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PICTURES OF KK-THEORY FOR REAL C^*-ALGEBRAS AND ALMOST COMMUTING MATRICES

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Abstract. We give a systematic account of the various pictures of KK-theory for real C^*-algebras, proving natural isomorphisms between the groups that arise from each picture. As part of this project, we develop the universal properties of KK-theory, and we use CRT-structures to prove that a natural transformation $F(A) \to G(A)$ between homotopy equivalent, stable, half-exact functors defined on real C^*-algebras is an isomorphism, provided it is an isomorphism on the smaller class of C^*-algebras. Finally, we develop E-theory for real C^*-algebras and use that to obtain new negative results regarding the problem of approximating almost commuting real matrices by exactly commuting real matrices.

1. Introduction and preliminaries

A real C^*-algebra is a Banach $*$-algebra A over the real numbers such that $\|x^*x\| = \|x\|^2$ holds for all x and such that every element of the form $1 + x^*x$ is invertible in the unitization of A (see [23]). In this paper, we will adopt the term R^*-*algebra* instead. As is well known, every R^*-algebra is isometrically isomorphic to a closed $*$-algebra of bounded operators on a Hilbert space over $\mathbb{R}$. In addition, every R^*-algebra is isomorphic to the $*$-algebra of fixed elements of a C^*-algebra with a conjugate linear involution.

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In his seminal paper [17] introducing KK-theory, Kasparov simultaneously considered both R^*-algebras and C^*-algebras. Since then, many alternate, but equivalent or closely related, pictures of KK-theory have been introduced and developed by various authors (see, e.g., [8], [9], [14], [15], [22], [25]). The ability to move among the various pictures has contributed immensely to the utility of KK-theory as a tool for solving problems. However, these authors have not consistently followed Kasparov's lead in considering the real case along with the complex.

In recent years, substantial progress has been made in developing the tools to study R^*-algebras, including the development of united K-theory and the universal coefficient theorem in [3] and [4]. This has led to a classification of purely infinite simple R^*-algebras (in [5]) and the classification of real forms of UHF-algebras that are stable over the CAR-algebra (in [24]).

Given the centrality of KK-theory for these projects, there has been a need to develop a systematic account of the various pictures of KK-theory for R^*-algebras. In this paper, we will develop several of the alternate pictures of KK-theory in the context of R^*-algebras and prove the appropriate equivalence theorem. In particular, in this paper we will consider the following pictures of KK-theory and prove appropriate equivalence theorems for each: the standard Kasparov bimodule picture of KK-theory, the Fredholm picture (both in Section 2), the universal property picture (Section 3), and suspended E-theory using asymptotic morphisms (Section 4). Since this aspect of the theory goes through in the real case very much like the complex case, we present these results in a survey-like overview.

As part of this, we prove a more general theorem which reduces the work required to replicate many of these equivalent theorems and promises to ease the way for similar projects in the future. Suppose that $\mu\colon F \to G$ is a natural transformation between homotopy invariant, stable, half-exact functors. We prove that if μ is an isomorphism for all C^*-algebras, then it is an isomorphism for R^*-algebras. This is accomplished in Section 3, where we develop the universal properties of KK-theory and K-theory for R^*-algebras.

In the last two sections, we will apply these ideas, using KK-theory to prove the existence of certain asymptotic morphisms, which, in turn, is used to obtain new results for the problem of approximating a set of almost commuting matrices over the field of real numbers. In particular, let the Halmos number be the largest integer d such that whenever d real self-adjoint matrices almost commute (pairwise) they can be approximated by d pairwise commuting matrices. More precisely, for all $\varepsilon > 0$, there should be a $\delta > 0$ such that if $\{H_i\}_{i=1}^d$ is a collection of d self-adjoint matrices such that

$$\|H_r\| \leq 1 \qquad \text{and} \qquad \big\|[H_r, H_s]\big\| \leq \delta,$$

for all r, s, then there exists a collection $\{K_i\}_{i=1}^d$ of self-adjoint matrices such that

$$\|K_r\| \leq 1 \qquad \text{and} \qquad \big\|[K_r, K_s]\big\| = 0 \qquad \text{and} \qquad \|H_r - K_r\| \leq \varepsilon.$$

Furthermore, the dependence of δ on ε must be uniform, independent of the dimension of the matrices H_r. It is shown in [20] that, in the context of real matrices, the statement is true for $d = 2$. We will show in Section 7 that the statement is false for $d = 5$. Therefore, the Halmos number for real matrices is between 2 and 4, inclusive.

2. The standard and Fredholm pictures of KK-theory

We start with the following definition of KK-theory. It is essentially the same as that in [17], where it was simultaneously developed for both R^*-algebras and C^*-algebras. It also appears in Section 2.3 of [23] for R^*-algebras, Section 2.1 of [16], and the Appendix of [14].

Definition 2.1. Let A and B be graded separable R^*-algebras with B assumed to be σ-unital.

(i) A Kasparov (A-B)-bimodule is a triple (E, ϕ, T), where E is a countably generated graded real Hilbert B-module, $\phi : A \to \mathcal{L}_{\mathbb{R}}(E)$ is a graded $*$-homomorphism, and T is an element of $\mathcal{L}_{\mathbb{R}}(E)$ of degree 1 such that
$$(T - T^*)\phi(a), (T^2 - 1)\phi(a) \qquad \text{and} \qquad \big[T, \phi(a)\big]$$
lie in $\mathcal{K}_{\mathbb{R}}(E)$ for all $a \in A$.

(ii) Two triples (E_i, ϕ_i, T_i) are *unitarily equivalent* if there is unitary U in $\mathcal{L}_{\mathbb{R}}(E_0, E_1)$, of degree zero, intertwining both ϕ_i and T_i in the appropriate way.

(iii) Let (E, ϕ, T) be a Kasparov (A-B)-bimodule, and let $\beta\colon B \to B'$ be a $*$-homomorphism of R^*-algebras. Then the pushed-forward Kasparov (A-B')-bimodule is defined by
$$\beta_*(E, \phi, T) = (E \hat{\otimes}_\beta B', \phi \hat{\otimes} 1, T \hat{\otimes} 1).$$

(iv) Two Kasparov (A-B)-bimodules (E_i, ϕ_i, T_i) for $i = 0, 1$ are *homotopic* if there is a Kasparov bimodule (A-IB), say (E, ϕ, T), such that $(\varepsilon_i)_*(E, \phi, T)$ and (E_i, ϕ_i, T_i) are unitarily equivalent for $i = 0, 1$, where $IB = C([0, 1], B)$ and ε_i denotes the evaluation map.

(v) A triple (E, ϕ, T) is *degenerate* if the elements
$$(T - T^*)\phi(a), (T^2 - 1)\phi(a) \qquad \text{and} \qquad \big[T, \phi(a)\big]$$
are zero for all $a \in A$. By Proposition 2.3.3 of [23], degenerate bimodules are homotopic to trivial bimodules.

(vi) $KK(A, B)$ is defined to be the set of homotopy equivalence classes of Kasparov (A-B)-bimodules.

The following theorem summarizes the principal properties of KK-theory for R^*-algebras from Chapter 2 of [23].

Proposition 2.2. *$KK(A, B)$ is an abelian group for separable A and σ-unital B. As a functor on separable R^*-algebras (contravariant in the first argument and covariant in the second argument), it is homotopy invariant, stable, and has split exact sequences in both arguments. Furthermore, there is a natural associate pairing (the intersection product)*
$$KK(A, C \otimes B) \otimes KK(C \otimes A', B') \to KK(A \otimes A', B \otimes B').$$

We now turn to the Fredholm picture of KK-theory, which was developed in [14] in the context of C^*-algebras. A simplified picture along these lines also appears in Chapter 4 of [16]. As we show, the approach goes through the same for R^*-algebras, as follows.

Definition 2.3. Let A and B be separable R^*-algebras.

