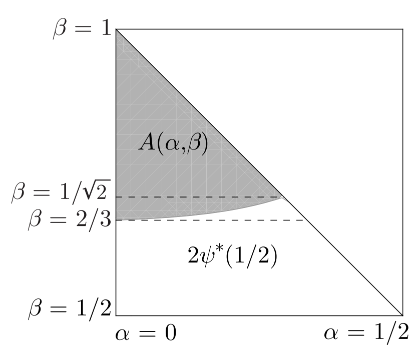

FIGURE 1. The behavior of $J((\mathbb{R}^2\|\cdot\|_{\alpha,\beta})^*)$.

2. The James constant of $(\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*$

By the definition of $\psi_{\alpha,\beta}$, we can write the function $\psi^*_{\alpha,\beta}$ as follows:

$$\psi^*_{\alpha,\beta}(t) = \begin{cases} 1 - \frac{1-2\alpha}{1-\alpha}t & (0 \le t \le \frac{1}{1+k_0}), \\ \frac{1-\beta}{\beta} + \frac{2\beta-1}{\beta}t & (\frac{1}{1+k_0} \le t \le 1), \end{cases}$$

where $k_0 = \frac{\beta(1-2\alpha)}{(1-\alpha)(2\beta-1)}$. The norm corresponding to $\psi^*_{\alpha,\beta}$ is

$$\big\|(x_1, x_2)\big\|^*_{\alpha,\beta} = \begin{cases} |x_1| + \frac{\alpha}{1-\alpha}|x_2| & (|x_1| \ge k_0|x_2|), \\ \frac{1-\beta}{\beta}|x_1| + |x_2| & (|x_1| \le k_0|x_2|). \end{cases}$$

Now, we shall calculate the James constants of $(\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*$. Let

$$x(\theta) = \frac{(\cos\theta, \sin\theta)}{\|(\cos\theta, \sin\theta)\|^*_{\alpha,\beta}} \quad (0 \le \theta < 2\pi).$$

Then we can rewrite the James constant of $(\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*$ as follows:

$$\begin{aligned} &J\big((\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*\big) \\ &\quad = \sup\Big\{\min\Big\{\big\|x(\theta) + x(\theta')\big\|^*_{\alpha,\beta}, \big\|x(\theta) - x(\theta')\big\|^*_{\alpha,\beta}\Big\} : 0 \le \theta \le \theta' < 2\pi\Big\}. \end{aligned}$$

Let $\theta_0 \in [0, 2\pi)$ be the angle satisfying $\tan\theta_0 = \frac{1}{k_0}$. Since $k_0 \ge 1$, we have $0 < \theta_0 < \frac{\pi}{4}$ (Figure 2).

Since the unit sphere of $(\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*$ is symmetric with respect to the x-axis, if $\theta \in [\pi, \frac{3}{2}\pi]$, then we consider $x(\theta - \pi) = -x(\theta)$. Similarly, if $\theta \in [\frac{3}{2}\pi, 2\pi]$, then we consider $x(2\pi - \theta) = -x(\pi - \theta)$. Since $\|\cdot\|^*_{\alpha,\beta}$ is absolute, the mapping $T(x_1, x_2) = (-x_1, x_2)$ is an isometry. Moreover, we have $T(x(\theta)) = x(\pi - \theta)$, $Ty \in S_X$ and

$$\big\|x(\theta) \pm y\big\|^*_{\alpha,\beta} = \big\|T\big(x(\theta) \pm y\big)\big\|^*_{\alpha,\beta} = \big\|x(2\pi - \theta) \pm Ty\big\|^*_{\alpha,\beta}.$$

From these facts, we may assume that $0 \le \theta \le \frac{\pi}{2}$. Thus it is enough to consider the following five cases:

(i) $0 \le \theta, \theta' \le \frac{\pi}{2}$ (Figure 3).

(ii) $0 \le \theta \le \frac{\pi}{4}$, $\frac{3}{4}\pi \le \theta' \le \pi$ (Figure 4).

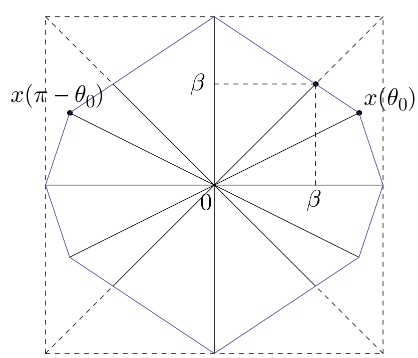

Figure 2. The unit sphere of $(\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*$.

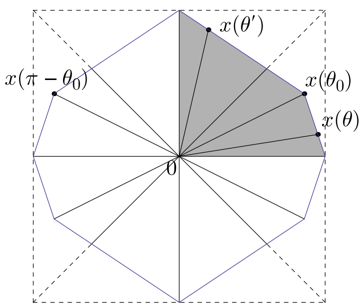

FIGURE 3. Case (i).

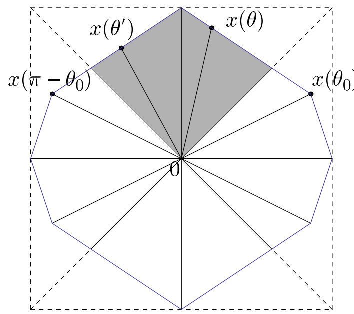

FIGURE 5. Case (iii).

