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ON JORDAN CENTRALIZERS OF TRIANGULAR ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a unital algebra over a number field $\mathbb{F}$. A linear mapping ϕ from $\mathcal{A}$ into itself is called a *Jordan-centralized mapping* at a given point $G \in \mathcal{A}$ if $\phi(AB+BA) = \phi(A)B + \phi(B)A = A\phi(B) + B\phi(A)$ for all A, $B \in \mathcal{A}$ with $AB = G$. In this paper, it is proved that each Jordan-centralized mapping at a given point of triangular algebras is a centralizer. These results are then applied to some non-self-adjoint operator algebras.

1. Introduction and preliminaries

Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{F}$, and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a linear mapping. Recall that Φ is a left (right) centralizer or multiplier if $\Phi(AB) = \Phi(A)B$ ($\Phi(AB) = A\Phi(B)$) holds for all A and B in $\mathcal{A}$. Φ is called a *centralizer* if it is both a left and right centralizer. More generally, Φ is a left (right) Jordan centralizer if $\Phi(A^2) = \Phi(A)A$ ($\Phi(A^2) = A\Phi(A)$) is fulfilled for all A in $\mathcal{A}$; equivalently, $\Phi(AB+BA) = \Phi(A)B+\Phi(B)A$ ($\Phi(AB+BA) = A\Phi(B)+B\Phi(A)$) is fulfilled for all A and B in $\mathcal{A}$. Φ is called a *Jordan centralizer* if it is both a left and right Jordan centralizer. It is well known that centralizers and Jordan centralizers are very important both in theory and applications, and so they have been studied intensively. For example, the theory of centralizers for C^*-algebras and some non-self-adjoint operator algebras have been relatively well studied in the literature (see [1], [2], [10], [11] and references therein). Centralizers and Jordan centralizers have also been studied in the general framework of prime rings

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and semiprime rings by Zalar [18] and more recently by Vukman and Kosi-Ulbl [8], [14], [16], [15].

In this paper, we mainly study Jordan centralizers by their local actions. In general there are two directions in the study of the local actions of mappings of operator algebras. One is the well-known local mappings problem, such as local derivations and local automorphisms (see, e.g., [6], [7], [9] and references therein). The other direction is to study conditions under which mappings of operator algebras can be completely determined by the action on some sets of operators. A linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be *left (right) centralized* at a given point $G \in \mathcal{A}$ if $\Phi(AB) = \Phi(A)B$ ($\Phi(AB) = A\Phi(B)$) for all A, $B \in \mathcal{A}$ with $AB = G$. Φ is called *centralized* at G if it is both left centralized and right centralized at G. Recently, Brešer [3] studied the centralizers of prime rings in this direction. It was proved in [3] that if $\mathcal{A}$ is a prime ring containing a nontrivial idempotent and Φ is right centralized at 0, then Φ is a right centralizer on $\mathcal{A}$. Later, Qi and Hou [13] discussed the same question on triangular rings.

Motivated by centralizers, we say a linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ is *left (right) Jordan-centralized* at a given point $G \in \mathcal{A}$ if $\Phi(AB + BA) = \Phi(A)B + \Phi(B)A$ ($\Phi(AB + BA) = A\Phi(B) + B\Phi(A)$) holds for all A, $B \in \mathcal{A}$ with $AB = G$. Φ is called *Jordan-centralized* at G if it is both left Jordan-centralized and right Jordan-centralized at G. It is natural to ask how to characterize the mappings which are Jordan-centralized at a general given point. The purpose of the present paper is to answer the question for triangular algebras. These results are then applied to some non-self-adjoint operator algebras.

The triangular algebras were first introduced in [4] and then studied by many authors (see, e.g., [5], [17]). Let $\mathcal{A}$ and $\mathcal{C}$ be two algebras over a field $\mathbb{F}$ with unit I_1 and I_2, respectively, and let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{C})$-bimodule; that is, $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{C})$-bimodule satisfying, for $X \in \mathcal{A}$, that if $X\mathcal{M} = \{0\}$, then $X = 0$ (i.e., $\mathcal{M}$ is a faithful left $\mathcal{A}$-module), and, for $Z \in \mathcal{C}$, that if $\mathcal{M}Z = \{0\}$, then $Z = 0$ (i.e., $\mathcal{M}$ is a faithful right $\mathcal{C}$-module). Recall that the algebra

$$\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{C}) = \left\{ \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} : X \in \mathcal{A}, Y \in \mathcal{M}, Z \in \mathcal{C} \right\},$$

under the usual matrix addition and formal matrix multiplication will be called a *triangular algebra*. Let $\mathcal{Z}(\mathcal{T})$ be the center of $\mathcal{T}$. It follows from [5, Proposition 3] that

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} : X \in \mathcal{Z}(\mathcal{A}), Z \in \mathcal{Z}(\mathcal{C}), XY = YZ \text{ for all } Y \in \mathcal{M} \right\}.$$

In Section 2, we characterize the structure of the left (right) Jordan centralizers by acting on a general point of triangular algebras (see Theorems 2.1, 2.3). We also prove that every Jordan-centralized mapping at a given point is a centralizer on triangular algebras (see Theorem 2.4). We ought perhaps to mention that our approach is simple but efficient.