(i) A triple (ϕ_+, ϕ_-, U), where $\phi_\pm : A \to \mathcal{M}(\mathcal{K}_\mathbb{R} \otimes B)$ are $*$-homomorphisms, and U is an element of $\mathcal{M}(\mathcal{K}_\mathbb{R} \otimes B)$ such that
$$U\phi_+(a) - \phi_-(a)U, \phi_+(a)(U^*U - 1), \text{ and } \phi_-(a)(UU^* - 1)$$
lie in $\mathcal{K}_\mathbb{R} \otimes B$ for all $a \in A$ is called a $KK(A, B)$-*cycle.*

(ii) Two $KK(A, B)$-cycles $(\phi_+^1, \phi_-^1, U^1)$ and $(\phi_+^2, \phi_-^2, U^2)$ are *homotopic* if there is a $KK(A, IB)$-cycle (ϕ_+, ϕ_-, U) such that
$$\big(\varepsilon_i\phi_+, \varepsilon_i\phi_-, \varepsilon_i(U)\big) = (\phi_+^i, \phi_-^i, U^i),$$
where $\varepsilon_i : \mathcal{M}(\mathcal{K}_\mathbb{R} \otimes IB) \to \mathcal{M}(\mathcal{K}_\mathbb{R} \otimes B)$ is induced by evaluation at i.

(iii) A $KK(A, B)$-cycle (ψ_+, ψ_-, V) is *degenerate* if the elements
$$V\psi_+(a) - \psi_-(a)V, \psi_+(a)(V^*V - 1), \text{ and } \psi_-(a)(VV^* - 1)$$
are zero for all $a \in A$.

(iv) The sum $(\phi_+, \phi_-, U) \oplus (\psi_+, \psi_-, V)$ of two $KK(A, B)$-cycles is the $KK(A, B)$-cycle
$$\left(\begin{pmatrix} \phi_+ & 0 \\ 0 & \psi_+ \end{pmatrix}, \begin{pmatrix} \phi_- & 0 \\ 0 & \psi_- \end{pmatrix}, \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\right),$$
where the algebra $M_2(\mathcal{M}(\mathcal{K}_\mathbb{R} \otimes B))$ is identified with $\mathcal{M}(\mathcal{K}_\mathbb{R} \otimes B)$ by means of some $*$-isomorphism $M_2(\mathcal{K}_\mathbb{R}) \cong \mathcal{K}_\mathbb{R}$, which is unique up to homotopy by [17, Section 1.17].

(v) Two cycles $(\phi_+^0, \phi_-^0, U^0)$ and $(\phi_+^1, \phi_-^1, U^1)$ are said to be *equivalent* if there exist degenerate cycles $(\psi_+^0, \psi_-^0, V^0)$ and $(\psi_+^1, \psi_-^1, V^1)$ such that
$$(\phi_+^0, \phi_-^0, U^0) \oplus (\psi_+^0, \psi_-^0, V^0) \qquad \text{and} \qquad (\phi_+^1, \phi_-^1, U^1) \oplus (\psi_+^1, \psi_-^1, V^1)$$
are homotopic.

(vi) $\boldsymbol{KK}(A, B)$ is defined to be the set of equivalence classes of $KK(A, B)$-cycles.

The following lemma is the real version of Lemma 2.3 of [14], with the same proof.

Proposition 2.4. *$\boldsymbol{KK}(A, B)$ is an abelian group, for separable R^*-algebras A and B. As a functor it is contravariant in the first argument and covariant in the second argument.*

Theorem 2.5. *Let A and B be separable R^*-algebras (with the trivial grading). Then $KK(A, B)$ is isomorphic to $\boldsymbol{KK}(A, B)$.*

Sketch of proof. Let $\mathbb{H}_B$ be the Hilbert B-module consisting of all sequences $\{b_n\}_{n=1}^\infty$ in B such that $\sum_{n=1}^\infty b_n^* b_n$ converges. Let $\hat{\mathbb{H}}_B = \mathbb{H}_B \oplus \mathbb{H}_B$ be the graded Hilbert B-module with $\hat{\mathbb{H}}_B^{(0)} = \mathbb{H}_B \oplus 0$ and $\hat{\mathbb{H}}_B^{(1)} = 0 \oplus \mathbb{H}_B$. This induces a grading on $\mathcal{L}_\mathbb{R}(\hat{\mathbb{H}}_B) \cong \mathcal{M}(\mathcal{K}_\mathbb{R} \otimes B)$.

Given a $KK(A,B)$-cycle $x = (\phi_+, \phi_-, U)$, we define

$$\alpha(x) = \left(\hat{\mathbb{H}}_B, \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}, \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \right).$$

It is verified that $\left(\begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix}\right)$ has degree 1 and that $\alpha(x)$ is indeed a Kasparov (A-B) bimodule. That α induces a well-defined isomorphism

$$\overline{\alpha} \colon \boldsymbol{KK}(A,B) \to KK(A,B)$$

can be shown by adapting the methods of [14, Appendix] or [16, Chapter 4]. □

3. The universal property of KK-theory

Let F be a functor from the category $\boldsymbol{C^*}\mathbb{R}$**-Alg** of separable R^*-algebras to the cateogry **Ab** of abelian groups. We say that F is

(i) *homotopy invariant* if $(\alpha_1)_* = (\alpha_2)_*$ whenever α_1 and α_2 are homotopic $*$-homomorphisms on the level of R^*-algebras;

(ii) *stable* if $(e_A)_* \colon F(A) \to F(\mathcal{K}_\mathbb{R} \otimes A)$ is an isomorphism for the inclusion $e_A \colon A \hookrightarrow \mathcal{K}_\mathbb{R} \otimes A$ defined via any rank 1 projection;

(iii) *split exact* if any split exact sequence of separable R^*-algebras

$$0 \to A \to B \to C \to 0$$

induces a split exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0;$$

(iv) *half-exact* if any short exact sequence of separable R^*-algebras

$$0 \to A \to B \to C \to 0$$

induces an exact sequence

$$F(A) \to F(B) \to F(C).$$

In what follows we will see that if F is homotopy invariant and half-exact, then it is split exact.

Proposition 3.1. *If F is a functor from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to **Ab** that is homotopy invariant, then the functor F_s defined by $F_s(A) = F(\mathcal{K}_\mathbb{R} \otimes A)$ is homotopy invariant and stable.*

Proof. Just as in the complex case (see [16, Theorem 4.1.13]), the map $e_{\mathcal{K}_\mathbb{R}} \colon \mathcal{K}_\mathbb{R} \to \mathcal{K}_\mathbb{R} \otimes \mathcal{K}_\mathbb{R}$ is homotopic to an isomorphism. □

The following theorem is the version for R^*-algebras of Theorem 3.7 of [14] and Theorem 22.3.1 of [2].

Theorem 3.2. *Let F be a functor from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to **Ab** that is homotopy invariant, stable, and split exact. Then there is a unique natural pairing $\alpha \colon F(A) \otimes KK(A,B) \to F(B)$ such that $\alpha(x \otimes 1_A) = x$ for all $x \in F(A)$ and where $1_A \in KK(A,A)$ is the class represented by the identity $*$-homomorphism.*

Furthermore, the pairing respects the intersection product on KK-theory in the sense that

$$\alpha\big(\alpha(x \otimes y) \otimes z\big) = \alpha\big(x \otimes (y \otimes_B z)\big) \colon F(A) \otimes KK(A,B) \otimes KK(B,C) \to F(C).$$

Proof. Let $\Phi \in KK(A, B)$. Using Theorem 2.5 we represent Φ with a $KK(A, B)$ cycle and, as in Lemma 3.6 of [14], we may assume that this cycle has the form $(\phi_+, \phi_-, 1)$. We use the same construction as in Definitions 3.3 and 3.4 of [14]. In that setting, F is assumed to be a functor from separable C^*-algebras, but the same construction applies for a functor from separable R^*-algebras to any abelian category. This construction produces a homomorphism $\Phi_*\colon F(A) \to F(B)$ and we then define $\alpha(x \otimes \Phi) = \Phi_*(x)$. The proofs of Theorems 3.7 and 3.5 of [14] carry over in the real case to show that α is natural, is well defined, satisfies $\alpha(x \otimes 1_A) = x$, and is unique.

That α respects the Kasparov product follows from the uniqueness statement. □

We also note the contravariant version of the result above. If F is a contravariant functor, otherwise satisfying the above hypotheses, then there is a pairing $\alpha\colon KK(A, B) \otimes F(B) \to F(A)$ such that $\alpha(1_A \otimes x) = x$ for all $x \in F(A)$.