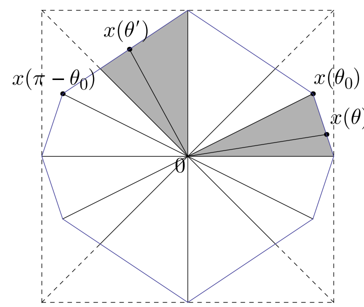

FIGURE 7. Case (v).

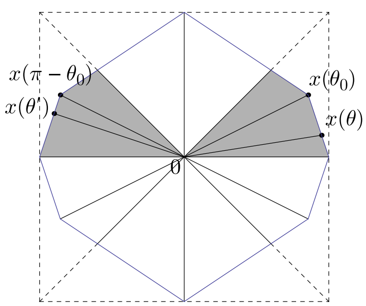

FIGURE 4. Case (ii).

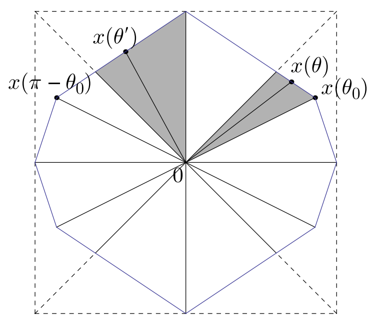

FIGURE 6. Case (iv).

(iii) $\frac{\pi}{4} \le \theta,\ \theta' \le \frac{3}{4}\pi$ (Figure 5).
(iv) $\theta_0 \le \theta \le \frac{\pi}{4},\ \frac{\pi}{2} \le \theta' \le \frac{3}{4}\pi$ (Figure 6).
(v) $0 \le \theta \le \theta_0,\ \frac{\pi}{2} \le \theta' \le \frac{3}{4}\pi$ (Figure 7).

The following lemma is due to Alonso and Martín [1, Lemma 2].

Lemma 2.1 (Alonso and Martín [1])**.** *Let* $\theta_1 < \theta_2 < \theta_3 < \theta_4 (\le \theta_1 + \pi)$. *Then*

$$\big\|x(\theta_2) - x(\theta_3)\big\| \le \big\|x(\theta_1) - x(\theta_4)\big\|$$

and

$$\big\|x(\theta_2) + x(\theta_3)\big\| \ge \big\|x(\theta_1) + x(\theta_4)\big\|.$$

Using this result, we can easily obtain the following propositions.

Proposition 2.2. *Put*

$$Q_1 = \sup\Big\{\min\{\big\|x(\theta) + x(\theta')\big\|^*_{\alpha,\beta}, \big\|x(\theta) - x(\theta')\big\|^*_{\alpha,\beta}\} : 0 \le \theta \le \theta' \le \frac{\pi}{2}\Big\}.$$

Then

$$Q_1 = 2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big).$$

Proof. Let $0 \leq \theta \leq \theta' \leq \frac{\pi}{2}$. By Lemma 2.1,

$$\begin{aligned}\min\{\|x(\theta)+x(\theta')\|^*_{\alpha,\beta}, \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\} &\leq \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\\ &\leq \|x(0)-x(\pi/2)\|^*_{\alpha,\beta}\\ &= 2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big).\end{aligned}$$

Thus we have $Q_1 = 2\psi^*_{\alpha,\beta}(\frac{1}{2})$. □

Proposition 2.3. *Put*

$$Q_2 = \sup\Big\{\min\{\|x(\theta)+x(\theta')\|^*_{\alpha,\beta}, \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\} : 0 \leq \theta \leq \frac{\pi}{4}, \frac{3}{4}\pi \leq \theta' \leq \pi\Big\}.$$

Then

$$Q_2 = \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}.$$

Proof. Let $0 \leq \theta \leq \frac{\pi}{4}$ and $\frac{3}{4}\pi \leq \theta' \leq \pi$. By Lemma 2.1,

$$\begin{aligned}\min\{\|x(\theta)+x(\theta')\|^*_{\alpha,\beta}, \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\} &\leq \|x(\theta)+x(\theta')\|^*_{\alpha,\beta}\\ &\leq \|x(\pi/4)+x(3\pi/4)\|^*_{\alpha,\beta}\\ &= \|(0,2\beta)\|^*_{\alpha,\beta}\\ &= \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}.\end{aligned}$$

Thus we have $Q_2 = \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}$. □

Proposition 2.4. *Put*

$$Q_3 = \sup\Big\{\min\{\|x(\theta)+x(\theta')\|^*_{\alpha,\beta}, \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\} : \frac{\pi}{4} \leq \theta \leq \theta' \leq \frac{3}{4}\pi\Big\}.$$

Then

$$Q_3 = \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}.$$

Proof. Let $\frac{\pi}{4} \leq \theta \leq \theta' \leq \frac{3}{4}\pi$. By Lemma 2.1,

$$\begin{aligned}\min\{\|x(\theta)+x(\theta')\|^*_{\alpha,\beta}, \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\} &\leq \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\\ &\leq \|x(\pi/4)-x(3\pi/4)\|^*_{\alpha,\beta}\\ &= \|(2\beta,0)\|^*_{\alpha,\beta}\\ &= \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}.\end{aligned}$$

Thus we have $Q_3 = \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}$. □

Therefore, we only calculate cases (iv) and (v). The following propositions are the main tasks of this argument.

