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INTERPOLATION OF S-NUMBERS AND ENTROPY NUMBERS OF OPERATORS

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To Professor Ronald G. Douglas

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Abstract. We introduce the notion of $\vec{s}$-numbers for operators in Banach couples. We investigate variants of the approximation, Gelfand, and Kolmogorov numbers. In particular, we derive upper estimates of these numbers for operators between spaces generated by interpolation functors on Banach couples satisfying interpolation variants of approximation properties. We also study two-sided interpolation of entropy numbers.

1. Introduction

The problem of the behavior of s-numbers and entropy numbers of operators between interpolation spaces has received considerable attention in recent years. This problem has a long history and some related problems in the area are still open. We note that these numbers are used in various areas of analysis including the theory of asymptotic geometric analysis (see [21]) and spectral theory of operators. In particular, these numbers are useful tools in the study of the eigenvalues of operators in Banach spaces (see [10], [20]).

The main aim of this article is to study interpolation variants of some important s-numbers and show applications to the above-mentioned problem. For motivation, we list some known remarkable results. We start with some fundamental notation and definitions. Fix Banach spaces X and Y. In the space of all real

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sequences modeled on the set of positive integers $\mathbb{N}$, we consider sequences of numbers associated with an operator $T\colon X \to Y$. Particularly important are the approximation numbers $a_n(T)$, Gelfand numbers $c_n(T)$, Kolmogorov numbers $d_n(T)$, and entropy numbers $\varepsilon_n(T)$ defined, for each $n \in \mathbb{N}$, by the formulas

$$\begin{aligned} a_n(T) &:= \inf\bigl\{\|T - S\|; S \in \mathcal{L}(X,Y), \operatorname{rank}(S) < n\bigr\},\\ c_n(T) &:= \inf\bigl\{\|T|_G\|; G \subset X, \operatorname{codim}(G) < n\bigr\},\\ d_n(T) &:= \inf\bigl\{\|Q_N^Y T\|_{X\to Y/N}; N \subset Y, \dim(N) < n\bigr\}, \end{aligned}$$

where $Q_N\colon T \to Y/N$ is the quotient map, and

$$\varepsilon_n(T) = \inf\Bigl\{\varepsilon > 0; T(B_X) \subset \bigcup_{i=1}^{n}(y_i + \varepsilon B_Y) \text{ for some } y_1, \dots, y_n \in Y\Bigr\}.$$

The approximation numbers, Gelfand numbers, and Kolmogorov numbers are special cases of the so-called *s-numbers of operators* introduced by Pietsch. (For basic properties of these numbers, more background, and applications, we refer to the monographs [3], [19]–[21].) It was mentioned that these numbers are powerful tools for estimating eigenvalues of operators in Banach spaces (see [10], [20]). We recall that when $T\colon X \to X$ is a Riesz (in particular, a compact) operator acting on a complex Banach space X, then we denote by $(\lambda_n(T))$ the sequence of eigenvalues of X rearranged according to their algebraic multiplicity and so that $(|\lambda_n(T)|)$ is a nonincreasing sequence. If T has only finitely many nonzero eigenvalues $\lambda_1, \dots, \lambda_k$, then we set, by convention, $\lambda_n = 0$ for each $n > k$.

Let us quote two famous formulas that hold for every Riesz operator T acting on a complex Banach space X. The König formula (see [4]) states that, for any s-number sequence,

$$\lim_{k\to\infty} s_n(T^k)^{1/k} = \bigl|\lambda_n(T)\bigr|, \quad n \in \mathbb{N}.$$

Another formula due to Carl and Triebel [4] gives an estimate of eigenvalues by single entropy numbers

$$\Bigl(\prod_{i=1}^{n}\bigl|\lambda_i(T)\bigr|\Bigr)^{1/n} \le k^{\frac{1}{2n}}\varepsilon_k(T), \quad k, n \in \mathbb{N}.$$

The entropy numbers or s-numbers of some specific operator T are usually rather difficult to calculate, or even to estimate. Various methods are required, with interpolation methods playing a particularly significant role. It should be pointed out that resolution of the behavior of entropy numbers or a given s-number under interpolation, when nontrivial couples of Banach spaces are involved at each endpoint, is a difficult problem. Note that some results for the interpolation of Gelfand and Kolmogorov numbers are available when one endpoint is formed by a trivial couple (see [19, pp. 150, 152]). However, when both endpoints can vary this is not true in general, as the following example of Carl shows. Consider finite-dimensional complex Banach couples $(A_0, A_1) := (\ell_1^{3n}, \ell_1^{3n})$ and $(B_0, B_1) = (\ell_1^{3n}, \ell_\infty^{3n})$. Then for the complex interpolation spaces with $\theta = 1/2$, we have $A := [A_0, A_1]_\theta \cong \ell_1^{3n}$ and $B := [B_0, B_1]_\theta \cong \ell_2^{3n}$ isometrically. Since the

Kolmogorov numbers satisfy $d_n(\mathrm{id}\colon A_0 \to B_0) = 1$, $d_n(\mathrm{id}\colon A_1 \to B_1) \asymp 1/\sqrt{n}$ (see [9]), and $d_{2n-1}(\mathrm{id}\colon A \to B) \asymp 1/\sqrt{3}$ (see [19, Lemma 11.11.8]), it follows that there is no $C > 0$ independent of n such that

$$d_{2n-1}(\mathrm{id}\colon A \to B) \le C d_n(\mathrm{id}\colon A_0 \to B_0)^{1-\theta} d_n(\mathrm{id}\colon A_1 \to B_1)^{\theta} \asymp n^{-\frac{1}{4}}.$$

We note that it is shown in [23] that, in contrast to the situation in Banach spaces, the s-numbers of operators do interpolate well between Hilbert spaces at least in the case of the complex method. Those results suggest posing the problem on the two-sided interpolation of s-numbers as the problem of finding conditions on the Banach couples $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ and the interpolation functor $\mathcal{F}$ of exponent $\theta \in (0,1)$ under which there exists a function $\varphi\colon \mathbb{N} \times \mathbb{N} \to (0, \infty)$ (that may depend on $\mathcal{F}$, $\vec{X}$, and $\vec{Y}$) such that, for an operator $T\colon \vec{X} \to \vec{Y}$ and each m, $n \in \mathbb{N}$, the estimate

$$s_{m+n-1}(T\colon X \to Y) \le \varphi(m,n) s_m(T\colon X_0 \to Y_0)^{1-\theta} s_n(T\colon X_1 \to Y_1)^{\theta}$$

is valid for a given s-number with $X := \mathcal{F}(\vec{X})$ and $Y := \mathcal{F}(\vec{Y})$ generated by an interpolation functor $\mathcal{F}$. We note that the aforementioned Carl example shows that, with $m = n$, the factor $\varphi(m,n)$, which appears on the right-hand side of the above estimate, must grow at least like $n^{\frac{1}{4}}$.

It is interesting to point out a recent result of Edmunds and Netrusov [6] who showed (this result solved a long-standing question) that there is no positive constant $C = C(\theta)$ such that an inequality

$$\begin{aligned} &e_{m+n-1}\big(T\colon (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}\big) \\ &\quad \le C e_m(T\colon X_0 \to Y_0)^{1-\theta} e_n(T\colon X_1 \to Y_1)^{\theta}, \quad m, n \in \mathbb{N} \end{aligned}$$

is true for every operator $T\colon (X_0, X_0) \to (Y_0, Y_1)$. Here (e_n) is the sequence of *dyadic entropy numbers* of an operator between Banach spaces given by $e_n := \varepsilon_{2^{n-1}}$ for each $n \in \mathbb{N}$, and $(\cdot)_{\theta,q}$ denotes the real method of interpolation with $\theta \in (0,1)$, $q \in [1, \infty]$. This fact motivates a similar problem for entropy numbers instead of s-numbers: namely, how the numbers of an operator $T\colon \vec{X} \to \vec{Y}$ between nontrivial couples of Banach spaces behave under interpolation. It is still not completely clear under which general conditions two-sided interpolation problems have positive answers. Motivated by applications, our aim in the present article is to study the above-mentioned problems and some of their variants.