For any R^*-algebra A, we define $SA = \{f \in C([0, 1], A) \mid f(0) = f(1) = 0\}$, or, equivalently up to $*$-isomorphism, $SA = C_0(\mathbb{R}, A)$. We similarly define $S^{-1}A = \{f \in C_0(\mathbb{R}, A_{\mathbb{C}}) \mid f(-x) = \overline{f(x)}\}$. By iteration, $S^n A$ is defined for all $n \in \mathbb{Z}$. Since $SS^{-1}\mathbb{R}$ is KK-equivalent to $\mathbb{R}$, the formula $S^n S^m A \equiv S^{n+m} A$ holds up to KK-equivalence for all $n, m \in \mathbb{Z}$. Then, for any functor F on $\boldsymbol{C^*}\mathbb{R}$**-Alg** and any integer n, we define $F_n(A) = F(S^n(A))$.

Corollary 3.3. *Let F be a functor from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to **Ab** that is homotopy invariant, stable, and split exact. Then $F_*(A)$ has the structure of a graded module over the ring $K_*(\mathbb{R})$. In particular, $F_n(A) \cong F_{n+8}(A)$ for all $n \in \mathbb{Z}$.*

Proof. For all separable A and σ-unital B, the pairing of Proposition 2.2 gives $KK_*(A, B)$ the structure of a module over $KK_*(\mathbb{R}, \mathbb{R})$. Taking $A = B$, we define a graded ring homomorphism β from $K_*(\mathbb{R}) \cong KK_*(\mathbb{R}, \mathbb{R})$ to $KK_*(A, A)$ by multiplication by $1_A \in KK(A, A)$. Then, for any $x \in F_m(A)$ and $y \in K_n(\mathbb{R})$, we define $x \cdot y = \alpha(x \otimes \beta(y)) \in F_{n+m}(A)$. □

Similarly, the pairing Theorem 3.2 extends to a well-defined graded pairing

$$\alpha\colon F_*(A) \otimes KK_*(A, B) \to F_*(B).$$

Let $\boldsymbol{KK}$ be the category whose objects are separable R^*-algebras, and the set of morphisms from A to B is $KK(A, B)$. There is a canonical functor KK from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to $\boldsymbol{KK}$ that takes an object A to itself and which takes a $*$-homomorphism $f\colon A \to B$ to the corresponding element $[f] \in KK(A, B)$.

Corollary 3.4. *Let F be a functor from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to **Ab** that is homotopy invariant, stable, and split exact. Then there exists a unique functor $\hat{F} : \boldsymbol{KK} \to \boldsymbol{A}$ such that $\hat{F} \circ KK = F$.*

Proof. This statement is proved as in Section 2.8 of [14]. □

Proposition 3.5. *Let F be a functor from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to **Ab** that is homotopy invariant and half-exact. Then, for any short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

there is a natural boundary map $\partial\colon F(SC) \to F(A)$ that fits into a (half-infinite) long exact sequence

$$\cdots \to F(SB) \xrightarrow{g_*} F(SC) \xrightarrow{\partial} F(A) \xrightarrow{f_*} F(B) \xrightarrow{g_*} F(C).$$

Proof. Use the mapping cone construction as in Section 21.4 of [2]. □

Corollary 3.6. *A functor F from **C*ℝ-Alg** to **Ab** that is homotopy invariant and half-exact is also split exact.*

Proof. The splitting implies that g_* is surjective. Thus, in the sequence of Proposition 3.5, $\partial = 0$ and f_* is injective. □

Proposition 3.7. *Let F be a functor from **C*ℝ-Alg** to **Ab** that is homotopy invariant, stable, and half-exact. Then, for any short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

there is a natural long exact sequence (with 24 distinct terms)

$$\cdots \to F_{n+1}(C) \xrightarrow{\partial} F_n(A) \xrightarrow{f_*} F_n(B) \xrightarrow{g_*} F_n(C) \xrightarrow{\partial} F_{n-1}(A) \to \cdots .$$

Proof. From Corollary 3.6 and Corollary 3.3, F is periodic; so Proposition 3.5 gives the long exact sequence. □

We say that a homotopy invariant, stable, half-exact functor F from **C*ℝ-Alg** to the category **Ab** of abelian groups:

(v) satisfies the *dimension axiom* if there is an isomorphism $F_*(\mathbb{R}) \cong K_*(\mathbb{R})$ as graded modules over $K_*(\mathbb{R})$;

(vi) is *continuous* if, for any direct sequence of R^*-algebras (A_n, ϕ_n), the natural homomorphism

$$\lim_{n\to\infty} F_*(A_n) \to F_*\big(\lim_{n\to\infty} (A_n)\big)$$

is an isomorphism.

Theorem 3.8. *Let F be a functor from **C*ℝ-Alg** to **Ab** that is homotopy invariant, stable, half-exact, and satisfies the dimension axiom. Then there is a natural transformation $\beta\colon K_n(A) \to F_n(A)$. If F is also continuous, then β is an isomorphism for all R^*-algebras in the smallest class of separable R^*-algebras which contains $\mathbb{R}$ and is closed under KK-equivalence, countable inductive limits, and the two-out-of-three rule for exact sequences.*

Proof. Let z be a generator of $F(\mathbb{R}) \cong \mathbb{Z}$, and for $x \in K_n(A) \cong KK(\mathbb{R}, S^n A)$ define a $K_*(\mathbb{R})$-module homomorphism $\beta\colon K_*(A) \to F_*(A)$ by $\beta(x) = \alpha(z \otimes x)$. Taking $A = \mathbb{R}$, Theorem 3.2 yields that $\beta(1_0) = z$ where 1_0 is the unit of the ring $K_*(\mathbb{R}) = KK_*(\mathbb{R}, \mathbb{R})$. Therefore, β is an isomorphism for $A = \mathbb{R}$. Then bootstrapping arguments show that β is an isomorphism for all R^*-algebras in the class described. □

For any homotopy invariant, stable, split exact functor F on **C*ℝ-Alg**, define the united F-theory of an R^*-algebra A to be

$$F^{CRT}(A) = \big\{ F_*(A), F_*(\mathbb{C} \otimes A), F_*(T \otimes A) \big\}$$

with the module-structure given by multiplication by elements of $KK_*(X,Y)$ with $X,Y \in \{\mathbb{R},\mathbb{C},T\}$ via the pairing of Theorem 3.2.

Proposition 3.9. *Let F be a homotopy invariant, stable, split exact functor from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to **Ab**, and let A be a separable R^*-algebra. Then $F^{CRT}(A)$ is a CRT-module. Moreover, if in addition F is half-exact, then $F^{CRT}(A)$ is acyclic.*

Proof. The first statement follows immediately, and the second statement follows from the exact sequences

$$\cdots \to F_n(A) \xrightarrow{\eta_O} F_{n+1}(A) \xrightarrow{c} F_{n+1}(\mathbb{C}\otimes A) \xrightarrow{r\beta_U^{-1}} F_{n-1}(A) \to \cdots$$
$$\cdots \to F_n(A) \xrightarrow{\eta_O^2} F_{n+2}(A) \xrightarrow{\varepsilon} F_{n+2}(T\otimes A) \xrightarrow{\tau\beta_T^{-1}} F_{n-1}(A) \to \cdots$$
$$\cdots \to F_{n+1}(\mathbb{C}\otimes A) \xrightarrow{\gamma} F_n(T\otimes A) \xrightarrow{\zeta} F_n(\mathbb{C}\otimes A) \xrightarrow{1-\psi_U} F_n(\mathbb{C}\otimes A) \to \cdots$$

which arise from the short exact sequences

$$0 \to S^{-1}\mathbb{R}\otimes A \to \mathbb{R}\otimes A \to \mathbb{C}\otimes A \to 0$$
$$0 \to S^{-2}\mathbb{R}\otimes A \to \mathbb{R}\otimes A \to T\otimes A \to 0$$
$$0 \to S\mathbb{C}\otimes A \to T\otimes A \to \mathbb{C}\otimes A \to 0$$

as in Sections 1.2 and 1.4 of [3]. □

Finally, we obtain the following reduction theorem, which is a formalization of a common argument used to reduce results from the complex case to the real case, such as those found in [1], [3], and [4].