Proposition 2.5. *Put*

$$Q_4 = \sup\left\{\min\{\left\|x(\theta)+x(\theta')\right\|_{\alpha,\beta}^*, \left\|x(\theta)-x(\theta')\right\|_{\alpha,\beta}^*\} : \begin{matrix}\theta_0 \le \theta \le \frac{\pi}{4} \\ \frac{\pi}{2} \le \theta' \le \frac{3}{4}\pi\end{matrix}\right\}.$$

(i) *If* $\frac{1}{2} \le \beta \le \frac{2}{3}$, *then* $Q_4 \le 2\psi_{\alpha,\beta}^*(\frac{1}{2})$.
(ii) *If* $\frac{2}{3} \le \beta \le 1$, *then*

$$A(\alpha,\beta) \le Q_4 \le \max\left\{2\psi_{\alpha,\beta}^*\left(\frac{1}{2}\right), A(\alpha,\beta)\right\},$$

where $A(\alpha,\beta) = \frac{2(1-\alpha)((2\beta-1)^2\alpha+(1-2\alpha)\beta)}{(\beta-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))}$.

Moreover, $A(\alpha,\beta) \ge \frac{1}{\psi_{\alpha,\beta}^*(\frac{1}{2})}$ *for each* $\frac{1}{2} \le \beta \le 1$.

Proof. Let $\theta_0 \le \theta \le \frac{\pi}{4}$ and $\frac{\pi}{2} \le \theta' \le \frac{3}{4}\pi$. Then we can write

$$x(\theta) = \left(s, -\frac{1-\beta}{\beta}s+1\right) \quad \left(\beta \le s \le \frac{\beta(1-2\alpha)}{\beta-\alpha}\right),$$
$$x(\theta') = \left(-t, -\frac{1-\beta}{\beta}t+1\right) \quad (0 \le t \le \beta).$$

Note that

$$\left\|x(\theta)+x(\theta')\right\|_{\alpha,\beta}^* = \left\|\left(s-t, -\frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t+2\right)\right\|_{\alpha,\beta}^*$$

and

$$\left\|x(\theta)-x(\theta')\right\|_{\alpha,\beta}^* = \left\|\left(s+t, \frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t\right)\right\|_{\alpha,\beta}^*.$$

To calculate the norms $\|x(\theta) \pm x(\theta')\|_{\alpha,\beta}^*$, we consider the following cases:

(4a) $s-t \ge k_0\{-\frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t+2\}$,
(4b) $s-t \le k_0\{-\frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t+2\}$,
(4c) $s+t \ge k_0\{\frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t\}$, and
(4d) $s+t \le k_0\{\frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t\}$.

Case (4a). This case does not occur. Indeed, one has

$$k_0\left\{-\frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t+2\right\} - (s-t) = \frac{H(s,t)}{(1-\alpha)(2\beta-1)},$$

where

$$H(s,t) = (\alpha-\beta)s + (-4\alpha\beta+3\alpha+3\beta-2)t - 4\alpha\beta+2\beta.$$

We remark that the function H is affine and decreasing with respect to the variable s. Moreover, we obtain

$$H(\beta,0) \ge H\left(\frac{\beta(1-2\alpha)}{\beta-\alpha}, 0\right) = \beta(1-2\alpha) \ge 0$$

and

$$H(\beta,\beta) \ge H\left(\frac{\beta(1-2\alpha)}{\beta-\alpha}, \beta\right) = \beta\big((2\beta-1)(1-2\alpha)+\beta-\alpha\big) \ge 0.$$

Hence it follows that $H \geq 0$ on the domain $[\beta, \frac{\beta(1-2\alpha)}{\beta-\alpha}] \times [0, \beta]$. As a consequence, we always have Case (4b).

Case (4d). Since $\beta \leq s \leq \frac{\beta(1-2\alpha)}{\beta-\alpha}$,

$$\begin{aligned}\left\|x(\theta) - x(\theta')\right\|^*_{\alpha,\beta} &= \frac{1-\beta}{\beta}(s+t) + \frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t \\ &= \frac{2(1-\beta)}{\beta}s \\ &\leq \frac{2(1-\beta)}{\beta} \cdot \frac{\beta(1-2\alpha)}{\beta-\alpha} \\ &= \frac{1}{\beta} - \frac{(2\beta-1)(\beta(1-2\alpha)+\alpha)}{\beta(\beta-\alpha)} \\ &< \frac{1}{\beta} = 2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big).\end{aligned}$$

Thus $\min\{\|x(\theta)+x(\theta')\|^*_{\alpha,\beta}, \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\} < 2\psi^*_{\alpha,\beta}(\frac{1}{2})$.

Therefore it is enough to consider (4b)–(4c).