Let us now describe the paper and its contents. In Section 2, we introduce the new notion of the $\vec{s} = (\vec{s}_n)$-number sequence defined on a class of all operators between Banach couples. The restriction $\vec{s}$ to a subclass of all trivial couples (i.e., a given couple is formed by the same Banach space) induces the classical s-number sequence in the sense of Pietsch for operators in Banach spaces. For an operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$ and each positive integer n, we define the nth approximation number $\vec{a}_n(T)$, Gelfand number $\vec{c}_n(T)$, and Kolmogorov number $\vec{d}_n(T)$. We examine the relations between $\vec{s}_n(T)$ and s-numbers $s_n(T_i\colon X_i \to Y_i)$ for $i = 0, 1$. The main question we consider is: Under which conditions for Banach couples $\vec{X}$ and $\vec{Y}$ can we find reasonable functions $g\colon \mathbb{N} \to \mathbb{N}$ and $\varphi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to$

$\mathbb{R}_+$ such that

$$\vec{s}_{g(m+n-1)}(T\colon \vec{X} \to \vec{Y}) \leq \varphi\big(s_m(T\colon X_0 \to Y_0), s_n(T\colon X_1 \to Y_1)\big)$$

for each $m, n \in \mathbb{N}$ and every operator $T\colon \vec{X} \to \vec{Y}$? The main results related to this question concern the sequences $(\vec{a}_n)$, $(\vec{c}_n)$, and $(\vec{d}_n)$ and involve the variants of approximation properties for Banach couples. We discuss these approximation properties and show connections with the notion of $\vec{s}$-numbers for operators in Banach couples.

In Section 3, we present some applications of our previous results. We derive upper estimates of the approximation and Kolmogorov numbers of operators between spaces generated by interpolation functors on Banach couples satisfying interpolation variants of approximation properties. Finally, in Section 4, we prove some upper estimates for entropy numbers of operators between interpolation spaces.

2. $\vec{s}$-numbers for operators in Banach couples

The axiomatic approach to s-numbers was developed by Pietsch in [18]. The concept of s-numbers generalizes the notion of singular numbers in Hilbert spaces to the Banach space setting. Following Pietsch's idea in the case of operators between Banach spaces, we introduce the notion of $\vec{s}$-numbers for operators acting between Banach couples.

Let $\vec{\mathcal{B}}$ be the class of all Banach couples. Following the standard notation in interpolation theory, for given Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$, we define an operator $T\colon \vec{X} \to \vec{Y}$ to be a linear mapping $T\colon X_0 + X_1 \to Y_0 + Y_1$ such that restrictions $T|_{X_i}$ are bounded operators from X_i to Y_i for each $i = 0, 1$. The space $\mathcal{L}(\vec{X}, \vec{Y})$ of all operators $T\colon \vec{X} \to \vec{Y}$ is a Banach space equipped with the norm $\|T\|_{\vec{X}\to\vec{Y}} = \max_{i=0,1} \|T|_{X_i}\|_{X_i\to Y_i}$ (see [1]). In what follows, the class $\bigcup_{\vec{X},\vec{Y}\in\vec{\mathcal{B}}} \mathcal{L}(\vec{X}, \vec{Y})$ of all operators defined between Banach couples is denoted by $\vec{\mathcal{L}}$. A rule $\vec{s} = (\vec{s}_n)\colon \vec{\mathcal{L}} \to [0, \infty)^{\mathbb{N}}$ assigning to every operator $T \in \vec{\mathcal{L}}$ a nonnegative scalar sequence $(\vec{s}_n(T))$ is called an *$\vec{s}$-number sequence* if the following conditions are satisfied for each positive integers m and n:

(i) *Monotonicity*: $\|T\| \geq \vec{s}_1(T) \geq \vec{s}_2(T) \geq \cdots \geq 0$ for all $T \in \mathcal{L}(\vec{X}, \vec{Y})$.
(ii) *Additivity*: $\vec{s}_{m+n-1}(S + T) \leq \vec{s}_m(S) + \vec{s}_n(T)$ for all $S, T \in \mathcal{L}(\vec{X}, \vec{Y})$.
(iii) *Ideal property*: $\vec{s}_n(STR) \leq \|S\|\vec{s}_n(T)\|R\|$ for all $R \in \mathcal{L}(\vec{Z}, \vec{X})$, $T \in \mathcal{L}(\vec{X}, \vec{Y})$, and $S \in \mathcal{L}(\vec{Y}, \vec{W})$.
(iv) *Rank property*: if $\operatorname{rank}(T) < n$, then $\vec{s}_n(T) = 0$.
(v) *Norming property*: $\vec{s}_n(\mathrm{id}\colon (\ell_2^n, \ell_2^n) \to (\ell_2^n, \ell_2^n)) = 1$, where id denotes the identity operator on the n-dimensional Hilbert space ℓ_2^n.

As in the linear case, the nth number $s_n(T)$ of an operator $T\colon \vec{X} \to \vec{Y}$ between Banach spaces is also denoted by $\vec{s}_n(T\colon \vec{X} \to \vec{Y})$.

If for every $S \in \mathcal{L}(\vec{X}, \vec{Y})$ and $T \in \mathcal{L}(\vec{Y}, \vec{Z})$,

$$\vec{s}_{m+n-1}(TS) \leq \vec{s}_m(T)\vec{s}_n(S),$$

then the $\vec{s}$-number sequence $\vec{s} = (\vec{s}_n)$ is said to be *multiplicative*. Following important examples for s-number sequences, we introduce variants of these numbers in the setting of operators between Banach couples. The nth approximation number $\vec{a}_n(T)$ of an operator $T\colon \vec{X} \to \vec{Y}$ between Banach couples is defined by

$$\vec{a}_n(T) := \inf\big\{\|T - S\|_{\vec{X}\to\vec{Y}}; \operatorname{rank}(S) < n\big\}.$$

As in the classical case of s-numbers, we show (see [2, p. 469]) that, for an arbitrary $\vec{s}$-number sequence $\vec{s} = (\vec{s}_n)$, the *mixing multiplicativity* property holds: for $S \in \mathcal{L}(\vec{X}, \vec{Y})$ and $T \in \mathcal{L}(\vec{Y}, \vec{Z})$,

$$\vec{s}_{m+n-1}(TS) \le \vec{s}_m(T)\vec{a}_n(S) \qquad \text{and} \qquad \vec{s}_{m+n-1}(TS) \le \vec{a}_m(T)\vec{s}_n(S).$$

In particular, the approximation number sequence $\vec{a} = (\vec{a}_n)$ is multiplicative.

To define a variant of Kolmogorov numbers in the setting of operators between Banach couples, we note that if $\vec{X} = (X_0, X_1)$ is a Banach couple and $N \subset X_0 \cap X_1$ is a closed subspace in X_0 and X_1, then it can be easily shown that $\vec{X}/N := (X_0/N, X_1/N)$ forms a Banach couple if and only if N is closed in $X_0 + X_1$ (see, e.g., [8]). Moreover, if $Q_N^{X_j}$ denotes the canonical surjection from the Banach space X_i onto the quotient space X_i/N for $i = 0, 1$, then the linear map $Q_N^{\vec{X}}\colon X_0 + X_1 \to X_0/N + X_1/N$ given by

$$\vec{Q}_N^{\vec{X}}(x) := Q_N^{X_0}(x_0) + Q_N^{X_1}(x_1), \quad x = x_0 + x_1 \in X_0 + X_1,$$

is well defined, and is independent of the decomposition $x = x_0 + x_1 \in X_0 + X_1$ with $x_0 \in X_0$ and $x_1 \in X_1$. Obviously, $\vec{Q}_N^{\vec{X}}\colon (X_0, X_1) \to (X_0/N, X_1/N)$ with $\|\vec{Q}_N\|_{\vec{X}\to\vec{X}/N} = 1$.

Now we are ready to give the definitions of the nth *Gelfand number* $\vec{c}_n(T)$ and *Kolmogorov number* $\vec{d}_n(T)$ of an operator $T\colon \vec{X} \to \vec{Y}$ between Banach couples:

$$\vec{c}_n(T) := \inf\big\{\|T|_M\|_{\vec{M}\to\vec{Y}}; M \subset X_0 \cap X_1, \operatorname{codim}(M) < n\big\}$$

and

$$\vec{d}_n(T) := \inf\big\{\|Q_N^{\vec{Y}}T\|_{\vec{X}\to\vec{Y}/N}; N \subset Y_0 \cap Y_1, \dim(N) < n\big\},$$

where $\vec{M} = (M_0, M_1)$ with $M_i := (M, \|\cdot\|_{X_i})$ for $i = 0, 1$.

