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COMPOSITE BERNSTEIN CUBATURE

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Abstract. We consider a sequence of composite bivariate Bernstein operators and the cubature formula associated with them. The upper bounds for the remainder term of a cubature formula are described in terms of moduli of continuity of order two. Also, we include some results showing how nonmultiplicative the integration functional is.

1. Introduction

We reconsider (composite) bivariate Bernstein approximation and the corresponding cubature formulas. This is motivated by a recent series of articles by Bărbosu et al. (see [3], [4]). However, some of these papers contain rather misleading statements and claims which can hardly be verified. The present article is written with the intention to clean up some of the bugs, to optimize and generalize certain estimates, and thus to further describe the situation at hand.

Our present contribution is a continuation of [9]. Historically, the origin of the method discussed seems to be in the article [13] by D. D. Stancu and A. Vernescu.

2. A general result

We first introduce some notation that will be needed to formulate the general result.

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Definition 2.1. Let I and J be compact intervals of the real axis, and let $L : C(I) \to C(I)$ and $M : C(J) \to C(J)$ be discretely defined operators; that is,

$$L(g;x) = \sum_{e\in E} g(x_e)A_e(x), \quad g \in C(I), x \in I,$$

where E is a finite index set, the $x_e \in I$ are mutually distinct, and $A_e \in C(I)$, $e \in E$.

Analogously,

$$M(h;y) = \sum_{f\in F} h(y_f)B_f(y), \quad h \in C(J), y \in J.$$

If L is of the form above, then its parametric extension to $C(I \times J)$ is given by

$$_xL(F;x,y) = L(F_y;x) = \sum_{e\in E} F_y(x_e)A_e(x) = \sum_{e\in E} F(x_e,y)A_e(x).$$

Here the F_y, $y \in J$, denote the partial functions of F given by $F_y(x) = F(x,y)$, $x \in I$.

Similarly,

$$_yM(F;x,y) = \sum_{f\in F} F(x,y_f)B_f(y).$$

The tensor product of L and M (or M and L) is given by

$$({}_xL \circ {}_yM)(F;x,y) = \sum_{e\in E}\sum_{f\in F} F(x_e,y_f)A_e(x)B_f(y).$$

The theorem below is given in terms of so-called partial moduli of smoothness of order r, given for the compact intervals $I, J \subset \mathbb{R}$, for $F \in C(I \times J)$, $r \in \mathbb{N}_0$, and $\delta \in \mathbb{R}_+$ by

$$\omega_r(F;\delta,0) := \sup\Bigg\{\Bigg|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}F(x+\nu h,y)\Bigg| : (x,y),(x+rh,y) \in I \times J, |h| \leq \delta\Bigg\}$$

and symmetrically by

$$\omega_r(F;0,\delta) := \sup\Bigg\{\Bigg|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}F(x,y+\nu h)\Bigg| : (x,y),(x,y+rh) \in I \times J, |h| \leq \delta\Bigg\}.$$

The total modulus of smoothness of order r is defined by

$$\omega_r(F;\delta_1,\delta_2) := \sup\Bigg\{\Bigg|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}F(x+\nu h_1,y+\nu h_2)\Bigg| : (x,y),(x+rh_1,y+rh_2) \in I \times J, |h_1| \leq \delta_1, |h_2| \leq \delta_2\Bigg\}.$$

We now formulate and prove a simplified form of [6, Theorem 37].

Theorem 2.1. *Let L and M be discretely defined operators as given above such that*

$$\big|(g-Lg)(x)\big| \le \sum_{\rho=0}^{r} \Gamma_{\rho,L}(x)\omega_\rho\big(g;\Lambda_{\rho,L}(x)\big), \quad g\in C(I), x\in I,$$

and

$$\big|(h-Mh)(y)\big| \le \sum_{\sigma=0}^{s} \Gamma_{\sigma,M}(y)\omega_\sigma\big(h;\Lambda_{\sigma,M}(y)\big), \quad h\in C(J), y\in J.$$

Here ω_ρ, $\rho=0,\dots,r$, denote the moduli of order ρ, and Γ and Λ are bounded functions; the notation is analogous for M. Then for $(x,y)\in I\times J$ and $F\in C(I\times J)$ the following holds:

$$\begin{aligned}\big|\big[F-({}_xL\circ {}_yM)F\big](x,y)\big| \le &\sum_{\rho=0}^{r} \Gamma_{\rho,L}(x)\omega_\rho\big(F;\Lambda_{\rho,L}(x),0\big)\\ &+\|L\|\sum_{\sigma=0}^{s} \Gamma_{\sigma,M}(y)\omega_\sigma\big(F;0,\Lambda_{\sigma,M}(y)\big),\end{aligned}$$

where $\|L\|$ denotes the operator norm of L, which is finite due to the form of L.

Proof. We have

$$\begin{aligned}\big|\big[F-({}_xL\circ {}_yM)F\big](x,y)\big| &= \big|\big[(Id-{}_xL)+{}_xL\circ(Id-{}_yM)\big](F;x,y)\big|\\ &\le \big|(Id-{}_xL)(F;x,y)\big| + \big|{}_xL\circ(Id-{}_yM)(F;x,y)\big|\\ &=: E_1(x,y)+E_2(x,y).\end{aligned}$$

Now, for $x\in I$,

$$\begin{aligned}E_1(x,y) = \big|(Id-L)(F_y;x)\big| &\le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\cdot\omega_\rho\big(F_y;\Lambda_{\rho,L}(x)\big)\\ &\le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\cdot\omega_\rho\big(F;\Lambda_{\rho,L}(x),0\big).\end{aligned}$$

Furthermore, with $G := (Id-{}_yM)F$, we have

$$E_2(x,y) = \big|{}_xL(G;x,y)\big| = \big|L(G_y;x)\big| \le \big\|L(G_y)\big\|_{\infty,x\in I}.$$

Here again, $G_y\in C(I)$ for all $y\in J$. By our assumption on L we have for any $g\in C(I)$ that

$$\|Lg\|_\infty \le \Big(1+\sum_{\rho=0}^{r}2^\rho\cdot\|\Gamma_{\rho,L}\|_\infty\Big)\cdot\|g\|_\infty.$$

Hence $\|L\|<\infty$.