Cases (4b)–(4c). If (4b) holds, then

$$\begin{aligned}\left\|x(\theta) + x(\theta')\right\|^*_{\alpha,\beta} &= \frac{1-\beta}{\beta}(s-t) - \frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t + 2 \\ &= -\frac{2(1-\beta)}{\beta}t + 2,\end{aligned}$$

and if (4c) holds, then

$$\begin{aligned}\left\|x(\theta) - x(\theta')\right\|^*_{\alpha,\beta} &= s + t + \frac{\alpha}{1-\alpha}\Big(\frac{1-\beta}{\beta}s - \frac{1-\beta}{\beta}t\Big) \\ &= \frac{\alpha+\beta(1-2\alpha)}{\beta(1-\alpha)}s + \frac{\beta-\alpha}{\beta(1-\alpha)}t.\end{aligned}$$

For any $0 \leq \alpha \leq \frac{1}{2} \leq \beta \leq 1$ with $\alpha + \beta \leq 1$, we define

$$f_1(t) = -\frac{2(1-\beta)}{\beta}t + 2$$

and

$$g_1(s,t) = \frac{\alpha+\beta(1-2\alpha)}{\beta(1-\alpha)}s + \frac{\beta-\alpha}{\beta(1-\alpha)}t.$$

If $f_1(t) \geq 2\beta$, then

$$\begin{aligned}f_1(t) \geq 2\beta &\Longleftrightarrow -\frac{2(1-\beta)}{\beta}t + 2 \geq 2\beta \\ &\Longleftrightarrow t \leq \beta.\end{aligned}$$

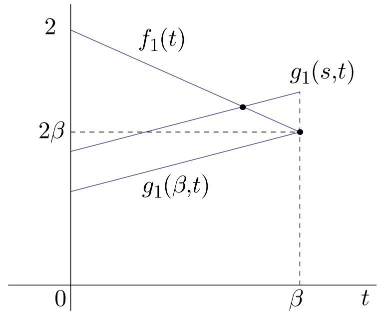

FIGURE 8. The graph of f_1 and g_1.

On the other hand, we have

$$\begin{aligned} g_1(s,t) &\le g_1\Big(\frac{\beta(1-2\alpha)}{\beta-\alpha}, t\Big) \\ &= \frac{1}{\beta(1-\alpha)}\Big\{\frac{\beta(1-2\alpha)(\alpha+\beta(1-2\alpha))}{\beta-\alpha} + (\beta-\alpha)t\Big\} \end{aligned}$$

and

$$g_1(s,\beta) \ge g_1(\beta,\beta) = 2\beta.$$

Thus, there exists a unique real number t_1 such that $f_1(t_1) = g_1(\frac{\beta(1-2\alpha)}{\beta-\alpha}, t_1)$ (Figure 8). Then

$$\begin{aligned} & f_1(t_1) = g_1\Big(\frac{\beta(1-2\alpha)}{\beta-\alpha}, t_1\Big) \\ \iff & -\frac{2(1-\beta)}{\beta}t_1 + 2 = \frac{1}{\beta(1-\alpha)}\Big\{\frac{\beta(1-2\alpha)(\alpha+\beta(1-2\alpha))}{\beta-\alpha} + (\beta-\alpha)t_1\Big\} \\ \iff & t_1 = \frac{\beta(-4\alpha^2\beta + 4\alpha^2 + 2\alpha\beta - 3\alpha + \beta)}{(\beta-\alpha)(2\alpha\beta - 3\alpha - \beta + 2)}, \end{aligned}$$

which implies that

$$\begin{aligned} f_1(t_1) &= -\frac{2(1-\beta)}{\beta}\cdot\frac{\beta(-4\alpha^2\beta + 4\alpha^2 + 2\alpha\beta - 3\alpha + \beta)}{(\beta-\alpha)(2\alpha\beta - 3\alpha - \beta + 2)} + 2 \\ &= \frac{2(1-\alpha)((2\beta-1)^2\alpha + (1-2\alpha)\beta)}{(\beta-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))}. \end{aligned}$$

Put

$$A(\alpha,\beta) = \frac{2(1-\alpha)((2\beta-1)^2\alpha + (1-2\alpha)\beta)}{(\beta-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))}.$$

Then we have $Q_4 \le \max\{2\psi^*_{\alpha,\beta}(\frac{1}{2}), A(\alpha,\beta)\}$. We remark here that $A(\alpha,\beta) \ge 2\beta = \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}$ for each $\beta \in [\frac{1}{2}, 1]$.

Now we shall show that $2\psi^*_{\alpha,\beta}(\frac{1}{2}) \geq A(\alpha,\beta)$ for each $\beta \in [\frac{1}{2},\frac{2}{3}]$. The difference is

$$\begin{aligned}
&2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big) - A(\alpha,\beta) \\
&\quad = \frac{1}{\beta} - \frac{2(1-\alpha)((2\beta-1)^2\alpha + (1-2\alpha)\beta)}{(\beta-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))} \\
&\quad = \frac{8\alpha^2\beta^3 - 12\alpha^2\beta^2 - 8\alpha\beta^3 + 16\alpha\beta^2 + 3\alpha^2 - 4\alpha\beta - 3\beta^2 - 2\alpha + 2\beta}{\beta(\beta-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))}.
\end{aligned}$$

We put

$$F(\alpha,\beta) = 8\alpha^2\beta^3 - 12\alpha^2\beta^2 - 8\alpha\beta^3 + 16\alpha\beta^2 + 3\alpha^2 - 4\alpha\beta - 3\beta^2 - 2\alpha + 2\beta.$$

Since $(\beta-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha)) > 0$, it is enough to show that $F(\alpha,\beta) \geq 0$ for each $\beta \in [\frac{1}{2},\frac{2}{3}]$.