Let H be a complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on H. A subspace lattice $\mathcal{L}$ on H is a collection of projections in $B(H)$ that is closed under the usual lattice operations $\vee$ (the closed linear span) and $\wedge$ (the set-theoretic intersection), and contains the zero operator 0 and the identity operator I. A totally ordered subspace lattice is called a *nest*. A subspace lattice $\mathcal{L}$ is called a *commutative subspace lattice*, or a CSL, if each pair of projections in $\mathcal{L}$ commute. For $E \in \mathcal{L}$, we define

$$E_- = \vee\{F \in \mathcal{L} : F \not\geq E\}, \quad E \neq 0$$

and

$$E_+ = \wedge\{F \in \mathcal{L} : F \not\leq E\}, \quad E \neq I.$$

A subspace lattice $\mathcal{L}$ is said to be *completely distributive* if $E = \vee\{F \in \mathcal{L} : F_- \not\geq E\}$ for every $E \in \mathcal{L}$ with $E \neq 0$, which is also equivalent to $E = \wedge\{F_- : F \in \mathcal{L}, F \not\leq E\}$ for every $E \in \mathcal{L}$ with $E \neq I$.

For a subspace lattice $\mathcal{L}$ on H, the associated subspace lattice algebra $\operatorname{alg}\mathcal{L}$ is the set of operators in $B(H)$ that leave invariant every projection in $\mathcal{L}$; that is,

$$\operatorname{alg}\mathcal{L} = \big\{T \in B(H) : TE = ETE, \forall E \in \mathcal{L}\big\}.$$

Obviously, $\operatorname{alg}\mathcal{L}$ is a unital weakly closed subalgebra of $B(H)$. We call a subspace lattice algebra $\operatorname{alg}\mathcal{L}$ a *CSL algebra* if $\mathcal{L}$ is a CSL, and a *CDC algebra* if $\mathcal{L}$ is a completely distributive CSL. Recall that a CDC algebra is irreducible if and only if its commutant is $\mathbb{C}I$. In particular, nest algebras are irreducible CDC algebras. In Section 3, we show that every Jordan-centralized mapping at a given point on irreducible CDC algebras is a centralizer. We also study Jordan centralizers on Banach space nest algebras.

2. Main results

In this section, we consider the question of characterizing the linear mappings on the triangular algebras which are Jordan-centralized at a given point. The result is based on the structure of the left (right) Jordan-centralized mappings at a given point.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{C}$ be two algebras over a number field $\mathbb{F}$ with unit I_1 and I_2, respectively, and let $\mathcal{M}$ be a faithful left $\mathcal{A}$-module. The triangular algebra* $\operatorname{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{C})$ *is written for $\mathcal{T}$. Suppose that we have the following:*

(i) *For every $X \in \mathcal{A}$, there is some integer n such that $nI_1 - X$ is invertible.*
(ii) *For every $Z \in \mathcal{C}$, there is some integer n such that $nI_2 - Z$ is invertible.*

If $\phi : \mathcal{T} \to \mathcal{T}$ is a left Jordan-centralized mapping at a given point $G = \left[\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right] \in \mathcal{T}$, then there exists an element $D \in \mathcal{A}$, two linear mappings $h_{12} : \mathcal{C} \to \mathcal{M}$ satisfying $h_{12}(ZW + WZ) = h_{12}(Z)W + h_{12}(W)Z$, and $h_{22} : \mathcal{C} \to \mathcal{C}$ satisfying $h_{22}(ZW + WZ) = h_{22}(Z)W + h_{22}(W)Z$ for all Z, $W \in \mathcal{C}$ with $ZW = C$ such that

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} DX & DY + h_{12}(Z) \\ 0 & h_{22}(Z) \end{bmatrix} \quad \textit{for all } \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathcal{T}.$$

We need the following basic fact, whose proof is easy and will be skipped.

Proposition 2.2. *Let V be a vector space over a number field $\mathbb{F}$. For any fixed $a_i \in V$, $i = 0, \pm1, \pm2, \ldots, \pm n$, if $\sum_{i=-n}^{n} a_i x^i = 0$, $x \in \mathbb{F}$, has at least $2n + 1$ distinct nonzero solutions in $\mathbb{F}$, then $a_i = 0$, $i = 0, \pm1, \pm2, \ldots, \pm n$.*

Proof of Theorem 2.1. Since ϕ is linear, for any $S = \left[\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right] \in \mathcal{T}$, we can write

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} f_{11}(X) + g_{11}(Y) + h_{11}(Z) & f_{12}(X) + g_{12}(Y) + h_{12}(Z) \\ 0 & f_{22}(X) + g_{22}(Y) + h_{22}(Z) \end{bmatrix},$$

where $f_{11} : \mathcal{A} \to \mathcal{A}$, $f_{12} : \mathcal{A} \to \mathcal{M}$, $f_{22} : \mathcal{A} \to \mathcal{C}$, $g_{11} : \mathcal{M} \to \mathcal{A}$, $g_{12} : \mathcal{M} \to \mathcal{M}$, $g_{22} : \mathcal{M} \to \mathcal{C}$, $h_{11} : \mathcal{C} \to \mathcal{A}$, $h_{12} : \mathcal{C} \to \mathcal{M}$, and $h_{22} : \mathcal{C} \to \mathcal{C}$ are linear mappings.