In the situation at hand we have

$$\begin{aligned}\|G_y\|_\infty &= \left\|\left[(Id - {}_yM)F\right]_y(\cdot)\right\|_\infty = \left\|(Id - {}_yM)F(\cdot,y)\right\|_\infty \\ &= \left\|(Id - {}_yM)F_x(y)\right\|_{\infty,x\in I} \le \left\|\sum_{\sigma=0}^{s}\Gamma_{\sigma,M}(y)\cdot\omega_\sigma\big(F_x;\Lambda_{\sigma,M}(y)\big)\right\|_\infty \\ &\le \sum_{\sigma=0}^{s}\Gamma_{\sigma,M}(y)\cdot\sup_{x\in I}\omega_\sigma\big(F_x;\Lambda_{\sigma,M}(y)\big) = \sum_{\sigma=0}^{s}\Gamma_{\sigma,M}(y)\cdot\omega_\sigma\big(F;0,\Lambda_{\sigma,M}(y)\big).\end{aligned}$$

Hence

$$\begin{aligned}E_1(x,y)+E_2(x,y) &\le \sum_{\rho=0}^{r}\Gamma_{\rho,L}(x)\cdot\omega_\rho\big(F;\Lambda_{\rho,L}(x),0\big) \\ &\quad + \|L\|\cdot\sum_{\sigma=0}^{s}\Gamma_{\sigma,M}(y)\cdot\omega_\sigma\big(F;0,\Lambda_{\sigma,M}(y)\big).\end{aligned}$$

□

3. Application to bivariate Bernstein operators

Example 3.1. If we take $L = B_{n_1}$ and $M = B_{n_2}$ with two classical Bernstein operators mapping $C[0,1]$ into $C[0,1]$, then, for $F \in C([0,1]\times[0,1])$ and $(x,y) \in [0,1]\times[0,1]$,

$$({}_xB_{n_1} \circ {}_yB_{n_2})(F;x,y) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} F\Big(\frac{i_1}{n_1},\frac{i_2}{n_2}\Big)p_{n_1,i_1}(x)p_{n_2,i_2}(y),$$

where $p_{n,i}(x) = \binom{i}{n}x^i(1-x)^{n-i}$, $x \in [0,1]$, and

$$\begin{aligned}&\left|\left[F - ({}_xB_{n_1}\circ {}_yB_{n_2})F\right](x,y)\right| \\ &\quad\le \frac{3}{2}\Big[\omega_2\Big(F;\sqrt{\frac{x(1-x)}{n_1}},0\Big) + \omega_2\Big(F;0,\sqrt{\frac{y(1-y)}{n_2}}\Big)\Big] \\ &\quad\le \frac{3}{2}\Big[\|F^{(2,0)}\|_\infty\frac{x(1-x)}{n_1} + \|F^{(0,2)}\|_\infty\frac{y(1-y)}{n_2}\Big], \quad F \in C^{2,2}\big([0,1]\times[0,1]\big).\end{aligned}$$

Proof. We apply Theorem 2.1 with $r = s = 2$, $\Gamma_{0,B_n} = \Gamma_{1,B_n} = 0$, $\Gamma_{2,B_n} = \frac{3}{2}$, and $\Lambda_{2,B_n}(z) = \sqrt{\frac{z(1-z)}{n}}$, for $n \in \{n_1,n_2\}$. The latter two choices are possible due to a well-known result of Păltănea (see [11]) showing that, for the univariate Bernstein operators, one has

$$\big|f(x) - B_n(f,x)\big| \le \frac{3}{2}\omega_2\Big(f,\sqrt{\frac{x(1-x)}{n}}\Big).$$

□

Remark 3.1. From the last inequality we get

$$\big|f(x) - B_n(f;x)\big| \le \frac{3}{2}\|f''\|_\infty\frac{x(1-x)}{n}, \quad f \in C^2[0,1].$$

This is worse than the known inequality

$$\big|f(x) - B_n(f;x)\big| \le \frac{1}{2}\|f''\|_\infty \frac{x(1-x)}{n}.$$

Our inequality was obtained from the more general statement in terms of ω_2 and well-known properties of the modulus.

However, we can use instead [7, Theorem 1] (take $p = q = 2$, $p' = q' = 0$, $r = s = 0$, $\Gamma_{0,0,B_{n_1}}(x) = \frac{1}{2} \cdot \frac{x(1-x)}{n_1}$ and $\Gamma_{0,0,B_{n_2}}(y) = \frac{1}{2} \cdot \frac{y(1-y)}{n_2}$) to arrive at

$$\begin{aligned}
&\big|\big[F - ({}_xB_{n_1} \circ {}_yB_{n_2})F\big](x,y)\big| \\
&\quad \le \frac{1}{2}\frac{x(1-x)}{n_1}\|F^{(2,0)}\|_\infty + \frac{1}{2}\frac{y(1-y)}{n_2}\|F^{(0,2)}\|_\infty + \frac{1}{4}\frac{x(1-x)y(1-y)}{n_1 n_2}\|F^{(2,2)}\|_\infty \\
&\quad \le \frac{1}{8n_1}\|F^{(2,0)}\|_\infty + \frac{1}{8n_2}\|F^{(0,2)}\|_\infty + \frac{1}{64 n_1 n_2}\|F^{(2,2)}\|_\infty.
\end{aligned}$$

An estimate of this kind can be found in [4, Theorem 2.3].