First, we show that $F(\alpha,\beta)$ is concave for each β. We have

$$\begin{aligned}
\frac{\partial}{\partial\beta}F(\alpha,\beta) &= -24\alpha(1-\alpha)\beta^2 + 2(-12\alpha^2 + 16\alpha - 3)\beta - 4\alpha + 2, \\
\frac{\partial^2}{\partial\beta^2}F(\alpha,\beta) &= -48\alpha(1-\alpha)\beta + 2(-12\alpha^2 + 16\alpha - 3).
\end{aligned}$$

Since $-48\alpha(1-\alpha) < 0$, we have that $\frac{\partial^2}{\partial\beta^2}F$ is nonincreasing for each $\beta \in [\frac{1}{2},1]$. Moreover, $\frac{\partial^2}{\partial\beta^2}F(\alpha,\frac{1}{2}) = -4(1-2\alpha) - 2 < 0$, which implies that the function $F(\alpha,\beta)$ is concave for each β. Since

$$\begin{aligned}
F\Big(\alpha,\frac{1}{2}\Big) &= \Big(\alpha - \frac{1}{2}\Big)^2 \geq 0, \\
F\Big(\alpha,\frac{2}{3}\Big) &= \frac{1}{27}\alpha(\alpha+2) \geq 0,
\end{aligned}$$

for each $\alpha \in [0,\frac{1}{2}]$ we have $F(\alpha,\beta) \geq 0$ when $\beta \in [\frac{1}{2},\frac{2}{3}]$. Thus we obtain that $\min\{\|x(\theta)+x(\theta')\|^*_{\alpha,\beta}, \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\} \leq 2\psi^*_{\alpha,\beta}(\frac{1}{2})$ when $\beta \in [\frac{1}{2},\frac{2}{3}]$.

Let us consider the case when $\frac{2}{3} \leq \beta \leq 1$. Put $s_1 = \frac{\beta(1-2\alpha)}{\beta-\alpha}$. Then s_1 and t_1 satisfy conditions (4b)–(4c) for each α and β. First, let us consider case (4b). Then

$$\begin{aligned}
&s_1 - t_1 \leq k_0\Big\{-\frac{1-\beta}{\beta}s_1 - \frac{1-\beta}{\beta}t_1 + 2\Big\} \\
&\quad\Longleftrightarrow \frac{2\beta(-8\alpha^2\beta^2 + 12\alpha^2\beta + 4\alpha\beta^2 - 3\alpha^2 - 8\alpha\beta + \beta^2 + 2\alpha)}{(\beta-\alpha)(2\beta-1)(\beta-\alpha+2(1-\beta)(1-\alpha))} \geq 0.
\end{aligned}$$

Thus it is enough to show that

$$\begin{aligned}
h_1(\alpha,\beta) &= -8\alpha^2\beta^2 + 12\alpha^2\beta + 4\alpha\beta^2 - 3\alpha^2 - 8\alpha\beta + \beta^2 + 2\alpha \\
&= (-8\beta^2 + 12\beta - 3)\alpha^2 + (4\beta^2 - 8\beta + 2)\alpha + \beta^2 \geq 0.
\end{aligned}$$

Since the discriminant D of the polynomial h_1 of second degree in α is given by

$$D/4 = 12\beta^4 - 28\beta^3 + 23\beta^2 - 8\beta + 1 = -(1-\beta)(3\beta-1)(2\beta-1)^2 \leq 0,$$

the equation $h_1(\alpha,\beta) = 0$ has at most one solution for each $\beta \in [\frac{1}{2}, 1]$, which together with $h_1(0,\beta) = \beta^2 \geq 0$, shows that $h_1(\alpha,\beta) \geq 0$. Thus, s_1 and t_1 satisfy case (4b).

Next, let us consider case (4c). Then

$$\begin{aligned} s_1 + t_1 \geq k_0\Big\{\frac{1-\beta}{\beta}s_1 - \frac{1-\beta}{\beta}t_1\Big\} \\ \iff \frac{2\beta(8\alpha^2\beta^2 - 16\alpha^2\beta - 8\alpha\beta^2 + 7\alpha^2 + 18\alpha\beta + \beta^2 - 8\alpha - 4\beta + 2)}{(\beta-\alpha)(2\beta-1)(\beta-\alpha+2(1-\beta)(1-\alpha))} \leq 0. \end{aligned}$$

Thus it is enough to show that

$$\begin{aligned} h_2(\alpha,\beta) &= 8\alpha^2\beta^2 - 16\alpha^2\beta - 8\alpha\beta^2 + 7\alpha^2 + 18\alpha\beta + \beta^2 - 8\alpha - 4\beta + 2 \\ &= (8\beta^2 - 16\beta + 7)\alpha^2 + 2(-4\beta^2 + 9\beta - 4)\alpha + \beta^2 - 4\beta + 2 \leq 0. \end{aligned}$$