Theorem 3.10. *Let F and G be homotopy invariant, stable, half-exact functors from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to **Ab** with a natural transformation $\mu_A\colon F(A) \to G(A)$. If μ_A is an isomorphism for all C^*-algebras A in $\boldsymbol{C^*}\mathbb{R}$**-Alg**, then μ_A is an isomorphism for all R^*-algebras in $\boldsymbol{C^*}\mathbb{R}$**-Alg**.*

Proof. Let A be a separable R^*-algebra. The natural transformation μ_A induces a homomorphism ${\mu_A}^{CRT}\colon F^{CRT}(A) \to G^{CRT}(A)$ of acyclic CRT-modules which is, by hypothesis, an isomorphism on the complex part. Then the results in [6, Section 2.3] imply that ${\mu_A}^{CRT}$ is an isomorphism. □

Corollary 3.11. *Any homotopy invariant, stable, half-exact functor from $\boldsymbol{C^*}\mathbb{R}$**-Alg** to **Ab** that vanishes on all C^*-algebras vanishes on all R^*-algebras.*

4. Asymptotic morphisms and E-theory

Since much of the theory of asymptotic morphisms and E-theory carries through in the real case exactly as in the complex case, we highlight the main definitions and results without proof. The goal of this section is to show that $KK(A,B)$ is naturally isomorphic to $E(A,B)$ for separable R^*-algebras A and B, when A is nuclear. At the end, we take advantage of Theorem 3.10 to obtain this theorem, saving us the need to check that all of the details in the complex case carry through to the real case. We note that asymptotic morphisms for R^*-algebras are also discussed in [5, Section 8].

Definition 4.1. Let A and B be R^*-algebras. An *asymptotic morphism* from A to B is a family $\langle \phi_t \rangle$ (for $t \in [1, \infty)$) of maps from A to B with the following properties:

(1) for all $a \in A$, $t \mapsto \phi_t(a)$ is bounded and continuous; and
(2) the set $\langle \phi_t \rangle$ is asymptotically $*$-linear and multiplicative; that is,
  (a) $\lim_{t\to\infty} \|\phi_t(\lambda a + b) - (\lambda \phi_t(a) + \phi_t(b))\| = 0$,
  (b) $\lim_{t\to\infty} \|\phi_t(a^*) - \phi_t(a)^*\| = 0$, and
  (c) $\lim_{t\to\infty} \|\phi_t(ab) - \phi_t(a)\phi_t(b)\| = 0$,
  for all $a, b \in A$ and $\lambda \in \mathbb{R}$.

Two asymptotic morphisms $\langle \phi_t \rangle, \langle \psi_t \rangle : A \to B$ are said to be *equivalent* if

$$\lim_{t\to\infty} \big(\phi_t(a) - \psi_t(a)\big) = 0$$

for all $a \in A$; and they are said to be *homotopic* if there exists an asymptotic morphism $\langle \Phi_t \rangle : A \to IB$ such that, for each $t \in [1, \infty)$ and $a \in A$,

$$\mathrm{ev}_0\big(\Phi_t(a)\big) = \phi_t(a) \qquad \text{and} \qquad \mathrm{ev}_1\big(\Phi_t(a)\big) = \psi_t(a),$$

where $\mathrm{ev}_i \colon IB \to B$ are the evaluation maps for $i = 0, 1$.

As in the complex case, equivalent asymptotic morphisms are always homotopic (see [2, Section 25.1.2(g)]). If ϕ is a $*$-homomorphism from A to B, then $\langle \phi \rangle$ denotes the corresponding constant asymptotic morphism. Also, as in Remark 25.1.4(a) of [2], there is a one-to-one correspondence between equivalence classes of asymptotic morphisms from A to B, and $*$-homomorphisms from A to $B_\infty = C_b([1, \infty), B)/C_0([1, \infty), B)$.

Definition 4.2. For R^*-algebras A and B, let $[[A, B]]$ denote the set of homotopy classes of asymptotic morphisms from A to B. For an asymptotic morphism $\langle \phi_t \rangle$, we will use $[\langle \phi_t \rangle]$ to denote the class in $[[A, B]]$ represented by $\langle \phi_t \rangle$. We define $E(A, B) = [[SA, \mathcal{K}_\mathbb{R} \otimes SB]]$.

There is a natural isomorphism between $[[A, \mathcal{K}_\mathbb{R} \otimes B]]$ and $[[\mathcal{K}_\mathbb{R} \otimes A, \mathcal{K}_\mathbb{R} \otimes B]]$ (see [2, Section 25.4.1]). Also as in the complex case, the set $[[A, \mathcal{K}_\mathbb{R} \otimes SB]]$ has the structure of an abelian group, defined using a chosen $*$-isomorphism $M_2(\mathcal{K}_\mathbb{R}) \cong \mathcal{K}_\mathbb{R}$.

Let $\mathfrak{e} : 0 \to B \xrightarrow{\iota} E \xrightarrow{\pi} A \to 0$ be an exact sequence of R^*-algebras. Suppose that B has a continuous approximate identity $\{u_t\}_{t\in[1,\infty)}$ which is quasicentral in E. Let σ be a bounded continuous cross section of π (the existence of which is given by the (real) Bartle–Graves selection theorem). Then the formula

$$\phi_t^\mathfrak{e}(f \otimes a) = f(u_t)\sigma(a), \quad a \in A, f \in S\mathbb{R},$$

defines an asymptotic morphism $\langle \phi_t^\mathfrak{e} \rangle : SA \to B$ and corresponding element $[\langle \phi_t^\mathfrak{e} \rangle] = \varepsilon_\mathfrak{e}$ in $[[SA, B]]$.

Proposition 4.3 ([2, Proposition 25.5.1])**.** *Let $\mathfrak{e} : 0 \to B \xrightarrow{\iota} E \xrightarrow{\pi} A \to 0$ be an exact sequence of R^*-algebras, where B is separable. Then the class $\varepsilon_\mathfrak{e}$ is independent of the choices of continuous approximate identity $\{u_t\}_{t\in[1,\infty)}$ and the cross section σ.*

In the real case, we also have the tensor product construction for asymptotic morphisms described in Lemma II.B.β.5 of [7]. Given two asymptotic morphisms $\langle\phi_t\rangle\colon A \to C$ and $\langle\psi_t\rangle\colon B \to D$, there is an asymptotic morphism $\langle(\phi\otimes\psi)_t\rangle\colon A\otimes_{\max} B \to C\otimes_{\max} D$ which satisfies

$$\lim_{t\to\infty}\big((\phi\otimes\psi)_t(a\otimes b) - \phi_t(a)\otimes\psi_t(b)\big) = 0,$$

for all $a \in A$ and $b \in B$.

As a special case, any asymptotic morphism $\langle\phi_t\rangle\colon A \to B$ yields a suspension asymptotic morphism from SA to SB, producing a natural element of $E(A,B)$. Similarly, the suspension construction produces a well-defined map $\Sigma\colon E(A,B) \to E(SA,SB)$, which can easily be shown to be a group homomorphism. Later in this section we show that Σ is an isomorphism.

The associative product structure on E-theory described in Proposition II.B.β.4 of [7] also carries over to the case of R^*-algebras. Given two asymptotic morphisms $\langle\phi_t\rangle\colon A \to B$ and $\langle\psi_t\rangle\colon B \to C$, there is a composition asymptotic morphism $\langle\psi\circ\phi\rangle_t\colon A \to C$, defined uniquely up to homotopy. In the special case that ψ or ϕ is an actual $*$-homomorphism, this product is a literal composition. In the general case, a reparameterization is necessary to construct an asymptotic morphism from the composition (see [2, Section 25.3]). The resulting product induces a natural homomorphism $E(A,B)\otimes E(B,C) \to E(A,C)$.