We note that in the case when $T\colon X \to Y$ is an operator between Banach spaces, then for the *trivial* Banach couples (X, X) and (Y, Y), we have $T\colon (X, X) \to (Y, Y)$ and

$$\vec{s}_n\big(T\colon (X, X) \to (Y, Y)\big) = s_n(T\colon X \to Y),$$

where (s_n) is a sequence of s-numbers in the setting of Banach spaces. In particular, we have

$$\vec{a}_n\big(T\colon (X, X) \to (Y, Y)\big) = a_n(T\colon X \to Y),$$
$$\vec{c}_n\big(T\colon (X, X) \to (Y, Y)\big) = c_n(T\colon X \to Y)$$

and

$$\vec{d}_n\big(T\colon (X, X) \to (Y, Y)\big) = d_n(T\colon X \to Y),$$

where (a_n), (c_n), and (d_n) are sequences of the classical approximation, Gelfand numbers, and Kolmogorov numbers, respectively.

It is natural to ask about the relationships between $\vec{s}$-number sequences and the classical s-number sequences. Let $\vec{s}$ be an $\vec{s}$-number sequence for operators in Banach couples, and let s be the induced s-number sequence for operators in Banach spaces (restricting $\vec{s}$ to all trivial couples). The main question we are interested in here is the following: Under what conditions on Banach couples $\vec{X}$ and $\vec{Y}$ can we find reasonable functions $g\colon \mathbb{N} \to \mathbb{N}$ and $\varphi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $m, n \in \mathbb{N}$ and every operator $T\colon \vec{X} \to \vec{Y}$, we have

$$\vec{s}_{g(m+n-1)}(T\colon \vec{X} \to \vec{Y}) \le \varphi\big(s_m(T\colon X_0 \to Y_0), s_n(T\colon X_1 \to Y_1)\big)?$$

We are interested in the case when $\vec{s} \in \{\vec{a}, \vec{c}, \vec{d}\}$, which involves certain variants of approximation properties for Banach couples. Before we state and prove our main results, we need to discuss these properties.

We recall that a Banach space X has the *bounded approximation property* if there exists $\lambda \ge 1$ such that, for every $\varepsilon > 0$ and every compact set $K \subset X$, there exists a finite-rank operator satisfying $\|T\| \le \lambda$, and

$$\|Tx - x\|_X \le \varepsilon \|x\|_X, \quad x \in K.$$

It is easily seen that in this definition we may replace compact sets by finite sets F in X. We also note that it is well known that the above property is equivalent to the following. There is a $\lambda \ge 1$ such that, for every finite-dimensional subspace E of X, there is a finite-dimensional operator $T\colon X \to X$ for which $\|T\| \le \lambda$ and $Tx = x$ for all $x \in E$.

The bounded approximation property allows the rank of an operator T to vary with the choice of a finite set F even if $\varepsilon > 0$ and $\operatorname{card}(F)$ remain fixed. A stronger quantitative version of the bounded approximation property is called the *uniform approximation property* (UAP for short). This property imposes an upper bound on the rank of T that depends on ε and $\operatorname{card}(F)$ but not on the specific choice of F. More precisely, a Banach space X has the UAP if there exists $\lambda \ge 1$ such that, for every $\varepsilon > 0$ and each $n \in \mathbb{N}$, there exists $f(\varepsilon, n)$ such that for every finite set $\{x_1, \ldots, x_n\}$ in X there exists a linear operator $T\colon X \to X$ satisfying $\|T\| \le \lambda$, $\operatorname{rank} T \le f(\varepsilon, n)$, and

$$\|Tx_i - x_i\|_X \le \varepsilon \|x_i\|_X, \quad 1 \le i \le n.$$

It can be shown that this property is connected with the property introduced by Pełczyński and Rosenthal [16]. Let $\lambda \ge 1$. A Banach space X has the λ-*uniform approximation property* (λ-UAP for short) if for every $\lambda' > \lambda$ there is a function $\phi_{\lambda'}\colon \mathbb{N} \to \mathbb{N}$ such that, and for every n-dimensional subspace E of X, there is an operator $T\colon X \to X$ such that $\|T\| < \lambda'$, $Te = e$ for all $e \in E$, and $\operatorname{rank} T < \phi_{\lambda'}(n)$. A Banach space X has the UAP if X has the λ-UAP for some $\lambda \ge 1$.

It should be pointed out that to check whether a given Banach space has the UAP is a subtle problem. We note that Szankowski [22] proved that the existence of an unconditional or even symmetric basis in a Banach space does not ensure that it has the UAP.

In the study of interpolation properties of compact operators, natural variants of approximation properties have appeared in the setting of Banach couples. We mention here that Lions [12] showed that for a wide class of Banach couples $\vec{X} = (X_0, X_1)$ there exists a sequence (P_n) of operators $P_n\colon X_0 + X_1 \to X_0 \cap X_1$ such that $P_n x \to x$ in X_i as $n \to \infty$ for each $x \in X_i$ $(i = 0, 1)$. We will use the special type approximation properties (H_0) (resp., (H)) in the setting of Banach couples, which to the best our knowledge appeared for the first time in [17] (resp., [24]) in the study of compactness (resp., the measure of noncompactness) of linear operators under interpolation.

A Banach couple $\vec{X} = (X_0, X_1)$ is said to have the (*bounded*) *approximation property* (H_0) (resp., (H)) if there exists a positive constant c such that for every $\varepsilon > 0$ and any finite set $F \subset X_0$ (resp., all finite sets $F_i \subset X_i$ for $i = 0, 1$), there exists a map $P\colon \vec{X} \to \vec{X}$ with $P(X_i) \subset X_0 \cap X_1$, $\|P\|_{\vec{X}\to\vec{X}} \le c$, and

$$\|Px - x\|_{X_i} < \varepsilon, \quad x \in F \text{ (resp., } x \in F_i,\ i = 0, 1).$$

Following the classical definitions for Banach spaces, we define variants of the UAP in the setting of Banach couples. Let $\vec{X} = (X_0, X_1)$ be a Banach couple, and let $\lambda \ge 1$. The couple $\vec{X} = (X_0, X_1)$ is said to have the λ-*UAP* if there exists a function $k_{\vec{X}}\colon (\lambda, \infty) \times \mathbb{N} \to [1, \infty)$ such that, for every $\lambda' > \lambda$ and every n-dimensional space E of $X_0 \cap X_1$, there is an operator $u\colon X_0 + X_1 \to X_0 \cap X_1$ such that $\|u\|_{\vec{X}\to\vec{X}} \le \lambda'$, $ux = x$, for all $x \in E$, and $\operatorname{rank} u < k_{\vec{X}}(\lambda', n)$. The function $k_{\vec{X}}$ is called the *uniformity function* of $\vec{X}$. A Banach couple $\vec{X}$ has the UAP if $\vec{X}$ has the λ-UAP for some $\lambda \ge 1$.

We start with the following observation, where in what follows I is the identity map.

Lemma 2.1. *Let $\vec{X} = (X_0, X_1)$ be a Banach couple, and let $\lambda \ge 1$. Suppose that there exists a function $\phi\colon (0, \infty) \times \mathbb{N} \to [1, \infty)$ such that, for every n-dimensional subspace E of $X_0 \cap X_1$ and every $\varepsilon > 0$, there is a finite-rank operator $T\colon X_0 + X_1 \to X_0 \cap X_1$ such that $\|T\|_{\vec{X}\to\vec{X}} < \lambda$, $\operatorname{rank} T < \phi(\varepsilon, n)$, and $\|Tx - x\|_{X_i} \le \varepsilon \|x\|_{X_i}$ for $i = 0, 1$ and for all $x \in E$. Then, for every $\lambda' > \lambda$ and every $E \subset X_0 \cap X_1$ with $\dim(E) = n < \infty$, there exist $\eta = \eta(\lambda', E) > 0$ and an operator $S\colon X_0 + X_1 \to X_0 \cap X_1$ such that $S|_E = I_E$, $\operatorname{rank} S < \phi(\eta, n) + n$, and $\|S\|_{\vec{X}\to\vec{X}} < \lambda'$.*