Such three-term expressions typically appear if one writes (I denoting the identity)

$$I - A \circ B = I - A + I - B - (I - A) \circ (I - B) = (I - A) \oplus (I - B);$$

that is, if one uses the fact that the remainder of the tensor product is the Boolean sum of the errors of the parametric extension. The approach behind Theorem 2.1 above invokes the decomposition

$$I - A \circ B = I - A + A \circ (I - B)$$

and therefore leads to the two-term bound.

4. The Bernstein-type cubature formula revisited

In this section we give a new upper bound for the approximation error of the cubature formula associated with the bivariate Bernstein operators. The bounds are described in terms of moduli of continuity of order two. The consideration of this cubature formula is motivated by Bărbosu and Pop's result in [5]. It is also necessary to correct some of the incorrect statements made there, in particular those with respect to Boolean sums.

Integrating the bivariate Bernstein polynomials for $F \in C([0,1] \times [0,1])$, one arrives at the following cubature formula,

$$\int_0^1 \int_0^1 F(x,y)\,dx\,dy = \frac{1}{(n_1+1)(n_2+1)} \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} F\Big(\frac{i_1}{n_1}, \frac{i_2}{n_2}\Big) + R_{n_1,n_2}[F], \quad (4.1)$$

where the remainder is bounded as follows:

$$\big|R_{n_1,n_2}[F]\big| \leq \frac{1}{12n_1}\|F^{(2,0)}\| + \frac{1}{12n_2}\|F^{(0,2)}\| + \frac{1}{144n_1n_2}\|F^{2,2}\|_\infty,$$

if $F \in C^{2,2}([0,1]\times[0,1])$.

This follows from the three-term upper bound of Remark 3.1. See [5], where the same integration error bound can be found.

The two-term bound from Example 3.1 leads to the following.

Theorem 4.1. *For the remainder term of the cubature formula (4.1), $n_1, n_2 \in \mathbb{N}$ and $F \in C([0,1]\times[0,1])$, it holds that*

$$\big|R_{n_1,n_2}[F]\big| \leq \frac{3}{2}\Big[\int_0^1 \omega_2\Big(F;\sqrt{\frac{x(1-x)}{n_1}},0\Big)\,dx + \int_0^1 \omega_2\Big(F;0,\sqrt{\frac{y(1-y)}{n_2}}\Big)\,dy\Big].$$

Moreover, if $F \in C^{2,2}([0,1]\times[0,1])$, then the above implies

$$\big|R_{n_1,n_2}[F]\big| \leq \frac{1}{4}\Big(\frac{1}{n_1}\|F^{(2,0)}\|_\infty + \frac{1}{n_2}\|F^{(0,2)}\|_\infty\Big).$$

Proof. All that needs to be observed is that a function of type $[0,1/2] \ni z \to \omega_2(F;z,0)$ (with F fixed and continuous) is continuous, and thus integrable. The mixed moduli of smoothness of order (k,l), with $k,l \in \mathbb{N}_0$, given for $\delta_1,\delta_2 \geq 0$ by

$$\omega_{k,l}(F;\delta_1,\delta_2) := \sup\Big\{\Big|\sum_{\nu=0}^{k}\sum_{r=0}^{l}(-1)^{\nu+\mu}\binom{k}{\nu}\binom{l}{\mu}F(x+\nu\cdot h_1, y+\mu\cdot h_2)\Big| : (x,y),(x+kh_1,y+lh_2) \in [0,1]^2, |h_i| \leq \delta_i, i=1,2\Big\},$$

is a positive, continuous, and nondecreasing function with respect to both variables (see [8], [14]). For continuous F these moduli are continuous in δ_1 and δ_2 and satisfy

$$\omega_k(F;\delta_1,0) = \omega_{k,0}(F;\delta_1,\delta_2) \qquad \text{and} \qquad \omega_k(F;0,\delta_2) = \omega_{0,k}(F;\delta_1,\delta_2).$$

The latter is only relevant to us for $k = 2$. □

5. The composite bivariate Bernstein operators

In this section we construct the bivariate composite Bernstein operators, and the order of convergence is considered involving the second modulus of continuity. Also, some inequalities of Chebyshev–Grüss type will be proved. These results are obtained using some general inequalities published in [1] and [12]. In order to give the main results of this section, we recall the following facts:

1. For $a,b \in \mathbb{R}$, $a < b$, and $f \in \mathbb{R}^{[a,b]}$ the Bernstein polynomial of degree $n \in \mathbb{N}$ associated to f is given, for $x \in [a,b]$, by

$$B_n^{[a,b]}(f;x) = \frac{1}{(b-a)^n}\sum_{i=0}^{n}\binom{n}{i}(x-a)^i(b-x)^{n-i}f\Big(a+i\frac{b-a}{n}\Big).$$

2. For $g \in C^2[a,b]$ one has

$$g(x) - B_n^{[a,b]}(g;x) = -\frac{(x-a)(b-x)}{2n} g''(\xi_x), \quad \xi_x \in (a,b).$$

If we divide $[0,1]$ into subintervals $[\frac{k-1}{m}, \frac{k}{m}]$, $k = 1, \dots, m \in \mathbb{N}$, then on $[\frac{k-1}{m}, \frac{k}{m}]$ we consider

$$\begin{aligned} B_{n,k}(f;x) &= B_n^{[\frac{k-1}{m}, \frac{k}{m}]}(f;x) \\ &= m^n \sum_{i=0}^{n} \binom{n}{i} \Big(x - \frac{k-1}{m}\Big)^i \Big(\frac{k}{m} - x\Big)^{n-i} f\Big(\frac{kn-n+i}{nm}\Big). \end{aligned}$$

Now we combine the $B_{n,k}$ to obtain the positive linear operator $\overline{B}_{n,m} : \mathbb{R}^{[0,1]} \to C[0,1]$,

$$\overline{B}_{n,m}(f;x) = B_{n,k}(f;x), \quad \text{if } x \in \Big[\frac{k-1}{m}, \frac{k}{m}\Big], 1 \le k \le m.$$