Theorem 4.4. *$E(A,B)$ is a bivariant functor from separable R^*-algebras to abelian groups. In both arguments, it is homotopy invariant, stable, half-exact, and has a degree 8 periodicity isomorphism.*

Proof. The homotopy invariance is immediate, and the stability follows from Proposition 3.1. By Proposition 4.3, any extension

$$\mathfrak{e} : 0 \to J \to A \xrightarrow{q} B \to 0$$

gives rise to a well-defined asymptotic morphism $\langle\phi_t^{\mathfrak{e}}\rangle$ from SB to J. Then the proofs leading up to, and including, Corollary 25.5.7 of [2] carry over to the real case to show that the functor $E(A,\cdot)$ is a split exact functor for fixed separable A. Then by Theorem 3.2 (of the present paper) there is a bilinear pairing $E(A,B)\otimes KK(B,C) \to E(A,C)$. Since this map is associative, multiplication by the Bott element in $KK(\mathbb{R}, S^8\mathbb{R})$ induces a periodicity isomorphism in the second argument of $E(\cdot,\cdot)$. Similarly, $E(\cdot,B)$ is also split exact, and so by the comments following Theorem 3.2 there is a natural pairing, $KK(A,B)\otimes E(B,C) \to E(A,C)$, proving periodicity in the first argument.

We postpone the proof of half-exactness until after the following lemma. $\square$

For any elements $x \in E(A,B)$ and $z \in KK(B,C)$, the pairing described in the proof above gives an element in $E(A,C)$ which we will denote by $x\otimes_\alpha z$. Taking $1_A \in E(A,A)$ we obtain a homomorphism $\varepsilon\colon KK(A,B) \to E(A,B)$. For $x \in E(A,B)$ and $y \in E(B,C)$, we let $x\otimes_E y$ denote the product in $E(A,C)$. Similarly for $z \in KK(A,B)$ and $w \in KK(B,C)$, we let $z\otimes_{KK} w$ denote the product in $KK(A,C)$. For $x \in KK(A,B)$ and $y \in KK(B,C)$ it can easily be shown that $\varepsilon(x\otimes_{KK} y) = \varepsilon(x)\otimes_E\varepsilon(y)$.

Lemma 4.5. *For any R^*-algebras, $\Sigma\colon E(A,B) \to E(SA,SB)$ is an isomorphism.*

Proof. Let $\alpha \in E(\mathbb{R}, S^8\mathbb{R})$ and $\beta \in E(S^8\mathbb{R}, \mathbb{R})$ be Bott elements arising from the corresponding Bott elements in KK-theory via ε. Since ε is multiplicative, it follows that these elements satisfy $\alpha \otimes_E \beta = 1_{\mathbb{R}}$ and $\beta \otimes \alpha = 1_{S^8\mathbb{R}}$. It follows that the map $z \mapsto (\alpha \otimes 1_A) \otimes_E z \otimes (\beta \otimes 1_B)$ is an isomorphism Θ from $E(S^8A, S^8B)$ to $E(A,B)$.

It can easily be shown that $\Theta \circ \Sigma^8 = \mathrm{id}_{E(A,B)}$ and that $\Sigma^8 \circ \Theta = \mathrm{id}_{E(S^8A,S^8B)}$. Hence $\Sigma\colon E(S^nA, S^nB) \to E(S^{n+1}A, S^{n+1}B)$ is an isomorphism if $n \geq 7$. Finally, $\Sigma\colon E(A,B) \to E(SA,SB)$ can be shown to be an isomorphism using the diagram

$$\begin{tikzcd}
E(A,B) \arrow[r, "\Sigma"] & E(SA,SB) \\
E(S^8A,S^8B) \arrow[u, "\Theta"] \arrow[r, "\Sigma"] & E(S^9A,S^9B) \arrow[u, "\Theta"']
\end{tikzcd}$$

The vertical maps and the lower map are all known to be isomorphisms. This diagram commutes modulo a homomorphism induced by the rearrangement of the order of the suspension factors of S^9A and S^9B. Since the rearrangement of the factors corresponds to an even permutation, it is homotopic to the identity and induces an identity homomorphism on E-theory. It follows that the $\Sigma\colon E(A,B) \to E(SA,SB)$ is an isomorphism. $\square$

Completion of the proof of Theorem 4.4. Finally, to prove half-exactness, let $0 \to J \xrightarrow{\iota} A \xrightarrow{q} B \to 0$ be an extension of separable R^*-algebras, and let D be an R^*-algebra. Suppose that h is an asymptotic morphism from $SD \otimes \mathcal{K}_{\mathbb{R}}$ to $SA \otimes \mathcal{K}_{\mathbb{R}}$ such that $[q \circ h] = [0]$. Lemma 25.5.12 of [2] and its proof carry over to the real case, so there exists an asymptotic morphism k from $S^2D \otimes \mathcal{K}_{\mathbb{R}}$ to $S^2J \otimes \mathcal{K}_{\mathbb{R}}$ such that $[S\iota \circ k] = [Sh]$. In the commutative diagram

$$\begin{tikzcd}
E(D,J) \arrow[r, "\iota_*"] \arrow[d, "\Sigma"] & E(D,A) \arrow[r, "q_*"] \arrow[d, "\Sigma"] & E(D,B) \arrow[d, "\Sigma"] \\
E(SD,SJ) \arrow[r, "\iota_*"] & E(SD,SA) \arrow[r, "q_*"] & E(SD,SB)
\end{tikzcd}$$

the vertical maps are isomorphisms, so there exists an asymptotic morphism g from $SD \otimes \mathcal{K}_{\mathbb{R}}$ to $SJ \otimes \mathcal{K}_{\mathbb{R}}$ such that $[\iota \circ g] = [h]$ in $E(D,A)$.

Half-exactness in the first argument is proved in a similar way. $\square$

Theorem 4.6. *Let A be a separable, nuclear R^*-algebra, and let B be a separable R^*-algebra. Then the homomorphism $\varepsilon\colon KK(A,B) \to E(A,B)$ is an isomorphism.*

Proof. By Theorem 3.10, it suffices to show that $KK(A, B) \to E(A, B)$ is an isomorphism when B is a C^*-algebra. In the diagram

$$\begin{tikzcd} KK^{\mathbb{C}}(A_{\mathbb{C}}, B) \arrow[r] \arrow[d] & E^{\mathbb{C}}(A_{\mathbb{C}}, B) \arrow[d] \\ KK(A, B) \arrow[r] & E(A, B) \end{tikzcd}$$

we use $KK^{\mathbb{C}}(-,-)$ and $E^{\mathbb{C}}(-,-)$ to denote the versions of these functors on C^*-algebras (e.g., $E^{\mathbb{C}}(-,-)$ consists of homotopy classes of asymptotic morphisms that are asymptotically linear over $\mathbb{C}$). Since $A_{\mathbb{C}}$ is nuclear, the top horizontal homomorphism is an isomorphism by Theorem 25.6.3 of [2]. The left vertical homomorphism is an isomorphism by Lemma 4.3 of [4]. The right vertical homomorphism is defined by restriction—a complex asymptotic morphism defined on $SA_{\mathbb{C}} \otimes \mathcal{K}$ restricts to a real asymptotic morphism defined on $SA \otimes \mathcal{K}_{\mathbb{R}}$—and it is an isomorphism, since every real asymptotic morphism on $SA \otimes \mathcal{K}_{\mathbb{R}}$ can be extended uniquely to a complex asymptotic morphism on $SA_{\mathbb{C}} \otimes \mathcal{K}$. The square commutes by Theorem 3.7 of [14], since the two directions around the square give natural transformations $KK^{\mathbb{C}}(A_{\mathbb{C}}, -) \to E(A, -)$ of functors defined on separable C^*-algebras, each of which sends $1_A \in KK^{\mathbb{C}}(A_{\mathbb{C}}, A_{\mathbb{C}})$ to the asymptotic morphism represented by the inclusion of A into $A_{\mathbb{C}}$. Therefore, the bottom row is an isomorphism, as desired. □

Finally, we mention that, with a similar application of Theorem 3.10, we can easily obtain the following real analogue of Theorem 5.8 of [22], showing that E-theory is a special case of KK-theory. Since $\mathcal{M}(B \otimes \mathcal{K}_{\mathbb{R}})$ is KK-trivial, it follows that $E(A, \mathcal{M}(B \otimes \mathcal{K}_{\mathbb{R}})) = 0$. Then we use the long exact sequence arising from

$$0 \to B \otimes \mathcal{K}_{\mathbb{R}} \to \mathcal{M}(B \otimes \mathcal{K}_{\mathbb{R}}) \to \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}}) \to 0$$

to get an isomorphism $E_0(A, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})) \cong E_{-1}(A, B \otimes \mathcal{K}_{\mathbb{R}})$. Combining this with stability and with Lemma 4.5, there is an isomorphism

$$\gamma \colon E\big(S^{-1}A, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})\big) \to E(A, B)$$

for R^*-algebras A and B.