Proof. Fix $\varepsilon > 0$, $\lambda' > \lambda$, and a subspace $E \subset X_0 \cap X_1$ with $n = \dim(E) < \infty$. Let $P\colon X_0 + X_1 \to X_0 + X_1$ be a bounded linear projection such that $P(X_0 + X_1) = E$. Clearly, $P\colon X_0 + X_1 \to X_0 \cap X_1$, and so $P\colon \vec{X} \to \vec{X}$. For $\eta = (\lambda' - \lambda)/\|P\|_{\vec{X}\to\vec{X}}$, we can find a finite-rank operator $T\colon X_0 + X_1 \to X_0 \cap X_1$ such that $\operatorname{rank} T < \phi(\eta, n)$, $\|T\|_{\vec{X}\to\vec{X}} < \lambda$, and

$$\|Tx - x\|_{X_i} \le \eta \|x\|_{X_i}, \quad x \in E,\ i = 0, 1.$$

We show that the operator S given by

$$S := T + P - TP = T + (I - T)P$$

satisfies the required conditions. To see this, observe that $S\colon X_0+X_1 \to X_0\cap X_1$ with $\operatorname{rank} S \le \operatorname{rank} T + \operatorname{rank} P < \phi(\eta,n)+n$, $Sx = x$ for all $x\in E$. Since

$$\|S\|_{\vec{X}\to\vec{X}} \le \|T\|_{\vec{X}\to\vec{X}} + \eta\|P\|_{\vec{X}\to\vec{X}} < \lambda',$$

this completes the proof. □

We provide an example of a Banach couple that has the approximation property (H_0). We will need the following simple fact (see [11, p. 286]). Let E be a k-dimensional subspace of a Banach space X, and let $\{e_1,\dots,e_k\}$ be an Auerbach basis in E—that is, $\|e_i\|_X = 1$ for each $1\le i\le k$ and

$$\Big\|\sum_{i=1}^{k} a_i e_i\Big\|_X \ge \max_{1\le i\le k} |a_i|$$

for all choices of scalars $\{a_1,\dots,a_k\}$. If for given $\eta>0$ there are $x_1,\dots,x_k\in X$ such that $\|e_i-x_i\|_X\le\eta$ for each $1\le i\le k$, then for every $e=\sum_{i=1}^k a_ie_i$ there exists $x\in\operatorname{span}\{x_1,\dots,x_k\}$ (e.g., we can take $x=\sum_{i=1}^k a_ix_i$) such that

$$\|e-x\|_X \le k\eta.$$

In what follows, we consider couples (X_0,X_1) of Banach lattices; that is, both X_0 and X_1 are Banach lattices which can be embedded as lattice ideals in some $L^0(\Omega,\mathcal{A},\mu)$, where $(\Omega,\mathcal{A},\mu)$ is a measure space. The set of all simple functions $\sum_{i=1}^n a_i\chi_{\Omega_i}$ with $n\in\mathbb{N}$, $a_i\in\mathbb{R}$ and pairwise disjoint sets $\Omega_i\in\mathcal{A}$ such that $\mu(\Omega_i)<\infty$ for each $1\le i\le n$ is denoted by $\mathcal{S}$. We will consider the Banach couple $(L_1(\mu),L_\infty(\mu))$ defined on any measure space $(\Omega,\mathcal{A},\mu)$. The closure of $L_1(\mu)\cap L_\infty(\mu)$ in $L_\infty(\mu)$ is denoted by $L_\infty^0(\mu)$.

Theorem 2.2. *Let $(\Omega,\mathcal{A},\mu)$ be any measure space, and let X be an exact interpolation space between $L_1(\mu)$ and $L_\infty(\mu)$. Then $\vec{X} := (L_\infty^0(\mu), X)$ has the approximation property* (H_0) *with $f_{\vec{X}}(\varepsilon,n)\le[\frac{4n}{\varepsilon}]^n$ for every $\varepsilon>0$ and for each $n\in\mathbb{N}$.*

Proof. We consider the family $\mathcal{P}$ of all finite-dimensional average operator (i.e., all operators P given by

$$Pf := \sum_{i=1}^{N}\Big(\frac{1}{\mu(\Omega_i)}\int_{\Omega_i} f\,d\mu\Big)\chi_{\Omega_i},\quad f\in L_1(\mu)+L_\infty(\mu),$$

where $\{\Omega_1,\dots,\Omega_N\}$ is any finite collection of pairwise disjoint measurable sets of finite measure). Obviously, for any $P\in\mathcal{P}$, we have $P\colon L_1(\mu)+L_\infty(\mu)\to L_1(\mu)\cap L_\infty^0(\mu)$ with $\|P\|_{L_1(\mu)\to L_1(\mu)}=1$ and $\|P\|_{L_\infty^0(\mu)\to L_\infty^0(\mu)}=1$. Since X is an exact interpolation space between $L_1(\mu)$ and $L_\infty(\mu)$, $P\colon X\to X$ with $\|P\|_{X\to X}\le 1$.

Let E be an n-dimensional subspace of $L_\infty^0(\mu)\cap X$. We choose an Auerbach basis $\{e_1,\dots,e_n\}$ in E equipped with the induced norm from $L_\infty^0(\mu)$. Since $|e_j|\le 1$ μ-a.e., for each $1\le j\le n$, we can choose $m=[\frac{4n}{\varepsilon}]$ pairwise disjoint sets with finite measure $A_1^j,\dots,A_m^j$ in $\mathcal{A}$ and finite sets of scalars $\{a_1^j,\dots,a_m^j\}$ such that

$$\Big\|e_j-\sum_{i=1}^m a_i^j\chi_{A_i^j}\Big\|_\infty < \frac{\varepsilon}{2k},\quad 1\le j\le n.$$

By a standard argument, we can construct pairwise disjoint measurable sets $\{\Omega_1, \dots, \Omega_N\}$ with finite measure and simple functions $f_1, \dots, f_n$ such that each one is a linear combination of characteristic functions $\chi_{\Omega_1}, \dots, \chi_{\Omega_N}$. Let $P \in \mathcal{P}$ be an average operator generated by the family $\{\Omega_1, \dots, \Omega_N\}$. Then, as mentioned in the paragraph preceding Theorem 2.2, it follows that, for every $e \in L_\infty^0$ with $\|e\|_\infty = 1$, there exists a simple function $f \in \operatorname{span}\{f_1, \dots, f_N\}$ such that

$$\|e - f\|_\infty \leq \varepsilon/2.$$

Combining these estimates, we conclude that $P\colon X_0 + X_1 \to L_1(\mu) \cap L_\infty(\mu) \hookrightarrow X_0 \cap X_1$ (with continuous inclusion) and $\|P\|_{\vec{X}\to\vec{X}} \leq 1$, $Pf = f$. This implies that

$$\|Pe - e\|_\infty \leq \big\|P(e - f)\big\|_\infty + \|e - f\|_\infty \leq \varepsilon.$$

Since $\dim P \leq N \leq [\frac{4n}{\varepsilon}]^n$, the proof is complete. □

The following lemma yields a characterization of the UAP of a Banach couple in terms of $\vec{s}$-approximation and Kolmogorov $\vec{s}$-numbers.

Lemma 2.3. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) \geq n$. Suppose that a Banach couple $\vec{X} = (X_0, X_1)$ has the λ-UAP for some $\lambda \geq 1$. Then the following conditions are equivalent.*

(i) *There exists a uniformity function $k_{\vec{X}}\colon [\lambda, \infty) \times \mathbb{N} \to [1, \infty)$ such that $k_{\vec{X}}(\lambda, n) = O(f(n))$.*

(ii) *There exists $c > 0$ such that, for every Banach couple $\vec{Y}$ and every operator $S\colon \vec{Y} \to \vec{X}$,*

$$\vec{a}_{[c(f(n-1)]+1}(S) \leq \lambda \vec{d}_n(S), \quad n \in \mathbb{N}.$$

Proof. (i) ⇒ (ii). Let $M \subset X_0 \cap X_1$ be a subspace with $\dim(M) < n$. For a given Banach couple $\vec{Y} = (Y_0, Y_1)$, let $S\colon \vec{Y} \to \vec{X}$ be any operator. Our hypothesis $k_{\vec{X}}(\lambda, n) \leq cf(n)$ for some $c > 0$ and all $n \in \mathbb{N}$ implies that there exists an operator $T\colon \vec{X} \to \vec{X}$ such that $\|T\|_{\vec{X}\to\vec{X}} < \lambda$, $Tx = x$, for all $x \in M$ and $\operatorname{rank}(T) < cf(n-1)$. By definition of the approximation numbers in the setting of Banach couples, we get that

$$\vec{a}_{[cf(n-1)]+1}(S) \leq \|S - TS\|_{\vec{Y}\to\vec{X}} = \big\|(I - T)S\big\|_{\vec{Y}\to\vec{X}}.$$