From now on (subscripted) symbols $n \dots$ will refer to a polynomial degree. (Subscripted) numbers $m \dots$ will be related to grids. Each function $\overline{B}_{n,m}(f)$ is a Schoenberg spline of degree n with respect to the knot sequence given as follows:

$$\begin{array}{ll} 0 = \dfrac{0}{m} & (n+1)\text{-fold} \\ \dfrac{1}{m} & n\text{-fold} \\ \vdots & \vdots \\ \dfrac{m-1}{m} & n\text{-fold} \\ 1 = \dfrac{m}{m} & (n+1)\text{-fold.} \end{array}$$

We renounce giving a precise numbering of the knots since this will not be needed below. Thus $\overline{B}_{n,m}$ reproduces linear functions, interpolates at $\frac{k}{m}$, $0 \le k \le m$, and has operator norm $\|\overline{B}_{n,m}\| = 1$.

For $n_1, n_2, m_1, m_2 \in \mathbb{N}$ we now consider the parametric extension ${}_x\overline{B}_{n_1,m_1}$ and ${}_y\overline{B}_{n_2,m_2}$ and their product ${}_x\overline{B}_{n_1,m_1} \circ {}_y\overline{B}_{n_2,m_2}$. For brevity the latter will be denoted by $\overline{\mathcal{B}}$.

For $(x,y) \in [\frac{k-1}{m_1}, \frac{k}{m_1}] \times [\frac{l-1}{m_2}, \frac{l}{m_2}]$, it follows that

$$\begin{aligned} \overline{\mathcal{B}}(f;x,y) = m_1^{n_1} \cdot m_2^{n_2} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i} \binom{n_2}{j} \Big(x - \frac{k-1}{m_1}\Big)^i \Big(\frac{k}{m_1} - x\Big)^{n_1-i} \\ \cdot \Big(y - \frac{l-1}{m_2}\Big)^j \Big(\frac{l}{m_2} - y\Big)^{n_2-j} f\Big(\frac{k-1}{m_1} + \frac{i}{m_1 n_1}, \frac{l-1}{m_2} + \frac{j}{n_2 m_2}\Big) \end{aligned}$$

and

$$\begin{aligned}&\left|f(x,y)-\overline{\mathcal{B}}(f;x,y)\right|\\&=\frac{(x-\frac{k-1}{m})(\frac{k}{m_1}-x)}{2n_1}\|f^{(2,0)}\|_\infty+\frac{(y-\frac{l-1}{m_2})(\frac{l}{m_2}-y)}{2n_2}\|f^{(0,2)}\|_\infty\\&\quad+\frac{(x-\frac{k-1}{m})(\frac{k}{m_1}-x)(y-\frac{l-1}{m_2})(\frac{l}{m_2}-y)}{4n_1n_2}\|f^{(2,2)}\|_\infty,\end{aligned}$$

where $f\in C^{2,2}([0,1]\times[0,1])$.

Using Theorem 2.1 again we get the following.

Theorem 5.1. *For $f\in C([0,1]\times[0,1])$, $n_1,n_2,m_1,m_2\in\mathbb{N}$, and $(x,y)\in[0,1]\times[0,1]$, it holds that*

$$\begin{aligned}\left|f(x,y)-\overline{\mathcal{B}}(f;x,y)\right|\le\frac{3}{2}\Big\{&\omega_2\Big(f;\sqrt{\frac{(x-\frac{k-1}{m_1})(\frac{k}{m_1}-x)}{n_1}},0\Big)\\&+\omega_2\Big(f;0,\sqrt{\frac{(y-\frac{l-1}{m_2})(\frac{l}{m_2}-y)}{n_2}}\Big)\Big\},\end{aligned}$$

if $(x,y)\in[\frac{k-1}{m_1},\frac{k}{m_1}]\times[\frac{l-1}{m_2},\frac{l}{m_2}]$, $1\le k\le m_1$, $1\le l\le m_2$.

Proof. For the univariate case we have

$$\left|\overline{B}_{n_1,m_1}(f;x)-f(x)\right|\le\frac{3}{2}\omega_2\Big(f;\sqrt{\frac{(x-\frac{k-1}{m_1})(\frac{k}{m_1}-x)}{n_1}}\Big),$$

for $x\in[\frac{k-1}{m_1},\frac{k}{m_1}]$, $1\le k\le m_1$. Here ω_2 is the second-order modulus over $[0,1]$. An analogous inequality holds for $\overline{B}_{n_2,m_2}$.

The theorem mentioned implies, with $r=s=2$, the inequality claimed. □

Remark 5.1. As mentioned earlier, for $g\in C^2[a,b]$ one has

$$\left|g(x)-B_n^{[a,b]}(g;x)\right|=\left|-\frac{(x-a)(b-x)}{2n}g''(\xi_x)\right|\le\frac{(b-a)^2}{8n}\|g''\|_{[a,b],\infty}.$$

For $[a,b]=[\frac{k-1}{m},\frac{k}{m}]$, the last expression equals $\frac{1}{8m^2n}\|g''\|_{[\frac{k-1}{m},\frac{k}{m}],\infty}$.

If $f\in C^{2,2}([0,1]\times[0,1])$ and $(x,y)\in[0,1]\times[0,1]$, using [7, Theorem 1], this leads to

$$\begin{aligned}&\left|f(x,y)-\overline{\mathcal{B}}(f;x,y)\right|\\&\le\frac{1}{8m_1^2n_1}\|f^{(2,0)}\|_\infty+\frac{1}{8m_2^2n_2}\|f^{(0,2)}\|_\infty+\frac{1}{64m_1^2n_1m_2^2n_2}\|f^{(2,2)}\|_\infty.\end{aligned}$$

For $m_1=m_2=1$ this is exactly the inequality in Remark 3.1.