Theorem 4.7. *Let A and B be separable R^*-algebras. Then there is an isomorphism*

$$\varepsilon' \colon KK\big(S^{-1}A, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})\big) \to E(A, B).$$

Proof. Let $\varepsilon' = \gamma \circ \varepsilon$, where ε is the isomorphism of Theorem 4.6. We know that ε' is an isomorphism in the complex case by Theorem 5.8 of [22]. So it suffices by Theorem 3.10 to show that ε' fits in a commutative square in the same way that ε does in the proof of Theorem 4.6. The square in question can be factored into two squares as follows:

$$\begin{tikzcd}
KK^{\mathbb{C}}\big(S^{-1}A_{\mathbb{C}}, \mathcal{Q}(B\otimes\mathcal{K}_{\mathbb{R}})\big) \arrow[r] \arrow[d] & E^{\mathbb{C}}\big(S^{-1}A_{\mathbb{C}}, \mathcal{Q}(B\otimes\mathcal{K}_{\mathbb{R}})\big) \arrow[r] \arrow[d] & E^{\mathbb{C}}(A_{\mathbb{C}}, B) \arrow[d] \\
KK\big(S^{-1}A, \mathcal{Q}(B\otimes\mathcal{K}_{\mathbb{R}})\big) \arrow[r] & E\big(S^{-1}A, \mathcal{Q}(B\otimes\mathcal{K}_{\mathbb{R}})\big) \arrow[r] & E(A, B)
\end{tikzcd}$$

The first square is just a specialization of the square in Theorem 4.6. The second square commutes, since the horizontal maps are homomorphisms that arise from stabilization, from long exact sequences, and from suspensions, all of which commute with the vertical restriction map. □

5. Application: Asymptotic morphisms on spheres

The goal of this section is to determine when there exist asymptotic morphisms from suspensions of $\mathbb{R}$ to $\mathcal{K}_{\mathbb{R}}$ and to $\mathcal{K}_{\mathbb{R}}\otimes\mathbb{H}$ that are detected by K-theory. For this we will use the results of the previous sections, the universal coefficient theorem for real C^*-algebras, and a united K-theory analysis. The main theorem of this section is the following.

Theorem 5.1. *Let $d \in \mathbb{N}$. There exists an asymptotic morphism $\langle \phi_t \rangle$ that induces a nontrivial homomorphism on K-theory of the form*

1. $S^d\mathbb{R} \to \mathcal{K}_{\mathbb{R}}$ *if and only if* $d \equiv 0, 4, 6, 7 \pmod 8$,
2. $S^d\mathbb{R} \to \mathcal{K}_{\mathbb{R}} \otimes \mathbb{H}$ *if and only if* $d \equiv 0, 2, 3, 4 \pmod 8$.

The correspondence $\langle \phi_t \rangle \mapsto \langle \phi_t \rangle_*$ gives a map

$$\big[[A, B]\big] \to \mathrm{Hom}_{\,CRT}\big(K^{CRT}(A), K^{CRT}(B)\big)$$

that respects compositions the expected way and, when B is stable, is a homomorphism of semigroups. Therefore, there is a group homomorphism γ' defined on $E(A, B)$ that can be shown, using Theorem 3.2, to commute in the following diagram:

$$\begin{tikzcd}
KK(A,B) \arrow[rr, "\gamma"] \arrow[dr, "\varepsilon"'] & & \mathrm{Hom}_{\,CRT}\big(K^{CRT}(A), K^{CRT}(B)\big) \\
& E(A,B) \arrow[ur, "\gamma'"'] &
\end{tikzcd}$$

Now, in the case that the first argument is a suspension algebra SA, we can further factor ε as in the following diagram:

$$\begin{tikzcd}
& KK(SA,B) \arrow[dl, "\varepsilon'"'] \arrow[dr, "\varepsilon"] & \\
\big[[SA, \mathcal{K}_{\mathbb{R}} \otimes B]\big] \arrow[rr, "\Sigma"] & & E(SA,B) = \big[[S^2A, \mathcal{K}_{\mathbb{R}} \otimes SB]\big]
\end{tikzcd}$$

where the map ε' is described as follows. An element of $KK(SA, B)$ is associated with an extension $\mathfrak{e} : 0 \to B \xrightarrow{\iota} E \xrightarrow{\pi} A \to 0$ using the isomorphism $KK(SA, B) \cong \mathrm{Ext}(A, B)^{-1}$ of Kasparov (see [17, Theorem 7.1]). Then the construction of Theorem 4.3 produces an asymptotic morphism $\varepsilon_{\mathfrak{e}}$ from SA to B, giving an element of $[[SA, \mathcal{K}_{\mathbb{R}} \otimes B]]$. Again with a little work, Theorem 3.2 can be used to show that the diagram commutes (see also the comment following Corollary 25.5.8 in [2]).

We are not claiming to have shown that ε' is an isomorphism, although that is the case in the complex setting (see [11, Corollary 5.3]). We hope to address unsuspended E-theory for R^*-algebras more thoroughly in future work. For our current purposes, it is enough to know that γ factors through ε'.

Theorem 5.2. *Let $d \in \mathbb{N}$. There exists a nontrivial CRT-homomorphism of degree 0 of the form*

(1) $K^{CRT}(S^d\mathbb{R}) \to K^{CRT}(\mathbb{R})$ *if and only if* $d \equiv 0, 4, 6, 7 \pmod 8$*;*
(2) $K^{CRT}(S^d\mathbb{R}) \to K^{CRT}(\mathbb{H})$ *if and only if* $d \equiv 0, 2, 3, 4 \pmod 8$*.*

Proof. The CRT-module $K^{CRT}(S^d\mathbb{R})$ is a free CRT-module with a single generator in the real part in degree $-d$. Hence there exists a nontrivial CRT-module homomorphism $K^{CRT}(S^d\mathbb{R}) \to M$ if and only if $M_O^{-d} \neq 0$. Now $K_*(\mathbb{R})$ is nonzero in and only in degrees $0, 1, 2,$ and $4 \pmod 8$; and $K_*(\mathbb{H})$ is nonzero in and only in degrees $0, 4, 5,$ and $6 \pmod 8$. Thus parts (1) and (2) follow, as well as the converse statements. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. For $d \geq 1$, where d is not in one of the required congruence classes, Theorem 5.2 implies that no asymptotic morphism can exist that induces a nontrivial homomorphism on K-theory.

Now suppose that $d \geq 1$, and suppose that $A = S^d\mathbb{R}$ and $B = \mathbb{R}$ or $B = \mathbb{H}$ are algebras such that the form $A \to B$ matches the conditions of the statement of Theorem 5.1. By Theorem 5.2, there is a nonzero homomorphism $K^{CRT}(A) \to K^{CRT}(B)$, and, by the universal coefficient theorem for real C^*-algebras (see [4, Theorem 1.1]), this CRT-module homomorphism is induced by a nonzero element $\xi \in KK(A, B)$. Then $\varepsilon'(\xi)$ is a class in $[[A, \mathcal{K}_{\mathbb{R}} \otimes B]]$, and when we choose a representative we obtain an asymptotic morphism that induces a nontrivial map on united K-theory. □

6. Almost commuting matrices

We now use this machinery to find novel examples of almost commuting real symmetric matrices. Our approach is to use commutative R^*-algebras and create asymptotic morphisms out of these. On the one hand, these carry K-theory data that can distinguish them from actual $*$-homomorphisms. On the other hand, in the image, the relations that make the R^*-algebra commutative get turned into the property of being almost commuting.

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, and let $d \in \mathbb{N}$. We generalize a question of Halmos [12, Section 4] to ask about d almost commuting self-adjoint matrices over $\mathbb{F}$.

Problem 6.1. For all $\varepsilon > 0$, does there exist $\delta > 0$ so that, for all n, given d self-adjoint contractions H_r in $\mathbf{M}_n(\mathbb{F})$ such that

$$\left\| [H_r, H_s] \right\| \leq \delta,$$

there exist d self-adjoint contractions K_r with $\|K_r - H_r\| \leq \varepsilon$ and

$$[K_r, K_s] = 0?$$

In the complex case, Lin [18] showed that in the anwer is "yes" for $d = 2$, while it was known much earlier (see [26]) that the answer is "no" for $d = 3$. For the quaternionic case, the result is the same: "yes" for $d = 2$ (see [20]) and "no" for $d = 3$ (see [13]).