We define a linear map $U\colon (X_0 + X_1)/M \to X_0 + X_1$ by setting

$$U[x] := x - Tx, \quad [x] \in (X_0 + X_1)/M = X_0/M + X_1/M.$$

Since $M \subset \ker(I - T)$, U is well defined. Clearly, $U\colon (X_0/M, X_1/M) \to (X_0, X_1)$ with $\|U\|_{\vec{X}/M\to\vec{X}} \leq \|I - T\|_{\vec{X}\to\vec{X}} \leq 1 + \lambda$; moreover, we have

$$I - T = UQ_M^{\vec{X}}.$$

In consequence,

$$\begin{aligned}\vec{a}_{[c(f(n-1)]+1}(S) &\leq \big\|(I - T)S\big\|_{\vec{Y}\to\vec{X}} = \|UQ_M^{\vec{X}}S\|_{\vec{Y}\to\vec{X}} \\ &\leq (1 + \lambda)\|Q_M^{\vec{X}}S\|_{\vec{Y}\to\vec{X}/M}.\end{aligned}$$

Since $M \subset X_0 \cap X_1$ with $\dim(M) < n$ was arbitrary, the desired estimate follows.

(ii) $\Rightarrow$ (i). Let $M \subset X_0 \cap X_1$ with $\dim(M) = n$, and let $P\colon X_0 + X_1 \to X_0 + X_1$ be any bounded linear projection such that $P(X_0 + X_1) = E$. Clearly, $P\colon X_0 + X_1 \to X_0 \cap X_1$ and so $P\colon \vec{X} \to \vec{X}$. Since $M = \ker(I - P)$, we can define an operator $V\colon (X_0/M, X_1/M) \to (X_0, X_1)$ by

$$V[x] = x - Px, \quad [x] \in (X_0 + X_1)/M = X_0/M + X_1/M,$$

which satisfy

$$I - P = VQ_M^{\vec{X}}, \quad Q_M^{\vec{X}}V = I_{X_0/M+X_1/M}.$$

Combining with (i), we get that there is an operator $R\colon (X_0/M, X_1/M) \to (X_0, X_1)$ with $\operatorname{rank}(R) \le cf(n)$ such that for some $\lambda \ge 1$,

$$\|V - R\|_{\vec{X}/M \to \vec{X}/M} \le \lambda d_{n+1}(V) \le \lambda \|Q_M^{\vec{X}}V\|_{\vec{X}/M \to \vec{X}/M} = \lambda.$$

Let us observe that now it is clear that an operator $T\colon \vec{X} \to \vec{X}$ given by $T := P + RQ_M^{\vec{X}}$ satisfies $Tx = x$ for all $x \in E$ and

$$\|T\|_{\vec{X} \to \vec{X}} = \big\|I - (V - R)Q_M^{\vec{X}}\big\|_{\vec{X} \to \vec{X}} \le 1 + \|V - R\|_{\vec{X}/M \to \vec{X}/M} \le 1 + \lambda.$$

This completes the proof. □

In what follows, we show the relationship between $\vec{s}$-number sequences and classical s-number sequences. We will use the following lemma.

Lemma 2.4. *Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ be Banach couples, and let $T\colon \vec{X} \to \vec{Y}$. Suppose that $\vec{Y}$ has the approximation property* (H). *Then, for every $\varepsilon > 0$ and any finite-dimensional space $N_i \subset Y_i$ with $\dim(N_i) < k_i$, $i = 0, 1$, there exists a finite-dimensional subspace $M \subset Y_0 \cap Y_1$ with $\dim(M) < k_0 + k_1 - 1$ such that*

$$\|Q_M^{Y_i}T\|_{X_i \to Y_i/M} \le \|Q_{N_i}^{Y_i}T\|_{X_i \to Y_i/N_i} + \varepsilon \|T\|_{X_i \to Y_i}, \quad i = 0, 1.$$

Proof. Note that if both N_0 and N_1 are trivial, then the inequality is obvious. Thus we may suppose that $N_i \ne \{0\}$ for $i = 0$ or $i = 1$. If $N_i \ne \{0\}$, then we let $B_i := \{y_1^i, \ldots, e_{n_i}^i\}$ with $n_i < k_i$ be a normalized basis of N_i, $i = 0, 1$. Then there is a constant $c > 0$ such that for all scalars $\lambda_1, \ldots, \lambda_{n_i}$,

$$\Big\|\sum_{j=1}^{n_i} \lambda_j y_j^i\Big\|_{Y_i} \ge c \sum_{j=1}^{n_i} |\lambda_j|.$$

Let $\delta = \varepsilon/(2 + \varepsilon)$. Since $\vec{Y}$ has the approximation property (H), there exists an operator $P\colon Y_0 + Y_1 \to Y_0 \cap Y_1$ such that

$$\|Py_j^i - y_j^i\|_{Y_i} \le c\delta, \quad 1 \le j \le n_i, i = 0, 1.$$

Combining the above inequalities, we conclude that for any $y = \sum_{j=1}^{n_i} \lambda_j y_j^i$,

$$\|Py - y\|_{Y_i} = \Big\|\sum_{j=1}^{n_i} \lambda_j (Py_j^i - y_j^i)\Big\|_{Y_i} < \delta \|y\|_{Y_i}, \quad i = 0, 1$$

and hence

$$\|Py\|_{Y_i} \geq \|y\|_{Y_0} - \|Py - y\|_{Y_0} \geq (1-\delta)\|y\|_{Y_0}.$$

This shows that $P\colon N_i \to M_i$ is an isomorphism with

$$\|P^{-1}\|_{M_i \to N_i} \leq (1-\delta)^{-1},$$

where $M_i := (P(N_i), \|\cdot\|_{Y_i})$. In particular, $\dim(M_i) = \dim(N_i)$ for each $i = 0, 1$.

We claim that the finite-dimensional space $M := M_0 + M_1 \subset Y_0 \cap Y_1$ satisfies the desired conditions. Obviously, $\dim(M) < k_0 + k_1 - 1$. To prove the required estimate, for $i = 0, 1$ we fix $x \in B_{X_i}$. Then we have

$$\|Q_{M_i}^{Y_i} Tx\|_{Y_i/M_i} = \inf_{y \in N_i} \|Tx - Py\|_{Y_i}, \quad i = 0, 1.$$

If for $i = 0, 1$ we let $A_i = \{y \in N_i; \|Py\|_{Y_i} \leq 2\|T\|_{X_i \to Y_i}\}$, then it is easy to see (by $\|Tx\|_{Y_i} \leq \|T\|_{X_i \to Y_i}$ and $y = 0 \in M_i$) that

$$\|Q_{M_i}^{Y_i} Tx\|_{Y_i/M_i} = \inf_{y \in A_i} \|Tx - Py\|_{Y_i}.$$

Since $\|P^{-1}\|_{M_i \to N_i} \leq (1-\delta)^{-1}$,

$$\|y\|_{N_i} \leq 2(1-\delta)^{-1}\|T\|_{X_i \to Y_i}, \quad y \in A_i, i = 0, 1.$$

By combining this estimate with the above inequality $\|Py - y\|_{Y_i} < \delta\|y\|_{Y_i}$ ($i = 0, 1$) applied for $y = \sum_{j=1}^{n_j} \lambda_j y_j^i$, we get

$$\begin{aligned}
\|Q_{M_i}^{Y_i} Tx\|_{Y_i/M_i} &= \inf_{y \in A_i} \|Tx - Py\|_{Y_i} \leq \inf_{y \in A_i} \big[\|Tx - y\|_{Y_i} + \|Py - y\|_{Y_i}\big] \\
&\leq \inf_{y \in A_i} \|Tx - y\|_{Y_i} + \frac{2\delta}{1-\delta}\|T\|_{X_i \to Y_i} \\
&= \inf_{y \in A_i} \|Tx - y\|_{Y_i} + \varepsilon\|T\|_{X_i \to Y_i}.
\end{aligned}$$

Now note that, due to $A_i \subset \{y \in N_i; \|y\|_{Y_i} \leq 2\|T\|_{X_i \to Y_i}\}$ for $i = 0, 1$, we have

$$\begin{aligned}
\|Q_{N_i}^{Y_i} Tx\|_{Y_i/N_i} &= \inf\big\{\|Tx - y\|_{Y_i}; y \in N_i, \|y\|_{Y_i} \leq 2\|T\|_{X_i \to Y_i}\big\} \\
&= \inf_{y \in A_i} \|Tx - y\|_{Y_i}.
\end{aligned}$$