6. A Chebyshev–Grüss-type inequality

In what follows we present an inequality for the bivariate composite Bernstein operators, expressed in terms of the least concave majorant of a modulus of continuity. Let $C(X)$ be the Banach lattice of real-valued continuous functions defined on the compact metric space (X, d).

Definition 6.1. Let $f \in C(X)$. If, for $t \in [0, \infty)$, the quantity

$$\omega_d(f;t) := \sup\bigl\{\bigl|f(x) - f(y)\bigr|, d(x,y) \le t\bigr\}$$

is the usual modulus of continuity, then its least concave majorant is given by

$$\tilde{\omega}_d(f,t) = \begin{cases} \sup_{0 \le x < t \le y \le d(X)} \frac{(t-x)\omega_d(f,y)+(y-t)\omega_d(f,x)}{y-x}, & 0 \le t \le d(X), \\ \omega_d(f, d(X)), & t > d(X), \end{cases}$$

and $d(X) < \infty$ is the diameter of the compact space X.

Denote

$$\mathrm{Lip}_r = \left\{ g \in C(X) \;\middle|\; |g|_{\mathrm{Lip}_r} := \sup_{d(x,y)>0} \frac{|g(x) - g(y)|}{d^r(x,y)} < \infty \right\}, \quad 0 < r \le 1.$$

Lip_r is a dense subspace of $C(X)$ equipped with the supremum norm $\|\cdot\|_\infty$, and $|\cdot|_{\mathrm{Lip}_r}$ is a seminorm on Lip_r.

The K-functional with respect to $(\mathrm{Lip}_r, |\cdot|_{\mathrm{Lip}_r})$ is given by

$$K\bigl(t, f; C(X), \mathrm{Lip}_r\bigr) := \inf_{g \in \mathrm{Lip}_r} \bigl\{\|f - g\|_\infty + t|g|_{\mathrm{Lip}_r}\bigr\}, \quad \text{for } f \in C(X) \text{ and } t \ge 0.$$

Lemma 6.1 (see [10]). *Every continuous function f on X satisfies*

$$K\Bigl(\frac{t}{2}, f; C(X), \mathrm{Lip}_1\Bigr) = \frac{1}{2}\tilde{\omega}_d(f,t), \quad 0 \le t \le d(X).$$

Let $H : C(X^2) \to C(X^2)$ be a positive linear operator reproducing a constant function, and define

$$T(f, g; x, y) = H(fg; x, y) - H(f; x, y) \cdot H(g; x, y).$$

In order to give an inequality of Chebyshev–Grüss type we recall a general result given by M. Rusu in [12].

From now on we consider the Euclidean metric d_2 derived from d.

Theorem 6.1 ([12, Theorem 3.3.1]). *If $f, g \in C(X^2)$ and $x, y \in X$ are fixed, then the inequality*

$$\begin{aligned}\bigl|T(f, g; x, y)\bigr| \le \frac{1}{4}&\tilde{\omega}_{d_2}\Bigl(f; 4\sqrt{H\bigl(d_2^2(\cdot,(x,y)); x, y\bigr)}\Bigr) \\ &\cdot \tilde{\omega}_{d_2}\Bigl(g; 4\sqrt{H\bigl(d_2^2(\cdot,(x,y)); x, y\bigr)}\Bigr)\end{aligned}$$

holds, where $H(d_2^2(\cdot,(x,y)); x, y)$ is the second moment of the bivariate operator H.

Proposition 6.1. *For $f,g\in C(X^2)$ and $x,y\in X$ fixed, the following Grüss-type inequality holds:*

$$\begin{aligned}&\left|\overline{\mathcal{B}}(fg;x,y)-\overline{\mathcal{B}}(f;x,y)\cdot\overline{\mathcal{B}}(g;x,y)\right|\\&\leq\frac{1}{4}\tilde{\omega}_{d_2}\big(f;4\sqrt{\Psi(x,y)}\big)\cdot\tilde{\omega}_{d_2}\big(g;4\sqrt{\Psi(x,y)}\big)\\&\leq\frac{1}{4}\tilde{\omega}_{d_2}\Big(f;2\sqrt{\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}}\Big)\cdot\tilde{\omega}_{d_2}\Big(g;2\sqrt{\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}}\Big),\end{aligned}$$

where $\Psi(x,y)=\frac{(x-\frac{k-1}{m_1})(\frac{k}{m_1}-x)}{n_1}+\frac{(y-\frac{l-1}{m_2})(\frac{l}{m_2}-y)}{n_2}$ *and* $(x,y)\in[\frac{k-1}{m_1},\frac{k}{m_1}]\times[\frac{l-1}{m_2},\frac{l}{m_2}]$.

7. A cubature formula based on $\overline{\mathcal{B}}$

In this section some upper bounds of the error of the cubature formula associated with the bivariate Bernstein operators are given. In [2], D. Bărbosu and D. Miclăuş introduced the following cubature formula:

$$\begin{aligned}\int_0^1\int_0^1 f(x,y)\,dx\,dy&=\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}f(x,y)\,dx\,dy\\&\approx\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\overline{\mathcal{B}}(f;x,y)\,dx\,dy\\&=\int_0^1\int_0^1\overline{\mathcal{B}}(f;x,y)\,dx\,dy:=\overline{\mathcal{I}}(f).\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\frac{k-1}{m_1}}^{\frac{k}{m}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\overline{\mathcal{B}}(f;x,y)\,dx\,dy\\&=m_1^{n_1}m_2^{n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{i}\binom{n_2}{j}\int_{\frac{k-1}{m_1}}^{\frac{k}{n_1}}\Big(x-\frac{k-1}{m_1}\Big)^i\Big(\frac{k}{m_1}-x\Big)^{n_1-i}dx\\&\quad\cdot\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\Big(y-\frac{l-1}{m_2}\Big)^j\Big(\frac{l}{m_2}-y\Big)^{n_2-j}dyf\Big(\frac{k-1}{m_1}+\frac{i}{m_1n_1},\frac{l-1}{m_2}+\frac{j}{n_2m_2}\Big)\\&=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}A_{n_1,n_2,m_1,m_2}f\Big(\frac{k-1}{m_1}+\frac{i}{m_1n_1},\frac{l-1}{m_2}+\frac{j}{n_2m_2}\Big),\end{aligned}$$

where $A_{n_1,n_2,m_1,m_2}=\frac{1}{m_1m_2(n_1+1)(n_2+1)}$.