We have that

$$h_2(\alpha,\beta) = (8\beta^2 - 16\beta + 7)\big(\alpha_1(\beta) - \alpha\big)\big(\alpha_2(\beta) - \alpha\big)$$

where,

$$\begin{aligned} \alpha_1(\beta) &= \frac{4\beta^2 - 9\beta + 4 + \sqrt{2}(2\beta-1)(1-\beta)}{8\beta^2 - 16\beta + 7}, \\ \alpha_2(\beta) &= \frac{4\beta^2 - 9\beta + 4 - \sqrt{2}(2\beta-1)(1-\beta)}{8\beta^2 - 16\beta + 7}. \end{aligned}$$

Moreover, one obtains $8\beta^2 - 16\beta + 7 < 0$ for each $\beta \in [\frac{2}{3}, 1]$, which implies that $\alpha_1(\beta) \leq \alpha_2(\beta)$ and

$$\alpha_1(\beta) - \frac{1}{2} = \frac{2\beta - 1}{1 + 2\sqrt{2}(1-\beta)} \geq 0,$$

since $8\beta^2 - 16\beta + 7 = (2\sqrt{2}\beta - 2\sqrt{2} - 1)(2\sqrt{2}\beta - 2\sqrt{2} + 1)$. It follows that $h_2(\alpha,\beta) \leq 0$ for each $\alpha \in [0, \frac{1}{2}]$ and each $\beta \in [\frac{2}{3}, 1]$, which shows that s_1 and t_1 satisfy case (4c). Thus we have

$$A(\alpha,\beta) \leq \min\{\|x(\theta) + x(\theta')\|^*_{\alpha,\beta}, \|x(\theta) - x(\theta')\|^*_{\alpha,\beta}\}.$$

Therefore we obtain

$$A(\alpha,\beta) \leq Q_4 \leq \max\Big\{2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big), A(\alpha,\beta)\Big\}$$

when $\frac{2}{3} \leq \beta \leq 1$. This completes the proof. □

Proposition 2.6. *Put*

$$Q_5 = \sup\left\{\min\{\|x(\theta) + x(\theta')\|^*_{\alpha,\beta}, \|x(\theta) - x(\theta')\|^*_{\alpha,\beta}\} : \begin{array}{c} 0 \leq \theta \leq \theta_0 \\ \frac{\pi}{2} \leq \theta' \leq \frac{3}{4}\pi \end{array}\right\}.$$

Then

$$Q_5 \leq \max\Big\{2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big), A(\alpha,\beta)\Big\}.$$

Proof. Let $0 \le \theta \le \theta_0$ and $\frac{\pi}{2} \le \theta' \le \frac{3}{4}\pi$. Then we can write

$$x(\theta) = \Big(s, \frac{1-\alpha}{\alpha}(1-s)\Big) \quad \Big(\frac{\beta(1-2\alpha)}{\beta-\alpha} \le s \le 1\Big),$$
$$x(\theta') = \Big(-t, -\frac{1-\beta}{\beta}t + 1\Big) \quad (0 \le t \le \beta).$$

Note that

$$\big\|x(\theta) + x(\theta')\big\|^*_{\alpha,\beta} = \Big\|\Big(s-t, \frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta}t + 1\Big)\Big\|^*_{\alpha,\beta}$$

and

$$\big\|x(\theta) - x(\theta')\big\|^*_{\alpha,\beta} = \Big\|\Big(s+t, 1 - \frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta}t\Big)\Big\|^*_{\alpha,\beta}.$$

To calculate the norms $\|x(\theta) \pm x(\theta')\|^*_{\alpha,\beta}$, we consider the following cases:

(5a) $s - t \ge k_0\{\frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta}t + 1\}$,
(5b) $s - t \le k_0\{\frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta}t + 1\}$,
(5c) $s + t \ge k_0\{1 - \frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta}t\}$, and
(5d) $s + t \le k_0\{1 - \frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta}t\}$.

Case (5a). We shall show that this case does not occur. Let

$$K(s,t) = k_0\Big\{\frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta}t + 1\Big\} - (s-t).$$

Then K is an affine function, and it satisfies

$$K(0,0) = k_0\Big(\frac{1-\alpha}{\alpha} + 1\Big) \ge 0,$$
$$K(0,\beta) = k_0\Big(\frac{1-\alpha}{\alpha} + \beta\Big) + \beta \ge 0,$$
$$K(1,0) = k_0 - 1 = \frac{1-\alpha-\beta}{(1-\alpha)(2\beta-1)} \ge 0,$$
$$K(1,\beta) = \frac{g(\alpha)}{(1-\alpha)(2\beta-1)},$$

where $g(\alpha) = (-4\beta^2+3\beta-1)\alpha + 3\beta^2 - 3\beta + 1$. We remark that $-4\beta^2+3\beta-1 < 0$ since it has complex roots. It follows from $\alpha + \beta \le 1$ that $g(\alpha) \ge g(1-\beta) = \beta(2\beta-1)^2 \ge 0$. Hence we have $K \ge 0$ on $[0,1] \times [0,\beta]$, which proves that (5b) always holds.