Since $M_i \subset M$ for each $i = 0, 1$,

$$\|Q_M^{Y_i} T\|_{X_i \to Y_i/M} \leq \|Q_{M_i}^{Y_i} T\|_{X_i \to Y_i/M_i} \leq \|Q_{N_i}^{Y_i} T\|_{X_i \to Y_i/N_i} + \varepsilon\|T\|_{X_i \to Y_i}.$$

This proves the claim and so the proof is complete. □

Theorem 2.5. *Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ be Banach couples, and let $T\colon \vec{X} \to \vec{Y}$ be an operator. Suppose that $\vec{Y}$ has the approximation property* (H). *Then for positive integers k_0, k_1,*

$$\vec{d}_{k_0+k_1-1}(T\colon \vec{X} \to \vec{Y}) \leq \max\big\{d_{k_0}(T\colon X_0 \to Y_0), d_{k_1}(T\colon X_1 \to Y_1)\big\}.$$

Proof. For a given $\varepsilon \in (0,1)$, we may find a finite-dimensional subspace N_i of Y_i with $\dim(N_i) < k_i$, such that

$$\|Q_{N_i}T\|_{X_i\to Y_i/N_i} \le (1+\varepsilon)d_{k_i}(X_i \to Y_i), \quad i=0,1.$$

From Lemma 2.4, it follows that for any $\varepsilon > 0$ there exists a subspace $M \subset Y_0 \cap Y_1$ with $\dim(M) < k_0 + k_1 - 1$ such that

$$\|Q_M^{Y_i}T\|_{X_i\to Y_i/M} \le \|Q_{N_i}^{Y_i}T\|_{X_i\to Y_i/N_i} + \varepsilon\|T\|_{X_i\to Y_i}, \quad i=0,1.$$

This implies that

$$\begin{aligned}\vec{d}_{k_0+k_1-1}(T\colon \vec{X}\to\vec{Y}) &\le \max_{i=0,1}\|Q_{M_i}^{Y_i}T\|_{X_i\to Y_i/M}\\ &\le (1+\varepsilon)\max_{i=0,1} d_{k_i}(T\colon X_i\to Y_i) + \varepsilon\|T\|_{X_i\to Y_i}.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the required estimate. □

In the case of the one-sided approximation property, the above proof gives the following corollary where $(\vec{d}_n^{\circ})$ is defined, for every $T\colon (X_0,X_1)\to(Y_0,Y_1)$ and for each $n\in\mathbb{N}$ by

$$\vec{d}_n^{\circ}(T) := \max\big\{\inf\big\{\|Q_M^{Y_0}T\|_{X_0\to Y_0/M}; M\subset Y_0\cap Y_1, \dim(M)<n\big\}, \|T\|_{X_1\to Y_1}\big\}.$$

Corollary 2.6. *Let $\vec{X} = (X_0,X_1)$, $\vec{Y} = (Y_0,Y_1)$ be Banach couples, and let $T\colon \vec{X}\to\vec{Y}$ be an operator. Suppose that $\vec{Y}$ has the approximation property* (H_0). *Then for each positive integer k, there exists a finite-dimensional subspace M of $Y_0\cap Y_1$, such that*

$$d_k(T\colon X_0\to Y_0) = \inf\big\{\|Q_M^{Y_0}T\|_{X_0\to Y_0/M}; M\subset Y_0\cap Y_1, \dim(M)<k\big\}.$$

In particular,

$$\vec{d}_k^{\circ}(T) = \max\big\{d_k(T\colon X_0\to Y_0), \|T\|_{X_1\to Y_1}\big\}.$$

3. Interpolation of the approximation and Kolmogorov numbers

We give applications of the above results to interpolation estimates of some approximation numbers of operators between interpolation spaces. We recall that a mapping $\mathcal{F}$ from the category of all couples of Banach spaces into the category of Banach spaces is said to be an *interpolation functor* (or an *interpolation method*) if for any couple $\vec{A}$, $\mathcal{F}(\vec{A})$ is a Banach space intermediate with respect to $\vec{A}$ and T maps $\mathcal{F}(\vec{A})$ into $\mathcal{F}(\vec{B})$ for all $T\colon \vec{A}\to\vec{B}$. If, additionally, there is a constant $C>0$ such that

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \le C\|T\|_{\vec{A}\to\vec{B}}$$

for every $T\colon \vec{A}\to\vec{B}$, then $\mathcal{F}$ is called *bounded* (and *exact* if $C=1$).

All interpolation functors considered in this article will be exact. For any such functor $\mathcal{F}$, we define a fundamental function $\varphi_{\mathcal{F}}$ by

$$\varphi_{\mathcal{F}}(s,t) := \sup\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})}, \quad s,t\ge 0,$$

where the supremum is taken over all $\vec{A}, \vec{B} \in \vec{\mathcal{B}}$ and all operators $T\colon \vec{A} \to \vec{B}$ such that $\|T\|_{A_0\to B_0} \le s$ and $\|T\|_{A_1\to B_1} \le t$. We remark that it follows immediately that

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \le \varphi_{\mathcal{F}}\big(\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\big)$$

for all Banach couples $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ and all $T\colon \vec{A} \to \vec{B}$.

An exact interpolation functor $\mathcal{F}$ is said to be of *exponent* θ with $\theta \in (0,1)$ whenever $\varphi_{\mathcal{F}}(s,t) \le s^{1-\theta}t^{\theta}$ for all $s, t > 0$. It is well known that the real method of interpolation $(\cdot)_{\theta,q}$ with $\theta \in (0,1)$ and $1 \le q \le \infty$ is exact of exponent θ.

Now we state and prove a result on interpolation estimates of approximation numbers of interpolated operators with the range couple satisfying interpolation variants of approximation properties.

Theorem 3.1. *Let $\mathcal{F}$ be an interpolation functor with fundamental function φ. Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be Banach couples. Suppose that $\vec{Y}$ has both the approximation property* (H) *and the λ-UAP with $k_{\vec{Y}}(\lambda, n) \le g(n)$ for each n, where $g\colon \mathbb{N} \to \mathbb{N}$ with $g(n) \ge n$. Then for every operator $T\colon \vec{X} \to \vec{Y}$ and for each positive integer k_0 and k_1,*

$$a_{g(k_0+k_1-1)+1}\big(T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) \le \varphi\big(d_{k_0}(T_0), d_{k_1}(T_1)\big).$$

In particular,

$$d_{g(k_0+k_1-1)+1}\big(T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) \le \varphi\big(d_{k_0}(T_0), d_{k_1}(T_1)\big).$$

Proof. We may assume without loss of generality that $\|T\|_{\vec{X}\to\vec{Y}} \le 1$. We let $d_{k_i}(T_i) := d_{k_i}(T\colon X_i \to Y_i)$. For a given $\varepsilon > 0$, we may find a finite-dimensional subspace N_i of Y_i with $\dim(N_i) < k_i$ such that

$$\|Q_{N_i}T\|_{X_i\to Y_i/N_i} \le (1+\varepsilon)d_{k_i}(T_i), \quad i = 0, 1.$$

Since $\vec{Y}$ has the approximation property (H), it follows from Lemma 2.4 that there exists a subspace $M \subset Y_0 \cap Y_1$ with $\dim(M) < k_0 + k_1 - 1$ such that

$$\|Q_M^{Y_i}T\|_{X_i\to Y_i/M} \le (1+\varepsilon)d_{k_i}(T_i) + \varepsilon, \quad i = 0, 1.$$

Our hypothesis $k_{\vec{X}}(\lambda, n) \le g(n)$ yields that there exists an operator $P\colon \vec{X} \to \vec{X}$ such that $\|P\|_{\vec{X}\to\vec{X}} \le \lambda$, $Px = x$ for all $x \in M$, and $\operatorname{rank}(P) \le f(k_0 + k_1 - 1)$. This implies that

$$\begin{aligned} &a_{f(k_0+k_1-1)+1}\big(T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) \\ &\quad \le \varphi\big(\|T - PT\|_{X_0\to Y_0}, \|T - PT\|_{X_1\to Y_1}\big) \\ &\quad = \varphi\big(\big\|(I-P)T\big\|_{X_0\to Y_0}, \big\|(I-P)T\big\|_{X_1\to Y_1}\big). \end{aligned}$$