We shall prove the theorem by checking several claims.

Claim 1. *$g_{11}(V) = h_{11}(W) = 0$ for all $V \in \mathcal{M}$ and $W \in \mathcal{C}$.*

For any invertible element $W \in \mathcal{C}$ and any real number $\lambda > 0$, we set

$$S = \begin{bmatrix} I_1 & 0 \\ 0 & \lambda C W^{-1} \end{bmatrix}, \qquad T = \begin{bmatrix} A & B \\ 0 & \lambda^{-1} W \end{bmatrix}.$$

Then $ST = G$, and we have

$$\begin{aligned}
&\begin{bmatrix} f_{11}(2A) + g_{11}(B + \lambda B C W^{-1}) + h_{11}(C + W C W^{-1}) & * \\ 0 & * \end{bmatrix} \\
&\quad = \phi(G + TS) \\
&\quad = \phi(S)T + \phi(T)S \\
&\quad = \begin{bmatrix} f_{11}(I_1) + \lambda h_{11}(C W^{-1}) & * \\ 0 & * \end{bmatrix} \begin{bmatrix} A & B \\ 0 & \lambda^{-1} W \end{bmatrix} \\
&\qquad + \begin{bmatrix} f_{11}(A) + g_{11}(B) + \lambda^{-1} h_{11}(W) & * \\ 0 & * \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ 0 & \lambda C W^{-1} \end{bmatrix} \\
&\quad = \begin{bmatrix} (f_{11}(I_1) + \lambda h_{11}(C W^{-1}))A + f_{11}(A) + g_{11}(B) + \lambda^{-1} h_{11}(W) & * \\ 0 & * \end{bmatrix}.
\end{aligned}$$

It follows from the matrix equation that

$$\begin{aligned}
&2 f_{11}(A) + g_{11}(B + \lambda B C W^{-1}) + h_{11}(C + W C W^{-1}) \\
&\quad = f_{11}(I_1)A + \lambda h_{11}(C W^{-1})A + f_{11}(A) + g_{11}(B) + \lambda^{-1} h_{11}(W).
\end{aligned}$$

By Proposition 2.2, the above equation implies that $h_{11}(W) = 0$ for all invertible elements $W \in \mathcal{C}$. For any W in $\mathcal{C}$, by hypothesis (ii) of Theorem 2.1, there exists some integer n such that $nI_2 - W$ is invertible in $\mathcal{C}$. It follows from the preceding case that $h_{11}(nI_2 - W) = 0$. Therefore, we have

$$h_{11}(W) = 0$$

for all $W \in \mathcal{C}$.

Moreover, for any $V \in \mathcal{M}$ and any real number $\lambda > 0$, taking

$$S = \begin{bmatrix} \lambda^{-1} I_1 & B - V \\ 0 & C \end{bmatrix}, \qquad T = \begin{bmatrix} \lambda A & \lambda V \\ 0 & I_2 \end{bmatrix},$$

we have $ST = G$. It follows from the fact $h_{11}(W) = 0$ that

$$\begin{aligned}
&\begin{bmatrix} f_{11}(2A) + g_{11}(B + \lambda A(B - V) + \lambda V C) & * \\ 0 & * \end{bmatrix} \\
&\quad = \phi(G + TS)
\end{aligned}$$

$$
\begin{aligned}
&= \phi(S)T + \phi(T)S \\
&= \begin{bmatrix} \lambda^{-1} f_{11}(I_1) + g_{11}(B - V) & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \lambda A & \lambda V \\ 0 & I_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} \lambda f_{11}(A) + \lambda g_{11}(V) & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \lambda^{-1} I_1 & B - V \\ 0 & C \end{bmatrix} \\
&= \begin{bmatrix} (f_{11}(I_1) + \lambda g_{11}(B - V))A + f_{11}(A) + g_{11}(V) & * \\ 0 & * \end{bmatrix}.
\end{aligned}
$$

The above matrix equation implies

$$
\begin{aligned}
&f_{11}(2A) + g_{11}\big(B + \lambda A(B - V) + \lambda VC\big) \\
&\quad = \big(f_{11}(I_1) + \lambda g_{11}(B - V)\big)A + f_{11}(A) + g_{11}(V).
\end{aligned}
$$

This leads to

$$
f_{11}(A) + g_{11}(B) - f_{11}(I_1)A = g_{11}(V).
$$

Taking $V = 0$, we get $f_{11}(A) + g_{11}(B) - f_{11}(I_1)A = 0$. Hence $g_{11}(V) = 0$ for all $V \in \mathcal{M}$.