Case (5d). Since $\frac{\beta(1-2\alpha)}{\beta-\alpha} \le s \le 1$,

$$\big\|x(\theta) - x(\theta')\big\|^*_{\alpha,\beta} = \frac{1-\beta}{\beta}(s+t) + 1 - \frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta}t$$
$$= \Big(\frac{1-\beta}{\beta} + \frac{1-\alpha}{\alpha}\Big)s + 1 - \frac{1-\alpha}{\alpha}$$

$$\begin{aligned}&\le \frac{1-\beta}{\beta} + \frac{1-\alpha}{\alpha} + 1 - \frac{1-\alpha}{\alpha}\\ &= \frac{1-\beta}{\beta} + 1 = \frac{1}{\beta} = 2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big).\end{aligned}$$

Hence $\min\{\|x(\theta)+x(\theta')\|^*_{\alpha,\beta}, \|x(\theta)-x(\theta')\|^*_{\alpha,\beta}\} < 2\psi^*_{\alpha,\beta}(\frac{1}{2})$. Thus, it is enough to consider (5b)–(5c).

Cases (5b)–(5c). If (5b) holds, then

$$\begin{aligned}\big\|x(\theta)+x(\theta')\big\|^*_{\alpha,\beta} &= \frac{1-\beta}{\beta}(s-t) + \frac{1-\alpha}{\alpha}(1-s) - \frac{1-\beta}{\beta} + 1\\ &= -\frac{\beta-\alpha}{\alpha\beta}s - \frac{2(1-\beta)}{\beta}t + \frac{1}{\alpha},\end{aligned}$$

and if (5c) holds, then

$$\begin{aligned}\big\|x(\theta)-x(\theta')\big\|^*_{\alpha,\beta} &= s+t+\frac{\alpha}{1-\alpha}\Big\{1-\frac{1-\alpha}{\alpha}(1-s)-\frac{1-\beta}{\beta}t\Big\}\\ &= 2s + \frac{\beta-\alpha}{\beta(1-\alpha)}t - \frac{1-2\alpha}{1-\alpha}.\end{aligned}$$

We define

$$f_2(s,t) = -\frac{\beta-\alpha}{\alpha\beta}s - \frac{2(1-\beta)}{\beta}t + \frac{1}{\alpha}$$

and

$$g_2(s,t) = 2s + \frac{\beta-\alpha}{\beta(1-\alpha)}t - \frac{1-2\alpha}{1-\alpha}.$$

We also have

$$\begin{aligned}f_2(s,t) &\le f_2\Big(\frac{\beta(1-2\alpha)}{\beta-\alpha}, t\Big)\\ &= 2 - \frac{2(1-\beta)}{\beta}t\end{aligned}$$

and

$$\begin{aligned}g_2(s,t) &\ge g_2\Big(\frac{\beta(1-2\alpha)}{\beta-\alpha}, t\Big)\\ &= \frac{\beta-\alpha}{\beta(1-\alpha)}t + \frac{2\beta(1-2\alpha)}{\beta-\alpha} - \frac{1-2\alpha}{1-\alpha}.\end{aligned}$$

Since $f_2(\frac{\beta(1-2\alpha)}{\beta-\alpha}, \beta) = 2\beta$ and

$$g_2\Big(\frac{\beta(1-2\alpha)}{\beta-\alpha}, \beta\Big) = \frac{(1-\alpha-\beta)(\alpha+\beta(1-2\alpha))}{(\beta-\alpha)(1-\alpha)} + 2\beta > 2\beta,$$

there exist $t(s)$ such that $f_2(s,t(s)) = g_2(s,t(s))$ for each s (Figure 9).

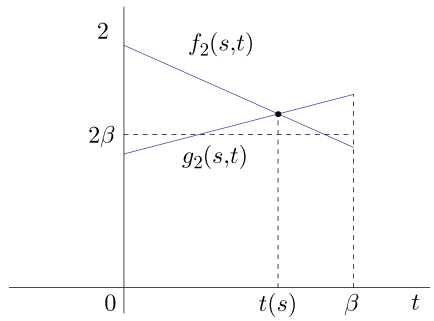

FIGURE 9. The graph of f_2 and g_2.

Hence we have

$$\begin{aligned}
&2s+\frac{\beta-\alpha}{\beta(1-\alpha)}t(s)-\frac{1-2\alpha}{1-\alpha}=-\frac{\beta-\alpha}{\alpha\beta}s-\frac{2(1-\beta)}{\beta}t(s)+\frac{1}{\alpha}\\
&\iff \Big(\frac{\beta-\alpha}{\beta(1-\alpha)}+\frac{2(1-\beta)}{\beta}\Big)t(s)=-\Big(\frac{\beta-\alpha}{\alpha\beta}+2\Big)s+\frac{1}{\alpha}+\frac{1-2\alpha}{1-\alpha}\\
&\iff \frac{\beta-\alpha+2(1-\beta)(1-\alpha)}{\beta(1-\alpha)}t(s)=-\frac{\alpha(2\beta-1)+\beta}{\alpha\beta}s+\frac{1-2\alpha^2}{\alpha(1-\alpha)}\\
&\iff t(s)=\frac{-(1-\alpha)(\alpha(2\beta-1)+\beta)s+\beta(1-2\alpha^2)}{\alpha(\beta-\alpha+2(1-\beta)(1-\alpha))}.
\end{aligned}$$