Theorem 7.1. *For $f\in C^{2,2}([0,1]\times[0,1])$, it follows that*

$$\begin{aligned}&\Big|\int_0^1\int_0^1 f(x,y)\,dx\,dy-\overline{\mathcal{I}}(f)\Big|\\&\leq\frac{1}{12n_1m_1^2}\|f^{(2,0)}\|_\infty+\frac{1}{12n_2m_2^2}\|f^{(0,2)}\|+\frac{1}{144n_1n_2m_1^2m_2^2}\|f^{(2,2)}\|_\infty.\end{aligned}$$

Proof. We have

$$
\begin{aligned}
&\Big|\int_0^1\int_0^1 f(x,y)\,dx\,dy-\overline{\mathcal{I}}(f)\Big|\\
&=\Big|\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}} f(x,y)\,dx\,dy\\
&\quad-\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}} \overline{\mathcal{B}}(f;x,y)\,dx\,dy\Big|\\
&\le\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\big|f(x,y)-\overline{\mathcal{B}}(f;x,y)\big|\,dx\,dy\\
&=\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\Big[\frac{(x-\frac{k-1}{m_1})(\frac{k}{m_1}-x)}{2n_1}\|f^{(2,0)}\|_\infty\\
&\quad+\frac{(y-\frac{l-1}{m_2})(\frac{l}{m_2}-y)}{2n_2}\|f^{(0,2)}\|_\infty\\
&\quad+\frac{(x-\frac{k-1}{m_1})(\frac{k}{m_1}-x)(y-\frac{l-1}{m_2})(\frac{l}{m_2}-y)}{4n_1n_2}\|f^{(2,2)}\|_\infty\Big]\,dx\,dy\\
&\le\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\Big[\frac{1}{12n_1m_1^3m_2}\|f^{(2,0)}\|_\infty+\frac{1}{12n_2m_2^3m_1}\|f^{(0,2)}\|_\infty\\
&\quad+\frac{1}{144n_1n_2m_1^3m_2^3}\|f^{(2,2)}\|_\infty\Big]\\
&=\frac{1}{12n_1m_1^2}\|f^{(2,0)}\|_\infty+\frac{1}{12n_2m_2^2}\|f^{(0,2)}\|+\frac{1}{144n_1n_2m_1^2m_2^2}\|f^{(2,2)}\|_\infty. \qquad\square
\end{aligned}
$$

One further estimate is given in the following.

Theorem 7.2. *For* $f\in C^{2,2}([0,1]\times[0,1])$*, it follows that*

$$\Big|\int_0^1\int_0^1 f(x,y)\,dx\,dy-\overline{\mathcal{I}}(f)\Big|\le\frac{1}{4}\Big\{\frac{1}{m_1^2n_1}\|f^{(2,0)}\|_\infty+\frac{1}{m_2^2n_2}\|f^{(0,2)}\|_\infty\Big\}.$$

Proof. Integrating the error given in Theorem 5.1 leads to

$$
\begin{aligned}
&\Big|\int_0^1\int_0^1 f(x,y)\,dx\,dy-\overline{\mathcal{I}}(f)\Big|\\
&\le\frac{3}{2}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\Big\{\frac{1}{m_2}\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\omega_2\Big(f;\sqrt{\frac{(x-\frac{k-1}{m_1})(\frac{k}{m_1}-x)}{n_1}},0\Big)\,dx\\
&\quad+\frac{1}{m_1}\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\omega_2\Big(f;0,\sqrt{\frac{(y-\frac{l-1}{m_2})(\frac{l}{m_2}-y)}{n_2}}\Big)\,dy\Big\}.
\end{aligned}
$$

And $f\in C^{2,2}([0,1]\times[0,1])$ leads to

$$\begin{aligned}
&\Big|\int_0^1\int_0^1 f(x,y)\,dx\,dy-\overline{\mathcal{I}}(f)\Big|\\
&\le \frac{3}{2}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\Big\{\frac{1}{m_2}\|f^{(2,0)}\|_\infty\int_{\frac{k-1}{m_1}}^{\frac{k}{m_1}}\frac{(x-\frac{k-1}{m_1})(\frac{k}{m_1}-x)}{n_1}\,dx\\
&\quad+\frac{1}{m_1}\|f^{(0,2)}\|_\infty\int_{\frac{l-1}{m_2}}^{\frac{l}{m_2}}\frac{(y-\frac{l-1}{m_2})(\frac{l}{m_2}-y)}{n_1}\,dy\Big\}\\
&=\frac{3}{2}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\Big\{\frac{1}{6m_1^3m_2n_1}\|f^{(2,0)}\|_\infty+\frac{1}{6m_1m_2^3n_2}\|f^{(0,2)}\|_\infty\Big\}\\
&=\frac{1}{4}\Big\{\frac{1}{m_1^2n_1}\|f^{(2,0)}\|_\infty+\frac{1}{m_2^2n_2}\|f^{(0,2)}\|_\infty\Big\}.
\end{aligned}$$

□

8. Nonmultiplicativity of the cubature formula

In this section we will give some results that suggest how nonmultiplicative the functional $\overline{\mathcal{I}}(f)=\int_0^1\int_0^1\overline{\mathcal{B}}(f;(x,y))\,dx\,dy$ is.