Claim 2. *$f_{22}(X) = g_{22}(Y) = 0$ for all $X \in \mathcal{A}$ and $Y \in \mathcal{M}$.*

Letting $S = \left[\begin{smallmatrix} \lambda X & B \\ 0 & C \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} \lambda^{-1} X^{-1} A & 0 \\ 0 & I_2 \end{smallmatrix}\right]$, where X is any invertible element in $\mathcal{A}$ and $\lambda > 0$ is any real number, we have $ST = G$. Since ϕ is left Jordan-centralized at G, we have

$$
\begin{aligned}
&\begin{bmatrix} * & * \\ 0 & f_{22}(A + X^{-1}AX) + g_{22}(B + \lambda^{-1}X^{-1}AB) + h_{22}(2C) \end{bmatrix} \\
&\quad = \phi(G + TS) \\
&\quad = \phi(S)T + \phi(T)S \\
&\quad = \begin{bmatrix} * & * \\ 0 & \lambda f_{22}(X) + g_{22}(B) + h_{22}(C) \end{bmatrix} \begin{bmatrix} \lambda^{-1} X^{-1} A & 0 \\ 0 & I_2 \end{bmatrix} \\
&\qquad + \begin{bmatrix} * & * \\ 0 & \lambda^{-1} f_{22}(X^{-1}A) + h_{22}(I_2) \end{bmatrix} \begin{bmatrix} \lambda X & B \\ 0 & C \end{bmatrix} \\
&\quad = \begin{bmatrix} * & * \\ 0 & \lambda f_{22}(X) + g_{22}(B) + h_{22}(C) + (\lambda^{-1} f_{22}(X^{-1}A) + h_{22}(I_2))C \end{bmatrix}.
\end{aligned}
$$

By Proposition 2.2, this matrix equation implies $f_{22}(X) = 0$ for all invertible elements $X \in \mathcal{A}$. For any X in $\mathcal{A}$, by hypothesis (i) of Theorem 2.1, there exists some integer n such that $nI_1 - A$ is invertible in $\mathcal{A}$. It follows from the preceding case that $f_{22}(nI_1 - X) = 0$. Therefore

$$
f_{22}(X) = 0
$$

for all $X \in \mathcal{A}$ can be obtained.

Moreover, for an arbitrary element $Y \in \mathcal{M}$ and any real number $\lambda > 0$, if we put $S = \left[\begin{smallmatrix} \lambda I_1 & -Y \\ 0 & \lambda^{-1} C \end{smallmatrix}\right]$, $T = \left[\begin{smallmatrix} \lambda^{-1} A & Y + \lambda^{-1} B \\ 0 & \lambda I_2 \end{smallmatrix}\right]$, then $ST = G$. Since we have proved

$f_{22}(X)=0$ for all $X\in\mathcal{A}$,

$$\begin{bmatrix} * & * \\ 0 & g_{22}(B-\lambda^{-1}AY+\lambda^{-1}YC+\lambda^{-2}BC)+h_{22}(2C)\end{bmatrix}$$
$$\begin{aligned}
&=\phi(G+TS)\\
&=\phi(S)T+\phi(T)S\\
&=\begin{bmatrix} * & * \\ 0 & -g_{22}(Y)+\lambda^{-1}h_{22}(C)\end{bmatrix}\begin{bmatrix}\lambda^{-1}A & Y+\lambda^{-1}B \\ 0 & \lambda I_2\end{bmatrix}\\
&\quad+\begin{bmatrix} * & * \\ 0 & g_{22}(Y+\lambda^{-1}B)+\lambda h_{22}(I_2)\end{bmatrix}\begin{bmatrix}\lambda I_1 & -Y \\ 0 & \lambda^{-1}C\end{bmatrix}\\
&=\begin{bmatrix} * & * \\ 0 & -\lambda g_{22}(Y)+h_{22}(C)+\lambda^{-1}g_{22}(Y+\lambda^{-1}B)C+h_{22}(I_2)C\end{bmatrix}.
\end{aligned}$$

The above matrix equation implies

$$\begin{aligned}
&g_{22}(B)-\lambda^{-1}g_{22}(AY)+\lambda^{-1}g_{22}(YC)+\lambda^{-2}g_{22}(BC)+2h_{22}(C)\\
&\quad=-\lambda g_{22}(Y)+h_{22}(C)+\lambda^{-1}g_{22}(Y)C+\lambda^{-2}g_{22}(B)C+h_{22}(I_2)C.
\end{aligned}$$

Thus $g_{22}(Y)=0$ for all $Y\in\mathcal{M}$.

Claim 3. $f_{12}(X)=0$ *for all* $X\in\mathcal{A}$.