Then we calculate that

$$\begin{aligned}
&g_2\big(s,t(s)\big)\\
&=2s+\frac{\beta-\alpha}{\beta(1-\alpha)}\frac{-((1-\alpha)(\alpha(2\beta-1)+\beta))s+\beta(1-2\alpha^2)}{\alpha(\beta-\alpha+2(1-\beta)(1-\alpha))}-\frac{1-2\alpha}{1-\alpha}\\
&=\Big(2-\frac{(\beta-\alpha)(\alpha(2\beta-1)+\beta)}{\alpha\beta(\beta-\alpha+2(1-\beta)(1-\alpha))}\Big)s\\
&\quad+\frac{(\beta-\alpha)(1-2\alpha^2)}{\alpha(\beta-\alpha+2(1-\beta)(1-\alpha))}-\frac{1-2\alpha}{1-\alpha}\\
&=\frac{4\alpha\beta(1-\alpha)(1-\beta)-(\alpha-\beta)^2}{\alpha\beta(\beta-\alpha+2(1-\beta)(1-\alpha))}s\\
&\quad+\frac{4\alpha^3\beta-4\alpha^3-6\alpha^2\beta+7\alpha^2+\alpha\beta-3\alpha+\beta}{\alpha(1-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))}.
\end{aligned}$$

We put $d(\alpha,\beta)=4\alpha\beta(1-\alpha)(1-\beta)-(\alpha-\beta)^2$.

(I) The case when $d(\alpha,\beta)\geq 0$. Since $\frac{\beta(1-2\alpha)}{\beta-\alpha}\leq s\leq 1$, the function g_2 takes the maximum at $s=1$. In this case,

$$\begin{aligned}
g_2\big(1,t(1)\big)<f_2(1,0)&=-\frac{\beta-\alpha}{\alpha\beta}+\frac{1}{\alpha}\\
&=\frac{1}{\beta}=2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big).
\end{aligned}$$

(II) The case when $d(\alpha,\beta) < 0$. Since $\frac{\beta(1-2\alpha)}{\beta-\alpha} \le s \le 1$, the function g_2 takes the maximum at $s = \frac{\beta(1-2\alpha)}{\beta-\alpha}$. Hence

$$\begin{aligned}
&g_2\Big(\frac{\beta(1-2\alpha)}{\beta-\alpha}, t\Big(\frac{\beta(1-2\alpha)}{\beta-\alpha}\Big)\Big)\\
&= \frac{4\alpha^2\beta^2 - 4\alpha^2\beta - 4\alpha\beta^2 - \alpha^2 + 6\alpha\beta - \beta^2}{\alpha\beta(\beta-\alpha+2(1-\beta)(1-\alpha))}\frac{\beta(1-2\alpha)}{\beta-\alpha}\\
&\quad + \frac{4\alpha^3\beta - 4\alpha^3 - 6\alpha^2\beta + 7\alpha^2 + \alpha\beta - 3\alpha + \beta}{\alpha(1-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))}\\
&= \frac{2(1-\alpha)((2\beta-1)^2\alpha + (1-2\alpha)\beta)}{(\beta-\alpha)(\beta-\alpha+2(1-\beta)(1-\alpha))}\\
&= A(\alpha,\beta).
\end{aligned}$$

Thus we have $Q_5 \le \max\{2\psi^*_{\alpha,\beta}(\frac{1}{2}), A(\alpha,\beta)\}$. □

Combining Propositions 2.2–2.6, let us show Theorem 1.1.

Proof of Theorem 1.1. As in the proof of Proposition 2.5, we have

$$2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big) \ge A(\alpha,\beta) \ge \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}$$

when $\frac{1}{2} \le \beta \le \frac{2}{3}$. Since $\psi^*_{\alpha,\beta}(\frac{1}{2}) = \frac{1}{2\beta}$, we have that

$$\frac{1}{2} \le \beta \le \frac{2}{3} \Longleftrightarrow \frac{3}{4} \le \psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big) \le 1.$$

Thus we have case (i). Let $\frac{2}{3} \le \beta \le 1$. By Propositions 2.2–2.6 and $A(\alpha,\beta) \ge \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})}$, we have

$$\begin{aligned}
\max\Big\{2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big), A(\alpha,\beta)\Big\} &\le J\big((\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*\big)\\
&= \max\{Q_1, Q_2, Q_3, Q_4, Q_5\}\\
&\le \max\Big\{2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big), A(\alpha,\beta)\Big\}.
\end{aligned}$$

Therefore we have

$$J\big((\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*\big) = \max\Big\{2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big), A(\alpha,\beta)\Big\}$$

when $\frac{2}{3} \le \beta \le 1$. In particular, if $\frac{1}{\sqrt{2}} \le \beta \le 1$, then

$$2\psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big) = \frac{1}{\beta} \le 2\beta = \frac{1}{\psi^*_{\alpha,\beta}(\frac{1}{2})} \le A(\alpha,\beta),$$

and so we have $J((\mathbb{R}^2, \|\cdot\|_{\alpha,\beta})^*) = A(\alpha,\beta)$. Since

$$\frac{1}{\sqrt{2}} \le \beta \le 1 \Longleftrightarrow \frac{1}{2} \le \psi^*_{\alpha,\beta}\Big(\frac{1}{2}\Big) \le \frac{1}{\sqrt{2}},$$

we have cases (ii) and (iii). This completes the proof. □

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