Let (X,d) be a compact metric space, and let $L: C(X)\to\mathbb{R}$ be a positive linear functional reproducing constants. We consider the positive bilinear functional

$$D(f,g):=L(fg)-L(f)L(g).$$

Theorem 8.1. *If $f,g\in C(X)$, (X,d) is a compact metric space, then the inequality*

$$\big|D(f,g)\big|\le\frac{1}{4}\tilde{\omega}_d\Big(f;2\sqrt{L^2\big(d^2(\cdot,\cdot)\big)}\Big)\tilde{\omega}_d\Big(g;2\sqrt{L^2\big(d^2(\cdot,\cdot)\big)}\Big)$$

holds.

Proof. Let $f,g\in C[a,b]$, and let $r,s\in \mathrm{Lip}_1$. Using the Cauchy–Schwarz inequality for a positive linear functional gives

$$\big|L(f)\big|\le L\big(|f|\big)\le\sqrt{L(f^2)\cdot L(1)}=\sqrt{L(f^2)},$$

and so we have

$$D(f,f)=L(f^2)-L(f)^2\ge 0.$$

Therefore, D is a positive bilinear form on $C(X)$. Using the Cauchy–Schwarz inequality for D, it follows that

$$\big|D(f,g)\big|\le\sqrt{D(f,f)D(g,g)}\le\|f\|_\infty\|g\|_\infty.$$

Since L is a positive linear functional, we can write

$$L(f):=\int_X f(t)\,d\mu(t),$$

where μ is a Borel probability measure on X; that is, $\int_X d\mu(t) = 1$. For $r \in \mathrm{Lip}_1$, it follows that

$$\begin{aligned}
D(r,r) = L(r^2) - L(r)^2 &= \int_X r^2(t)\,d\mu(t) - \Big(\int_X r(u)\,d\mu(u)\Big)^2 \\
&= \int_X \Big(r(t) - \int_X r(u)\,d\mu(u)\Big)^2\,d\mu(t) \\
&= \int_X \Big(\int_X \big(r(t) - r(u)\big)\,d\mu(u)\Big)^2\,d\mu(t) \\
&\le \int_X \Big(\int_X \big(r(t) - r(u)\big)^2\,d\mu(u)\Big)\,d\mu(t) \\
&\le |r|^2_{\mathrm{Lip}_1} \int_X \Big(\int_X d^2(t,u)\,d\mu(u)\Big)\,d\mu(t) \\
&= |r|^2_{\mathrm{Lip}_1} L^t\big[L\big(d^2(t,\cdot)\big)\big] = |r|^2_{\mathrm{Lip}_1} L^2\big(d^2(\cdot,\cdot)\big).
\end{aligned}$$

For $r, s \in \mathrm{Lip}_1$, we have

$$\big|D(r,s)\big| \le \sqrt{D(r,r)D(s,s)} \le |r|_{\mathrm{Lip}_1}|s|_{\mathrm{Lip}_1} L^2\big(d(\cdot,\cdot)\big).$$

Moreover, for $f \in C(X)$ and $s \in \mathrm{Lip}_1$, we have the estimate

$$\big|D(f,s)\big| \le \sqrt{D(f,f)D(s,s)} \le \|f\|_\infty |s|_{\mathrm{Lip}_1} \sqrt{L^2\big(d(\cdot,\cdot)\big)}.$$

In a similar way, if $r \in \mathrm{Lip}_1$ and $g \in C(X)$, we have

$$\big|D(r,g)\big| \le \sqrt{D(r,r)D(g,g)} \le \|g\|_\infty |r|_{\mathrm{Lip}_1} \sqrt{L^2\big(d(\cdot,\cdot)\big)}.$$

Let $f, g \in C(X)$ be fixed, and let $r, s \in \mathrm{Lip}_1$ be arbitrary; then

$$\begin{aligned}
\big|D(f,g)\big| &= \big|D(f-r+r, g-s+s)\big| \\
&\le \big|D(f-r, g-s)\big| + \big|D(f-r,s)\big| + \big|D(r,g-s)\big| + \big|D(r,s)\big| \\
&\le \|f-r\|_\infty \cdot \|g-s\|_\infty + \|f-r\|_\infty \cdot |s|_{\mathrm{Lip}_1}\sqrt{L^2\big(d^2(\cdot,\cdot)\big)} \\
&\quad + \|g-s\|_\infty \cdot |r|_{\mathrm{Lip}_1}\sqrt{L^2\big(d^2(\cdot,\cdot)\big)} + |r|_{\mathrm{Lip}_1}|s|_{\mathrm{Lip}_1} L^2\big(d^2(\cdot,\cdot)\big) \\
&= \Big\{\|f-r\|_\infty + |r|_{\mathrm{Lip}_1}\sqrt{L^2\big(d^2(\cdot,\cdot)\big)}\Big\}\Big\{\|g-s\|_\infty + |s|_{\mathrm{Lip}_1}\sqrt{L^2\big(d^2(\cdot,\cdot)\big)}\Big\}.
\end{aligned}$$

Passing to the infimum over r and s, respectively, leads to

$$\begin{aligned}
\big|D(f,g)\big| &\le K\Big(\sqrt{L^2\big(d^2(\cdot,\cdot)\big)}, f; C(X), \mathrm{Lip}_1\Big) \cdot K\Big(\sqrt{L^2\big(d^2(\cdot,\cdot)\big)}, g; C(X), \mathrm{Lip}_1\Big) \\
&\le \frac{1}{4}\tilde{\omega}\Big(f; 2\sqrt{L^2\big(d^2(\cdot,\cdot)\big)}\Big)\tilde{\omega}\Big(g; 2\sqrt{L^2\big(d^2(\cdot,\cdot)\big)}\Big). \qquad \square
\end{aligned}$$

Applying Theorem 8.1 for $L(f) = \overline{\mathcal{I}}(f)$ we obtain the following result.