For any invertible element $X\in\mathcal{A}$ and any real number $\lambda>0$, putting

$$S=\begin{bmatrix}X & \lambda B\\ 0 & \lambda C\end{bmatrix},\qquad T=\begin{bmatrix}X^{-1}A & 0\\ 0 & \lambda^{-1}I_2\end{bmatrix},$$

we get $ST=G$. By Claims 1–2, it follows that

$$\begin{bmatrix} * & f_{12}(A+X^{-1}AX)+g_{12}(B+\lambda X^{-1}AB)+h_{12}(2C) \\ 0 & * \end{bmatrix}$$
$$\begin{aligned}
&=\phi(G+TS)\\
&=\phi(S)T+\phi(T)S\\
&=\begin{bmatrix} f_{11}(X) & f_{12}(X)+\lambda g_{12}(B)+\lambda h_{12}(C) \\ 0 & \lambda h_{22}(C)\end{bmatrix}\begin{bmatrix}X^{-1}A & 0 \\ 0 & \lambda^{-1} I_2\end{bmatrix}\\
&\quad+\begin{bmatrix} f_{11}(X^{-1}A) & f_{12}(X^{-1}A)+\lambda^{-1}h_{12}(I_2) \\ 0 & \lambda^{-1}h_{22}(I_2)\end{bmatrix}\begin{bmatrix}X & \lambda B \\ 0 & \lambda C\end{bmatrix}\\
&=\begin{bmatrix} * & \Delta \\ 0 & *\end{bmatrix},
\end{aligned}$$

where $\Delta=\lambda^{-1}f_{12}(X)+g_{12}(B)+h_{12}(C)+\lambda f_{11}(X^{-1}A)B+\lambda f_{12}(X^{-1}A)C+h_{12}(I_2)C$. The above matrix equation leads to

$$\begin{aligned}
&f_{12}(A)+f_{12}(X^{-1}AX)+g_{12}(B)+\lambda g_{12}(X^{-1}AB)+2h_{12}(C)\\
&\quad=\lambda^{-1}f_{12}(X)+g_{12}(B)+h_{12}(C)+\lambda f_{11}(X^{-1}A)B+\lambda f_{12}(X^{-1}A)C+h_{12}(I_2)C.
\end{aligned}$$

By Proposition 2.2, we obtain $f_{12}(X)=0$ for all invertible elements $X\in\mathcal{A}$. For any X in $\mathcal{A}$, by hypothesis (i) of Theorem 2.1, there exists some integer

n such that $nI_1 - A$ is invertible in $\mathcal{A}$. It follows from the preceding case that $f_{12}(nI_1 - X) = 0$. Therefore

$$f_{12}(X) = 0$$

for all $X \in \mathcal{A}$ can be obtained.

Claim 4. *$f_{11}(X) = f_{11}(I_1)X$, $g_{12}(V) = f_{11}(I_1)V$ for all $X \in \mathcal{A}$ and $V \in \mathcal{M}$.*

For any arbitrary $V \in \mathcal{M}$ and invertible element $X \in \mathcal{A}$, taking

$$S = \begin{bmatrix} \lambda X & B - \lambda XV \\ 0 & C \end{bmatrix}, \qquad T = \begin{bmatrix} \lambda^{-1}X^{-1}A & V \\ 0 & I_2 \end{bmatrix},$$

we have $ST = G$. So we obtain

$$\begin{aligned}
&\begin{bmatrix} * & g_{12}(B + \lambda^{-1}X^{-1}A(B - \lambda XV) + VC) + h_{12}(2C) \\ 0 & * \end{bmatrix} \\
&= \phi(G + TS) \\
&= \phi(S)T + \phi(T)S \\
&= \begin{bmatrix} \lambda f_{11}(X) & g_{12}(B - \lambda XV) + h_{12}(C) \\ 0 & h_{22}(C) \end{bmatrix} \begin{bmatrix} \lambda^{-1}X^{-1}A & V \\ 0 & I_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} \lambda^{-1} f_{11}(X^{-1}A) & g_{12}(V) + h_{12}(I_2) \\ 0 & h_{22}(I_2) \end{bmatrix} \begin{bmatrix} \lambda X & B - \lambda XV \\ 0 & C \end{bmatrix} \\
&= \begin{bmatrix} * & \triangle \\ 0 & * \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\triangle = {}& \lambda f_{11}(X)V + g_{12}(B - \lambda XV) + h_{12}(C) \\
&+ \lambda^{-1} f_{11}(X^{-1}A)(B - \lambda XV) + \big(g_{12}(V) + h_{12}(I_2)\big)C.
\end{aligned}$$

By Proposition 2.2, the above matrix equation implies that $g_{12}(XV) = f_{11}(X)V$ for all $V \in \mathcal{M}$ and all invertible elements $X \in \mathcal{A}$. For any $X \in \mathcal{A}$, by hypothesis (i) of Theorem 2.1, there exists some integer n such that $nI_1 - A$ is invertible in $\mathcal{A}$. It follows from the preceding case that $g_{12}((nI_1 - X)V) = f_{11}(nI_1 - X)V$. Therefore, we have

$$g_{12}(XV) = f_{11}(X)V \tag{2.1}$$

for all $X \in \mathcal{A}$ and $V \in \mathcal{M}$. For any X, $U \in \mathcal{A}$, by (2.1), we get

$$g_{12}(XUV) = f_{11}(XU)V.$$

On the other hand,

$$g_{12}(XUV) = f_{11}(X)UV.$$

Comparing these two equalities, we have $f_{11}(XU)V = f_{11}(X)UV$ for all $V \in \mathcal{M}$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we obtain

$$f_{11}(XU) = f_{11}(X)U,$$

which is equivalent to $f_{11}(X) = f_{11}(I_1)X$ for all $X \in \mathcal{A}$.