These leave the real case, which is arguably the most important case. We know the answer is "yes" for $d = 2$ (see [20]). We will show that the answer is "no" for $d = 5$, leaving open the cases $d = 3, 4$. The proof techniques used for a negative result for $d = 3$ in the complex and quaternionic cases rely on the fact that $K_{-2}(\mathbb{H}) \neq 0$ and $K_{-2}(\mathbb{C}) \neq 0$ and so will not work for $\mathbb{F} = \mathbb{R}$ since $K_{-2}(\mathbb{R}) = 0$. However, since $K_{-4}(\mathbb{R})$ is nontrivial, we will see that these methods apply for $d = 5$.

We start by connecting this problem to a problem couched in the theory of R^*-algebras. For any sequence B_n of R^*-algebras, let π be the quotient map from the product $\prod_{n=1}^{\infty} B_n$ to its quotient by the sum $\prod_{n=1}^{\infty} B_n / \bigoplus_{n=1}^{\infty} B_n$.

Problem 6.2. Does every $*$-homomorphism of the form

$$\psi\colon S^{d-1}\mathbb{R} \to \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}),$$

where $\{m(n)\}_{n=1}^{\infty}$ is a sequence of integers, lift to a $*$-homomorphism

$$\tilde{\psi}\colon S^{d-1}\mathbb{R} \to \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

such that $\psi = \pi \circ \widetilde{\psi}$?

Theorem 6.3. *For a fixed positive integer d and division algebra $\mathbb{F}$, if the answer to Problem 6.1 is "yes," then the answer to Problem 6.2 is also "yes."*

We prove Theorem 6.3 below, following this basic lemma stating the universal properties of $C(S^{d-1}, \mathbb{R})$. This lemma can be proved using the techniques of Chapter 3 of [19].

Lemma 6.4. *If $h_1, \ldots, h_d$ are commuting self-adjoint contractions in an R^*-algebra A that satisfy $\sum_{j=1}^{d} h_j^2 = 1$, then there is a unique $*$-homomorphism $\psi\colon C(S^{d-1}, \mathbb{R}) \to A$ sending the jth coordinate function f_j to h_j.*

Proof of Theorem 6.3. Suppose that the answer to Problem 6.1 is "yes" for some d and $\mathbb{F}$, and let

$$\psi\colon C(S^{d-1}, \mathbb{F}) \to \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

be a $*$-homomorphism. Then taking the images of the coordinate functions in $C(S^{d-1}, \mathbb{R})$ and lifting them to representatives in $\prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$, we find that there exist sequences of matrices $H_{in} \in \mathbf{M}_{m(n)}(\mathbb{R})$ ($i \in \{1, \ldots, d\}$, $n \in \mathbb{N}$) that are asymptotically (as $n \to \infty$) self-adjoint contractions, that satisfy $\sum_{i=1}^{d} H_{in}^2 = 1$

asymptotically, and such that H_{in} and H_{jn} asymptotically commute for each $i, j \in \{1, \ldots, d\}$. We may assume that each H_{in} is exactly self-adjoint by replacing H_{in} by $\frac{1}{2}(H_{in} + H_{in}^*)$.

By our hypothesis, there exist sequences of self-conjugate contractions $K_{in} \in \mathbf{M}_{m(n)}(\mathbb{F})$ that exactly commute and satisfy $\lim_{n\to\infty}(H_{in} - K_{in}) = 0$ for each i. Furthermore, by normalizing, we may assume that $\sum_{i=1}^{d} K_{in}^2 = 1$ holds for each n.

Then, by Lemma 6.4, there exist $*$-homomorphisms $\psi_n'\colon C(S^{d-1}, \mathbb{F}) \to \mathbf{M}_{m(n)}(\mathbb{F})$ that map the d coordinate functions to K_{in}, which together form the desired lift of ψ. □

The rest of the section is devoted to showing that the answer to Problem 6.2 is "no" when $d = 5$, using a K-theoretic obstruction.

Lemma 6.5. *Given an asymptotic homomorphism $\langle\phi\rangle$ from A to $\mathcal{K}_{\mathbb{F}}$ such that $\langle\phi\rangle_*$ is nonzero, there exists a homomorphism $\psi\colon A \to \prod_{n=1}^{\infty} \mathcal{K}_{\mathbb{F}} / \bigoplus_{n=1}^{\infty} \mathcal{K}_{\mathbb{F}}$ such that ψ_* is nonzero.*

Proof. Given an asymptotic homomorphism $\langle\phi\rangle$ from A to $B = \mathcal{K}_{\mathbb{F}}$, there is a corresponding homomorphism from A to $B_\infty = C_b([1,\infty))/C_0([1,\infty))$, as discussed in Section 4. By evaluating at the positive integers, we also obtain a discrete version, that is, a map ψ from A to $B_\infty^{\mathrm{d}} = \prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B$. Note that, as in the complex case (see [10, Section 3.2]), we have $K^{CRT}(B_\infty^{\mathrm{d}}) \cong \prod_{n=1}^{\infty} K^{CRT}(B) / \bigoplus_{n=1}^{\infty} K^{CRT}(B)$. But, as each evaluation map is homotopic to each other evaluation map, the map

$$\psi_*\colon K^{CRT}(A) \to \prod_{n=1}^{\infty} K^{CRT}(B) \Big/ \bigoplus_{n=1}^{\infty} K^{CRT}(B)$$

is given by

$$x \mapsto \pi \circ \Delta \circ \langle\phi\rangle_*(x),$$

where Δ is the diagonal map and π is the quotient map. In particular, if $\langle\phi\rangle_*$ is nonzero, then so is ψ_*. □

Lemma 6.6. *Suppose that B is a separable R^*-algebra, and suppose that A is a finitely generated R^*-subalgebra of*

$$A \subseteq \prod_{n=1}^{\infty} (B \otimes \mathcal{K}_{\mathbb{R}}) \Big/ \bigoplus_{n=1}^{\infty} (B \otimes \mathcal{K}_{\mathbb{R}}).$$

Then there is a sequence $m(1) < m(2) < \cdots$ on natural numbers so that

$$A \subseteq \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(B) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(B).$$

Proof. Given a single element $a \in A$, write $a = [(a_1, a_2, \ldots)]$, where $a_i \in B \otimes \mathcal{K}_{\mathbb{R}}$. We can choose an increasing sequence $p_1, p_2, \ldots$ of standard projections in $1 \otimes \mathcal{K}_{\mathbb{R}}$ (with 1 in $\tilde{B}$ if needed) so that

$$\|p_n a_n p_n - a_n\| \leq \frac{1}{n}$$

and so $[(a_1, a_2, \dots)] = [(p_1 a_1 p_1, p_2 a_2 p_2, \dots)]$. More generally, for a finite set of elements in A, we can use a single sequence of projections as above to show that

$$A \subseteq \prod_{n=1}^{\infty} p_n(B \otimes \mathcal{K}_{\mathbb{R}})p_n \Big/ \bigoplus_{n=1}^{\infty} p_n(B \otimes \mathcal{K}_{\mathbb{R}})p_n.$$

□

We are ready to prove our main theorem.

Theorem 6.7. *Suppose that $d \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ satisfy the hypotheses of Theorem 5.1. Then there is a sequence of integers $m(1), m(2), \dots$ and a unital $*$-homomorphism*

$$\phi : C(S^d, \mathbb{R}) \to \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

that cannot be lifted to a unital $*$-homomorphism to $\prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$.

Proof. Let d be as above. By Theorem 5.1, there exists an asymptotic morphism

$$\langle \phi_t \rangle : S^d\mathbb{R} \to \mathcal{K}_{\mathbb{F}}$$

that induces a nonzero map on K-theory. Then, by Lemma 6.5, we obtain a $*$-homomorphism of the form

$$\phi' \colon S^d\mathbb{R} \to \prod_{n=1}^{\infty} \mathcal{K}_{\mathbb{F}} \Big/ \bigoplus_{n=1}^{\infty} \mathcal{K}_{\mathbb{F}}$$

that is nonzero on K-theory. By Lemma 6.6, this $*$-homomorphism factors through a $*$-homomorphism of the form

$$\phi : S^d\mathbb{R} \to \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

that must also be nonzero on united K-theory.