Corollary 8.1. *If $f,g\in C([0,1]\times[0,1])$, then*

$$\begin{aligned}\big|\overline{\mathcal{I}}(fg)-\overline{\mathcal{I}}(f)\overline{\mathcal{I}}(g)\big| &\le \frac{1}{4}\tilde{\omega}_{d_2}\Big(f;2\sqrt{\frac{1}{3}\Big(1+\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}\Big)}\Big)\\ &\quad\cdot\tilde{\omega}_{d_2}\Big(g;2\sqrt{\frac{1}{3}\Big(1+\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}\Big)}\Big).\end{aligned}\tag{8.1}$$

Proof. We have

$$\begin{aligned}\overline{\mathcal{I}}\big(d_2^2(\cdot,\cdot)\big) &= \sum_{k,k_1=1}^{m_1}\sum_{l,l_1=1}^{m_2}\sum_{i,i_1=0}^{n_1}\sum_{j,j_1=0}^{n_2}\frac{1}{m_1^2m_2^2(n_1+1)^2(n_2+1)^2}\\ &\quad\cdot\Big[\Big(\frac{k_1-1}{m_1}+\frac{i_1}{m_1n_1}-\frac{k-1}{m_1}-\frac{i}{m_1n_1}\Big)^2\\ &\quad+\Big(\frac{l_1-1}{m_2}+\frac{j_1}{n_2m_2}-\frac{l-1}{m_2}-\frac{j}{n_2m_2}\Big)^2\Big]\\ &=\frac{1}{m_1^2(n_1+1)^2}\sum_{k,k_1=1}^{m_1}\sum_{i,i_1=0}^{n_1}\Big(\frac{k_1-k}{m_1}+\frac{i_1-i}{m_1n_1}\Big)^2\\ &\quad+\frac{1}{m_2^2(n_2+1)^2}\sum_{l,l_1=1}^{m_2}\sum_{j,j_1=0}^{n_2}\Big(\frac{l_1-l}{m_2}+\frac{j_1-j}{m_2n_2}\Big)^2\\ &=\frac{1}{3}\Big(1+\frac{1}{m_1^2n_1}+\frac{1}{m_2^2n_2}\Big).\end{aligned}$$

Therefore, using Theorem 8.1 it follows that

$$\begin{aligned}\big|\overline{\mathcal{I}}(fg)-\overline{\mathcal{I}}(f)\overline{\mathcal{I}}(g)\big| &\le \frac{1}{4}\tilde{\omega}_{d_2}\Big(f;2\sqrt{\frac{1}{3}\Big(1+\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}\Big)}\Big)\\ &\quad\cdot\tilde{\omega}_{d_2}\Big(g;2\sqrt{\frac{1}{3}\Big(1+\frac{1}{n_1m_1^2}+\frac{1}{n_2m_2^2}\Big)}\Big).\end{aligned}$$

□

Finally, we will give a Chebyshev–Grüss-type inequality that involves oscillations of functions. This result is obtained using a general inequality published in [1]. Let Y be an arbitrary set, and let $B(Y^2)$ be the set of all real-valued, bounded functions on Y^2. Take $a_n,b_n\in\mathbb{R}$, $n\ge 0$, such that $\sum_{n=0}^{\infty}|a_n|<\infty$, $\sum_{n=0}^{\infty}a_n=1$ and $\sum_{n=0}^{\infty}|b_n|<\infty$, $\sum_{n=0}^{\infty}b_n=1$. Furthermore, let $x_n\in Y$, $n\ge 0$, and let $y_m\in Y$, $m\ge 0$, be arbitrary mutually distinct points. For $f\in B(Y^2)$ set $f_{n,m}:=f(x_n,y_m)$. Now consider the functional $L:B(Y^2)\to\mathbb{R}$, $Lf=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_nb_mf_{n,m}$. The functional L is linear and reproduces constant functions.

Theorem 8.2 (see [1])**.** *The Chebyshev–Grüss-type inequality for the above linear functional L is given by*

$$\big|L(fg)-L(f)\cdot L(g)\big|\le\frac{1}{2}\cdot\operatorname{osc}_L(f)\cdot\operatorname{osc}_L(g)\cdot\sum_{n,m,i,j=0,(n,m)\neq(i,j)}^{\infty}|a_nb_ma_ib_j|,$$

where $f, g \in B(Y^2)$ and we define the oscillations to be

$$\operatorname{osc}_L(f) := \sup\{|f_{n,m} - f_{i,j}| : n, m, i, j \geq 0\}.$$

Theorem 8.3 (see [1]). *In particular, if $a_n \geq 0$, $b_m \geq 0$, $n, m \geq 0$, then L is a positive linear functional and we have*

$$\left|L(fg) - L(f) \cdot L(g)\right| \leq \frac{1}{2} \cdot \Big(1 - \sum_{n=0}^{\infty} a_n^2 \cdot \sum_{m=0}^{\infty} b_m^2\Big) \cdot \operatorname{osc}_L(f) \cdot \operatorname{osc}_L(g),$$

for $f, g \in B(Y^2)$ and the oscillations given as above.

The following result gives us the nonmultiplicativity of the functional $\mathcal{I}$ using discrete oscillations. This result is better than (8.1) in the sense that the oscillations of functions are relative only to certain points, while in (8.1) the oscillations, expressed in terms of $\tilde{\omega}$, are relative to the whole interval $[0, 1]$.

Corollary 8.2. *If $f, g \in B([0, 1]^2)$, then*

$$\left|\overline{\mathcal{I}}(fg) - \overline{\mathcal{I}}(f)\overline{\mathcal{I}}(g)\right| \leq \frac{1}{2}\Big(1 - \frac{1}{m_1 m_2 (n_1 + 1)(n_2 + 1)}\Big) \operatorname{osc}(f) \operatorname{osc}(g).$$

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