At the same time, equation (2.1) gives $g_{12}(V) = f_{11}(I_1)V$ for all $V \in \mathcal{M}$.

Claim 5. *For any Z, $W \in \mathcal{C}$ with $ZW = C$, the following statements hold:*

(i) $h_{12}(ZW + WZ) = h_{12}(Z)W + h_{12}(W)Z$,
(ii) $h_{22}(ZW + WZ) = h_{22}(Z)W + h_{22}(W)Z$.

For arbitrary $Z, W \in \mathcal{C}$ with $ZW = C$, taking

$$S = \begin{bmatrix} \lambda^{-1}I_1 & 0 \\ 0 & \lambda Z \end{bmatrix}, \qquad T = \begin{bmatrix} \lambda A & \lambda B \\ 0 & \lambda^{-1}W \end{bmatrix},$$

we have $ST = G$. We have

$$\begin{aligned}
&\begin{bmatrix} f_{11}(2A) & g_{12}(B + \lambda^2 BZ) + h_{12}(C + WZ) \\ 0 & h_{22}(C + WZ) \end{bmatrix} \\
&\quad = \varphi(G + TS) \\
&\quad = \phi(S)T + \phi(T)S \\
&\quad = \begin{bmatrix} f_{11}(\lambda^{-1}I_1) & h_{12}(\lambda Z) \\ 0 & h_{22}(\lambda Z) \end{bmatrix} \begin{bmatrix} \lambda A & \lambda B \\ 0 & \lambda^{-1}W \end{bmatrix} \\
&\qquad + \begin{bmatrix} f_{11}(\lambda A) & g_{12}(\lambda B) + h_{12}(\lambda^{-1}W) \\ 0 & h_{22}(\lambda^{-1}W) \end{bmatrix} \begin{bmatrix} \lambda^{-1}I_1 & 0 \\ 0 & \lambda Z \end{bmatrix} \\
&\quad = \begin{bmatrix} f_{11}(I_1)A + f_{11}(A) & f_{11}(I_1)B + h_{12}(Z)W + \lambda^2 g_{12}(B)Z + h_{12}(W)Z \\ 0 & h_{22}(Z)W + h_{22}(W)Z \end{bmatrix}.
\end{aligned}$$

It follows from the above matrix equation and Claim 4 that

$$h_{22}(C + WZ) = h_{22}(Z)W + h_{22}(W)Z$$

and

$$h_{12}(C + WZ) = h_{12}(Z)W + h_{12}(W)Z$$

for all $Z, W \in \mathcal{C}$ with $ZW = C$.

Therefore, by Claims 1–5, we have

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} DX & DY + h_{12}(Z) \\ 0 & h_{22}(Z) \end{bmatrix} \quad \text{for all } \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathcal{T},$$

where $D = f_{11}(I_1) \in \mathcal{A}$, and $h_{12} : \mathcal{M} \to \mathcal{C}$ and $h_{22} : \mathcal{C} \to \mathcal{C}$ are linear mappings satisfying $h_{12}(ZW + WZ) = h_{12}(Z)W + h_{12}(W)Z$, $h_{22}(ZW + WZ) = h_{22}(Z)W + h_{22}(W)Z$ for all Z, $W \in \mathcal{C}$ with $ZW = C$. □

By using a similar argument to that of Theorem 2.1, we can get a characterization of right Jordan-centralized mappings at a given point.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{C}$ be two algebras over a number field $\mathbb{F}$ with unit I_1 and I_2, respectively, and let $\mathcal{M}$ be a faithful right $\mathcal{C}$-module. The triangular algebra* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{C})$ *is written for $\mathcal{T}$. Suppose that we have the following:*

(i) *For every $X \in \mathcal{A}$, there is some integer n such that $nI_1 - X$ is invertible.*
(ii) *For every $Z \in \mathcal{C}$, there is some integer n such that $nI_2 - Z$ is invertible.*

If $\phi : \mathcal{T} \to \mathcal{T}$ is a right Jordan-centralized mapping at a given point $G = \left[\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right] \in \mathcal{T}$, then there exists an element $E(= h_{22}(I_2)) \in \mathcal{C}$, two linear mappings $f_{11} : \mathcal{A} \to \mathcal{A}$ satisfying $f_{11}(XU + UX) = Xf_{11}(U) + Uf_{11}(X)$, and $f_{12} : \mathcal{A} \to \mathcal{M}$ satisfying $f_{12}(XU + UX) = Xf_{12}(U) + Uf_{12}(X)$ for all X, $U \in \mathcal{A}$ with $XU = A$ such that