We define a linear map $L\colon (Y_0 + Y_1)/M \to Y_0 + Y_1$ by setting

$$L[y] := y - Py, \quad [y] \in (Y_0 + Y_1)/M = Y_0/M + Y_1/M.$$

Since $M \subset \ker(I - P)$, L is well defined. Obviously,

$$L\colon (Y_0/M, Y_1/M) \to (Y_0, Y_1)$$

with $\|L\|_{\vec{Y}/M\to\vec{Y}} \le \|I-P\|_{\vec{Y}\to\vec{Y}} \le 1+\lambda$, and moreover, we have

$$I - P = LQ_M^{\vec{Y}}.$$

Combining these together yields, with $C = 1+\lambda$,

$$\begin{aligned} &a_{f(k_0+k_1-1)+1}\big(T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) \\ &\quad\le \varphi\big(\|LQ_M^{Y_0}T\|_{X_0\to Y_0}, \|LQ_M^{Y_1}T\|_{X_1\to Y_1}\big) \\ &\quad\le C\varphi\big(\|Q_M^{Y_0}T\|_{X_0\to Y_0/M}, \|Q_M^{Y_1}T\|_{X_1\to Y_1/M}\big) \\ &\quad\le C\varphi\big((1+\varepsilon)d_{k_0}(T_0)+\varepsilon, (1+\varepsilon)d_{k_1}(T_1)+\varepsilon\big). \end{aligned}$$

Since φ is continuous, letting $\varepsilon \to 0$ yields the required estimate. □

4. Entropy estimates in interpolation spaces

We recall that the nth dyadic entropy number of an operator $T\colon X \to Y$ between Banach spaces is given by $e_n(T) := \varepsilon_{2^{n-1}}(T)$ for each $n \in \mathbb{N}$. We remark that in the setting of trivial Banach couples the sequence of entropy numbers (ε_n) satisfies properties (i)–(iii) and (v) in the definition of $(\vec{s}_n)$-number sequence (see Section 2) with $(\vec{s}_n)$ replaced by (ε_n); however, it does not satisfy property (iv). According to Pietsch [19], the sequence (ε_n) forms the so-called *pseudo-s-number sequence*.

From the point of view of the theory of operators on Banach spaces as well as applications, it is useful to identify other important properties that behave well under interpolation by some method. We note that for many years it was an open question whether entropy numbers behave well under real interpolation. More precisely, does there exist a constant C depending only on $\theta \in (0,1)$ and $q \in [1,\infty]$ such that the entropy estimate

$$\begin{aligned} &e_{m+n-1}\big(T\colon (X_0,X_1)_{\theta,q} \to (Y_0,Y_1)_{\theta,q}\big) \\ &\qquad\le Ce_m(T\colon X_0 \to Y_0)^{1-\theta} e_n(T\colon X_1 \to Y_1)^{\theta} \end{aligned}$$

is true for every operator $T\colon (X_0,X_1) \to (Y_0,Y_1)$ and each m, $n \in \mathbb{N}$? Edmunds and Netrusov [6] answered this question negatively. Let us remark that by using simple interpolation tricks it is possible to generate a counterexample with $(X_0, X_1) = (Y_0, Y_1)$ (see [14]).

The goal of this section is to derive some interpolation estimates for entropy numbers of operators between interpolation spaces. We begin by recalling some known results, which will be used in this section. Before we state these results, we need to introduce some definitions. We recall that a Banach space X is of (*Gaussian*) *type* 2 provided that there is $C > 0$ such that for every finite sequence $\{x_1, \ldots, x_n\}$ in X,

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n g_i x_i\Big\|^2\Big)^{1/2} \le C\Big(\sum_{i=1}^n \|x_i\|^2\Big)^{1/2},$$

where $\mathbb{E}$ denotes the expectation, and $(g_n)_{n\in\mathbb{N}}$ is a sequence of independent standard Gaussian variables on some probability space; that is, each g_n has the density function

$$(2\pi)^{-1/2}\exp(-t^2/2).$$

The best type 2 constant C is denoted by $T_2(X)$.

For the reader's convenience, we collect below two known results, which will be used in this section. An application of Maurey's extension theorem yields the following result (see [7, Lemma 1.4]).

Lemma 4.1. *Let X and Y be Banach spaces such that X and Y^* are of type 2. Then for any operator $T\colon X \to Y$ and $b := T_2(X)T_2(Y^*)$,*

$$c_n(T) \le a_n(T) \le bc_n(T), \qquad d_n(T) \le a_n(T) \le bd_n(T), \quad n \in \mathbb{N}.$$

This lemma may be combined with Carl's result stating that there exists an absolute constant $a > 0$ that, under the same assumptions of type 2 for Banach spaces X and Y^*, satisfies for any operator $T\colon X \to Y$,

$$\Big(\prod_{i=1}^{n} \big|c_i(T)\big|\Big)^{1/n} \le aT_2(X)T_2(Y^*)e_n(T).$$

The combination immediately implies the following corollary (see [7, Corollary 1.6]).

Corollary 4.2. *Let X and Y be Banach Spaces such that X and Y^* are of type 2. Then for any operator $T\colon X \to Y$, we have*

$$a_n(T) \le de_n(T),$$

where $d = a(T_2(X)T_2(Y^*))^2$ and a is the absolute constant from the Carl inequality above.

We will use the well-known entropy estimate true for any real n-dimensional Banach space X (see [3]):

$$e_k(\mathrm{Id}\colon X \to X) \le 4 \cdot 2^{-\frac{k-1}{n}}, \quad k \in \mathbb{N}.$$

To simplify notation, we will continue to write T_i instead of $T\colon X_i \to Y_i$ $(i = 0, 1)$ for an operator $T\colon \vec{X} \to \vec{Y}$.

Theorem 4.3. *Let $\mathcal{F}$ be an exact interpolation functor with fundamental function φ. Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be Banach couples. Suppose that $\vec{Y}$ has both the approximation properties* (H) *and UAP with $k_{\vec{Y}}(\lambda, n) \le g(n)$, where $g\colon \mathbb{N} \to \mathbb{N}$ is such that $g(n) \ge n$. Then for any operator $T\colon \vec{X} \to \vec{Y}$ and for each positive integer m, n, we have with $C = 1 + \lambda$,*

$$e_m\big(T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) \le C\varphi\big(d_n(T_0), d_n(T_1)\big) + 4\lambda 2^{-\frac{m-1}{g(2n-1)}}\|T\|_{\vec{X}\to\vec{Y}}.$$

If in addition X_i and Y_i^* are of type 2 for $i = 0, 1$, then

$$e_m\big(T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) \le \widetilde{C}\varphi\big(e_n(T_0), e_n(T_1)\big) + 4\lambda 2^{-\frac{m-1}{g(2n-1)}}\|T\|_{\vec{X}\to\vec{Y}},$$

where $\widetilde{C}$ depends on the type 2 constants and λ.

Proof. We follow the proof of Theorem 3.1. We may assume without loss of generality that $\|T\|_{\vec{X}\to\vec{Y}} \le 1$. For a given $\varepsilon > 0$ and $m \in \mathbb{N}$, we may find a finite-dimensional subspace N_i of Y_i with $\dim(N_i) < n$ such that

$$\|Q_{N_i}T\|_{X_i\to Y_i/N_i} \le (1+\varepsilon)d_n(T_i), \quad i = 0, 1.$$

Since $\vec{Y}$ has the approximation property (H), it follows from Lemma 2.4 that there exists a subspace $M \subset Y_0 \cap Y_1$ with $\dim(M) < 2n-1$ such that

$$\|Q_M^{Y_i}T\|_{X_i\to Y_i/M} \le (1+\varepsilon)d_n(T_i) + \varepsilon, \quad i = 0, 1.$$

By our hypothesis $k_{\vec{X}}(\lambda, n) \le g(n)$, it follows that there exists an operator $P\colon \vec{X} \to \vec{X}$ such that $\|P\|_{\vec{X}\to\vec{X}} \le \lambda$, $Px = x$ for all $x \in M$ and $\operatorname{rank}(P) \le g(2n-1)$.