Now, if ϕ could be lifted to a $*$-homomorphism with values in $\prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$, then such a lift would have to be nonzero on K-theory. However, as any homomorphism from S^d to $\mathcal{K}_F$ vanishes on K-theory, no such lift of ϕ exists.

Finally, extend unitally to form a $*$-homomorphism

$$\phi : C(S^d, \mathbb{R}) \to \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

that similarly cannot be lifted. □

Corollary 6.8. *For $d = 5$, $\mathbb{F} = \mathbb{R}$, and for $d = 3$, $\mathbb{F} = \mathbb{H}$, the answer to Problem 6.2, and hence also to Problem 6.1, is "no."*

Compare the result above for $d = 3$, $\mathbb{F} = \mathbb{H}$ to Theorem 1.3 of [20], which states that the answer is "yes" when the appropriate obstruction vanishes in $KO_{-2}(\mathbb{H}) \cong \mathbb{Z}_2$.

7. Pictures of K-theory

It is standard practice to represent K_0 classes by projections and K_1 classes by unitaries. Beginning with the publication of [13], a picture has been developing in which all ten of the real and complex K-theory groups of an R^*-algebra can be represented concretely in terms of homotopy classes of unitary elements with certain symmetries. We finish this paper showing how our methods partially extend this picture to $K_{-1}(A)$ and $K_3(A)$, allowing us to represent any such element with a specific class of unitaries. We expect that a complete study of unsuspended E-theory in the case of R^*-algebras can be used to complete this picture, including a concrete description of all of the interrelating natural transformations and the boundary maps.

The table below summarizes this picture and extends Tables 7 and 8 of [13], showing the symmetries that are used to represent each K-group. Here A is an R^*-algebra and τ is the associated anti-automorphism of the complexification $A_{\mathbb{C}}$. Thus, the first line indicates that $KU_0(A)$ is isomorphic to the group of homotopy classes of self-adjoint unitaries in $M_\infty(A_{\mathbb{C}})$, while $KO_0(A)$ is isomorphic to the group of homotopy classes of self-adjoint unitaries satisfying $\tau(u) = u^*$ (a priori, u is in $A_{\mathbb{C}}$, but the second condition restricts it to A).

K-group	Unitary classes
$KU_0(A)$	$u = u^*$
$KU_1(A)$	–
$KO_{-1}(A)$	$u^\tau = u$
$KO_0(A)$	$u = u^*$, $u^\tau = u^*$
$KO_1(A)$	$u^\tau = u^*$
$KO_2(A)$	$u = u^*$, $u^\tau = -u$
$KO_3(A)$	$u^{\tau\otimes\sharp} = u$
$KO_4(A)$	$u = u^*$, $u^{\tau\otimes\sharp} = u^*$
$KO_5(A)$	$u^{\tau\otimes\sharp} = u^*$
$KO_6(A)$	$u = u^*$, $u^{\tau\otimes\sharp} = -u$

In this table, the line with $KO_{-1}(A)$ comes from Theorem 7.2 below. The line for $KO_3(A)$ then arises via the isomorphism $KO_{n+4}(A) \cong KO_n(\mathbb{H} \otimes_{\mathbb{R}} A)$. Then the anti-automorphism of $A_{\mathbb{C}} \otimes_{\mathbb{C}} M_2(\mathbb{C})$ associated to $\mathbb{H} \otimes_{\mathbb{R}} A$ is $\sharp \otimes \tau\otimes$, where

$$\sharp\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

All the other lines of the table are discussed in [13].

Let A be an R^*-algebra, and let $A_{\mathbb{C}}$ be the complexification with anti-automorphism τ. Let $G(A)$ be the group of homotopy classes of unitaries $u \in M_\infty(\widetilde{A_{\mathbb{C}}})$ that satisfy $u^\tau = u$. The associated anti-automorphism on $M_n(A_{\mathbb{C}}) \cong M_n(\mathbb{C}) \otimes A_{\mathbb{C}}$ is $\mathrm{tr} \otimes \tau$, and we have $M_n(\widetilde{A})$ mapping into $M_{n+1}(\widetilde{A})$ by $u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 7.1. *For an R^*-algebra A, $G(A)$ is an abelian group.*

Proof. Let $R(t) = \left(\begin{smallmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{smallmatrix}\right)$ be the rotation matrix for $t \in [0,1]$. Since $R^{\mathrm{tr}} = R^*$, for any unitaries $u_1, u_2 \in A$ satisfying $u_i^\tau = u_i$, we have that $U(t) = R(t)\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)R(t)^*$ is a path of unitaries satisfying $U(t)^\tau = U(t)$ from $\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} u_2 & 0 \\ 0 & u_1 \end{smallmatrix}\right)$ showing that $G(A)$ is a commutative semigroup.

To show that there are inverses in $G(A)$, it suffices to show that $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$ is homotopic to 1_2 in $C(M_2(S^1))$, through homotopies satisfying $u^\tau = u$, taking the trivial involution on $C(S^1, \mathbb{C})$. Indeed, $z \in C(S^1, \mathbb{C})$ is the universal unitary satisfying $u^\tau = u$. It is easy to see that $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$ is homotopic to

$$\begin{cases} \left(\begin{smallmatrix} z^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \operatorname{Im} z \geq 0, \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & \bar{z}^2 \end{smallmatrix}\right) & \operatorname{Im} z < 0. \end{cases}$$

Using a variation of the rotation argument of the first paragraph, this is homotopic to

$$\begin{cases} \left(\begin{smallmatrix} z^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \operatorname{Im} z \geq 0, \\ \left(\begin{smallmatrix} \bar{z}^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \operatorname{Im} z < 0, \end{cases}$$

which is clearly homotopic to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. $\square$

Theorem 7.2. *There is a natural group homomorphism $\phi_A \colon K_{-1}(A) \to G(A)$ with a left inverse.*

This theorem will be proved easily from the machinery already established in this paper. It follows that any unitary u satisfying $u^\tau = u$ determines a class in $K_{-1}(A)$, that any class in $K_{-1}(A)$ arises from such a unitary in this way, and that two distinct K_{-1} classes must arise from distinct homotopy classes of unitaries. That ϕ_A is an isomorphism will be left to further work.

Proof of Theorem 7.2. We use the homomorphism

$$KK(S\mathbb{R}, A) \xrightarrow{\varepsilon'} \big[[S\mathbb{R}, \mathcal{K}_\mathbb{R} \otimes_\mathbb{R} A]\big]$$

discussed in Section 5, which was shown there to have a left inverse. As there is a natural isomorphism $K_{-1}(A) \cong KK(S\mathbb{R}, A)$, it only remains to establish an isomorphism between $[[S\mathbb{R}, K_\mathbb{R} \otimes_\mathbb{R} A]]$ and $G(A)$.

From Section 3 of [21] we know that $S\mathbb{R}$ is semiprojective, which gives us the first isomorphisms in the following chain:

$$\begin{aligned} \big[[S\mathbb{R}, K_\mathbb{R} \otimes_\mathbb{R} A]\big] &\cong [S\mathbb{R}, \mathcal{K}_\mathbb{R} \otimes_\mathbb{R} A] \\ &\cong [\widetilde{S\mathbb{R}}, \widetilde{\mathcal{K}_\mathbb{R} \otimes_\mathbb{R} A}]^+ \\ &\cong \big[(\widetilde{S\mathbb{C}}, \mathrm{id}), (\widetilde{\mathcal{K} \otimes A_\mathbb{C}}, \mathrm{id} \otimes \tau)\big]^+ \\ &\cong G(A). \end{aligned}$$

The third isomorphism comes from the categorical equivalence between (unital) R^*-algebras and (unital) C^*-algebras with anti-automorphism. The fourth isomorphism arises from the fact that $\widetilde{S\mathbb{C}} = C(S^1, \mathbb{C})$ is the universal C^*-algebra generated by a unitary u satisfying $u^\tau = u$, as in the proof of Lemma 7.1. $\square$

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