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} f_{11}(X) & YE + f_{12}(X) \\ 0 & ZE \end{bmatrix} \quad \text{for all } \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathcal{T}.$$

The following result states that the Jordan centralizers on triangular algebras can be determined by their action on a given point.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{C}$ be two algebras over a number field $\mathbb{F}$ with unit I_1 and I_2, respectively, and let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{C})$-bimodule. The triangular algebra* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{C})$ *is written for $\mathcal{T}$. Suppose that we have the following:*

(i) *For every $X \in \mathcal{A}$, there is some integer n such that $nI_1 - X$ is invertible.*
(ii) *For every $Z \in \mathcal{C}$, there is some integer n such that $nI_2 - Z$ is invertible.*

If $\phi : \mathcal{T} \to \mathcal{T}$ is a Jordan-centralized mapping at a given point $G \in \mathcal{T}$, then ϕ is a centralizer.

Proof of Theorem 2.4. By Theorems 2.1 and 2.3, we have

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} DX & DY + h_{12}(Z) \\ 0 & h_{22}(Z) \end{bmatrix} = \begin{bmatrix} f_{11}(X) & YE + f_{12}(X) \\ 0 & ZE \end{bmatrix}$$

for all $\left[\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right] \in \mathcal{T}$, where there exist $D \in \mathcal{A}$, $E \in \mathcal{C}$, four linear mappings $h_{12} : \mathcal{C} \to \mathcal{M}$ satisfying $h_{12}(ZW + WZ) = h_{12}(Z)W + h_{12}(W)Z$, $h_{22} : \mathcal{C} \to \mathcal{C}$ satisfying $h_{22}(ZW+WZ) = h_{22}(Z)W+h_{22}(W)Z$ for all $Z, W \in \mathcal{C}$ with $ZW = C$, $f_{11} : \mathcal{A} \to \mathcal{A}$ satisfying $f_{11}(XU + UX) = Xf_{11}(U) + Uf_{11}(X)$, and $f_{12} : \mathcal{A} \to \mathcal{M}$ satisfying $f_{12}(XU+UX) = Xf_{12}(U)+Uf_{12}(X)$ for all $X, U \in \mathcal{A}$ with $XU = A$. This leads to

$$DX = f_{11}(X), \qquad ZE = h_{22}(Z)$$

and

$$DY + h_{12}(Z) = YE + f_{12}(X). \tag{2.2}$$

Let $X = Z = 0$ in (2.2); we get $DY = YE$ for all $Y \in \mathcal{M}$. It follows from [5, Proposition 3] that $\left[\begin{smallmatrix} D & 0 \\ 0 & E \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{T})$, and then $D \in \mathcal{Z}(\mathcal{A})$ and $E \in \mathcal{Z}(\mathcal{C})$. Furthermore, let $X = 0$ and $Z = 0$ in equation (2.2), and we get $h_{12}(Z) = 0$ and $f_{12}(X) = 0$. Hence

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} DX & DY \\ 0 & ZE \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix}\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

and

$$\phi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} DX & YE \\ 0 & ZE \end{bmatrix} = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix}$$

for all $\left[\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right] \in \mathcal{T}$ hold. So ϕ is a centralizer of $\mathcal{T}$. $\square$

We remark here that a little more can be said about the result. We in fact have

$$\begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix} = \begin{bmatrix} f_{11}(I_1) & 0 \\ 0 & h_{22}(I_2) \end{bmatrix} = \phi(I).$$

Then $\phi(S) = \phi(I)S = S\phi(I)$ for all $S \in \mathcal{T}$.

3. Applications

In this section, we shall apply Theorem 2.4 to some non-self-adjoint operator algebras.

For irreducible CDC algebras, we have the following theorem.

Theorem 3.1. *Let* $\operatorname{alg}\mathcal{L}$ *be an irreducible CDC algebra on a complex Hilbert space* H*. Then every linear mapping from* $\operatorname{alg}\mathcal{L}$ *into itself Jordan-centralized at a given point* $G \in \operatorname{alg}\mathcal{L}$ *is a centralizer.*

Proof of Theorem 3.1. By [12, Theorem 3.4], there exists a nontrivial projection P in $\mathcal{L}$ such that $TP(\operatorname{alg}\mathcal{L})(I-P) = \{0\}$ implies $TP = 0$ and $P(\operatorname{alg}\mathcal{L})(I-P)T = \{0\}$ implies $(I-P)T = 0$. It follows that

$$\operatorname{alg}\mathcal{L} = \begin{bmatrix} P(\operatorname{alg}\mathcal{L})P|_{\operatorname{ran} P} & P(\operatorname{alg}\mathcal{L})(I-P)|_{\ker P} \\ 0 & (I-P)(\operatorname{alg}\mathcal{L})(I-P)|_{\ker P} \end{bmatrix}.$$

One can easily check that $\operatorname{alg}\mathcal{L}$ meets all of the hypotheses of Theorem 2.4. So by Theorem 2.4, the theorem holds. □

Note that nest algebras are irreducible CDC algebras. Hence, as a consequence of Theorem 3.1, we get the following corollary.

Corollary 3.2. *Let* $\mathcal{N}$ *be a nest on a complex Hilbert space* H*, and let* $\operatorname{alg}\mathcal{N}$ *be the associated nest algebra. Then every linear mapping from* $\operatorname{alg}\mathcal{N}$ *into itself Jordan-centralized at a given point* $G \in \operatorname{alg}\mathcal{N}$ *is a centralizer.*

Let H be a Euclidean n-dimensional space, and let $\{e_i : i = 1, 2, \ldots, n\}$ be its normal orthogonal basis. We may regard an $n \times n$ matrix as an operator on Euclidean n-dimension space H, naturally. We use the symbols $\mathcal{TM}_n$ to denote the algebra of all $n \times n$ upper triangular matrices. Thus $\mathcal{TM}_n$ is a nest algebra associated with nest $\mathcal{N}$, where $\mathcal{N} = \{N_i : 1 \le i \le n\}$ and $N_i = \operatorname{span}\{e_j : 1 \le j \le i\}$. By Corollary 3.2, we have the following corollary.

Corollary 3.3. *Let* $\mathcal{TM}_n$ *be* $n \times n$ *upper triangular matrices algebras. Then every linear mapping from* $\mathcal{TM}_n$ *into itself Jordan-centralized at a given point* $G \in \mathcal{TM}_n$ *is a centralizer.*

Next, we consider the same question on the Banach space nest algebras.

Let X be a Banach space over the complex field $\mathbb{C}$, and let $B(X)$ denote the algebra of all bounded linear operators on X. A nest $\mathcal{N}$ in X is a chain of norm-closed linear subspaces of X containing $\{0\}$ and X, which is closed under the formation of an arbitrary closed linear span and intersection. A nest is said to be nontrivial if $\mathcal{N} \neq \{\{0\}, X\}$. The nest algebra associated to a nest $\mathcal{N}$, denoted by $\operatorname{alg}\mathcal{N}$, is the set

$$\operatorname{alg}\mathcal{N} = \big\{T \in B(X) : TN \subseteq N, \forall N \in \mathcal{N}\big\}.$$

Theorem 3.4. *Let $\mathcal{N}$ be a nest on a complex Banach space. Suppose that there exists a nontrivial element in $\mathcal{N}$ which is complemented in X. If $\phi : \operatorname{alg}\mathcal{N} \to \operatorname{alg}\mathcal{N}$ is a Jordan-centralized mapping at a given point $G \in \operatorname{alg}\mathcal{N}$, then ϕ is a centralizer.*

Proof of Theorem 3.4. If N_0 is the nontrivial element in $\mathcal{N}$ which is complemented in X, then there exists an idempotent operator $P \in B(X)$ with $\operatorname{ran}(P) = N_0$, such that $X = \operatorname{ran}(P) \oplus \ker(P)$.

Moreover, it is easy to see $P \in \operatorname{alg}\mathcal{N}$. We set $\mathcal{N}_1 = \{N \cap \operatorname{ran} P : N \in \mathcal{N}\}$ and $\mathcal{N}_2 = \{N \cap \ker(P) : N \in \mathcal{N}\}$. Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are nests on Banach spaces $\operatorname{ran}(P)$ and $\ker(P)$, respectively. One can check that $PB(X)(I-P) \subseteq \operatorname{alg}\mathcal{N}$, which leads to $PB(X)(I-P) = P\operatorname{alg}\mathcal{N}(I-P)$. So we denote

$$\operatorname{alg}\mathcal{N} = \left\{ \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} : X \in \operatorname{alg}\mathcal{N}_1, Y \in B\big(\ker(P), \operatorname{ran}(P)\big), Z \in \operatorname{alg}\mathcal{N}_2 \right\}.$$

This means $\operatorname{alg}\mathcal{N}$ can be decomposed into a triangular algebra. We claim that $B(\ker(P), \operatorname{ran}(P))$ is a faithful $(\operatorname{alg}\mathcal{N}_1, \operatorname{alg}\mathcal{N}_2)$-bimodule. Indeed, for $X \in \operatorname{alg}\mathcal{N}_1$, if $XY = 0$ for any $Y \in B(\ker(P), \operatorname{ran}(P))$, we have $XPB(X)(I-P) = \{0\}$. Since $B(X)$ is prime, we get $X = 0$. So $B(\ker(P), \operatorname{ran}(P))$ is a faithful $(\operatorname{alg}\mathcal{N}_1)$-left module. Similarly, one can prove that $B(\ker(P), \operatorname{ran}(P))$ is a faithful $(\operatorname{alg}\mathcal{N}_2)$-right module. Hence, by Theorem 2.4, the theorem is obtained. ☐

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