By the definition of φ, we have

$$\|T - PT\|_{\mathcal{F}(\vec{X})\to\mathcal{F}(\vec{Y})} \le \varphi\big(\big\|(I-P)T\big\|_{X_0\to Y_0}, \big\|(I-P)T\big\|_{X_1\to Y_1}\big).$$

As in the proof of Theorem 3.1, we have

$$I - P = LQ_M^{\vec{Y}},$$

where $L\colon (Y_0/M, Y_1/M) \to (Y_0, Y_1)$ with $\|L\|_{\vec{Y}/M\to\vec{Y}} \le C = 1 + \lambda$.

We combine the above estimates to obtain

$$\begin{aligned}\|T - PT\|_{\mathcal{F}(\vec{X})\to\mathcal{F}(\vec{Y})} &\le \varphi\big(\|LQ_M^{Y_0}T\|_{X_0\to Y_0}, \|LQ_M^{Y_1}T\|_{X_1\to Y_1}\big)\\ &\le C\varphi\big(\|Q_M^{Y_0}T\|_{X_0\to Y_0/M}, \|Q_M^{Y_1}T\|_{X_1\to Y_1/M}\big)\\ &\le C\varphi\big((1+\varepsilon)d_n(T_0) + \varepsilon, (1+\varepsilon)d_n(T_1) + \varepsilon\big).\end{aligned}$$

By the additivity and monotonicity of entropy numbers, we have

$$\begin{aligned}e_m\big(T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) &\le \|T - PT\|_{\mathcal{F}(\vec{X})\to\mathcal{F}(\vec{Y})}\\ &\quad + e_m\big(PT\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big).\end{aligned}$$

Since $\operatorname{rank}(PT) \le \operatorname{rank} P \le g(2n-1)$,

$$\begin{aligned}e_m\big(PT\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) &\le 4 \cdot 2^{-\frac{m-1}{g(2n-1)}}\|PT\|_{\mathcal{F}(\vec{X})\to\mathcal{F}(\vec{Y})}\\ &\le 4\lambda 2^{-\frac{m-1}{g(2n-1)}}\|T\|_{\vec{X}\to\vec{Y}}.\end{aligned}$$

Combining the above estimates yields

$$\begin{aligned}e_m\big(T\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) &\le C\varphi\big((1+\varepsilon)d_n(T_0) + \varepsilon, (1+\varepsilon)d_n(T_1) + \varepsilon\big)\\ &\le 4\lambda 2^{-\frac{m-1}{g(2n-1)}}\|T\|_{\vec{X}\to\vec{Y}}.\end{aligned}$$

Since φ is continuous, letting $\varepsilon \to 0$ yields the desired estimate. This estimate combined with Corollary 4.2 concludes the proof. □

In the last part of this section we will prove a kind of two-sided estimation of entropy numbers of interpolated operators under weaker assumptions. In what follows, we will consider a special type of functions from the class Φ consisting of all functions $\varphi\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ which are nondecreasing in each variable and positively homogeneous (i.e., $\varphi(\lambda s, \lambda t) = \lambda\varphi(s, t)$ for all $\lambda, s, t > 0$).

For any $\varphi \in \Phi$, we define an involution of φ by $\varphi_*(s,t) := 1/\varphi(s^{-1}, t^{-1})$ for all $s, t > 0$, and $\overline{\varphi}$ given by

$$\overline{\varphi}(s,t) := \sup\Big\{\frac{\varphi(su,tv)}{\varphi(u,v)}; u, v > 0\Big\}, \quad s, t > 0.$$

Note that the fundamental function $\varphi_{\mathcal{F}}$ of an exact interpolation functor $\mathcal{F}$ belongs to Φ. Moreover, we have $\overline{\varphi}_{\mathcal{F}} = \varphi_{\mathcal{F}}$ by the easily verified fact that $\varphi_{\mathcal{F}}$ is submultiplicative; that is, for all $s, t, u, v > 0$,

$$\varphi_{\mathcal{F}}(su, tv) \le \varphi_{\mathcal{F}}(s,t)\varphi_{\mathcal{F}}(u,v).$$

Let $\vec{X} = (X_0, X_1)$ be a Banach couple. For every $s, t > 0$, we put

$$K(s,t,a;\vec{X}) = \inf\big\{s\|x_0\|_{X_0} + t\|x_1\|_{X_1}; x = x_0 + x_1\big\}, \quad x \in X_0 + X_1.$$

Let X be an intermediate Banach space with respect to $\vec{X}$. For each $s, t > 0$, set

$$\psi(s,t) = \psi_X(s,t;\vec{X}) := \sup\big\{K(s,t,x;\vec{X}); \|x\|_X = 1\big\}$$

and in the case $X_0 \cap X_1 \neq \{0\}$,

$$\phi(s,t) = \phi_X(s,t;\vec{X}) := \sup\big\{\|x\|_X; x \in X_0 \cap X_1, \|x\|_{X_0} \le s, \|x\|_{X_1} \le t\big\}.$$

To show some general examples of functions in Φ generated by interpolation functors, we recall, following [15], that the function φ, which corresponds to an exact interpolation functor $\mathcal{F}$ by the equality

$$\mathcal{F}(s\mathbb{R}, t\mathbb{R}) = \varphi(s,t)\mathbb{R}, \quad s, t > 0$$

is called the *characteristic function* of the functor $\mathcal{F}$. Here $\alpha\mathbb{R}$ denotes $\mathbb{R}$ equipped with the norm $\|\cdot\|_{\alpha\mathbb{R}} = \alpha|\cdot|$ for $\alpha > 0$.

We note that, for any exact interpolation functor $\mathcal{F}$ (see [15, p. 372]) and for any Banach couple $\vec{X} = (X_0, X_1)$, we have

$$\|x\|_{\mathcal{F}(\vec{X})} \le \varphi\big(\|x\|_{X_0}, \|x\|_{X_1}\big), \quad x \in X_0 \cap X_1;$$

moreover, by [15, Lemma 7.7.1], for all $s, t > 0$,

$$K(s,t,x;\vec{X}) \le \varphi_*(s,t)\|x\|_{\mathcal{F}(\vec{X})}, \quad x \in \mathcal{F}(\vec{X}),$$

where $\varphi_*(s,t) := 1/\varphi(s^{-1}, t^{-1})$. Hence, for a Banach space $X := F(\vec{X})$,

$$\varphi_X(s,t) \le \varphi(s,t), \qquad \psi_X(s,t) \le \varphi_*(s,t), \quad s, t > 0.$$

We will use the following one-sided estimate (see [13]).

Proposition 4.4. *Let (A_0, A_1) be a Banach couple, let A be a Banach space such that $A \hookrightarrow A_0 + A_1$, and let B be any Banach space. Then following estimate holds for any operator $T\colon (A_0, A_1) \to (B, B)$ and each $m, n \in \mathbb{N}$*

$$e_{m+n-1}(T\colon A \to B) \le \psi_B\big(e_m(T\colon A_0 \to B), e_n(T\colon A_1 \to B)\big).$$

Proposition 4.5. *Let (B_0, B_1) be a Banach couple, let B be an intermediate Banach space between B_0 and B_1, and let A be any Banach space. The following estimate holds for any operator $T\colon (A, A) \to (B_0, B_1)$ and each $m, n \in \mathbb{N}$*

$$e_{m+n-1}(T\colon A \to B) \le 2C\phi_Y\big(e_m(T\colon A \to B_0), e_n(T\colon A \to B_1)\big).$$

Theorem 4.6. *Let X and Y be intermediate Banach spaces with respect to Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$, respectively. Then for any operators $T\colon \vec{X} \to \vec{Y}$ and $S\colon \vec{Y} \to \vec{Y}$ with $S\colon Y_0 + Y_1 \to Y_0 \cap Y_1$, we have*

$$e_{2m+2n-3}(ST\colon X \to Y) \le f(S)\overline{\psi}_X\big(e_m(T_0), e_n(T_1)\big),$$

where $f(S) = \overline{\psi}_X(\phi_Y(\|S\|_{Y_0\to Y_0}, \|S\|_{Y_0\to Y_1}), \phi_Y(\|S\|_{Y_1\to Y_0}, \|S\|_{Y_1\to Y_1})$.

Proof. For simplicity of notation, we put $T_j = T|_{X_j}\colon X_j \to Y_j$ for $j \in \{0, 1\}$. By the closed graph theorem, it follows that $S\colon Y_0 + Y_1 \to Y_0 \cap Y_1$ is bounded and so the restrictions $S\colon Y_0 \to Y_1$ and $S\colon Y_1 \to Y_0$ are bounded operators. Thus we have that $ST\colon X_0 + X_1 \to Y_0 \cap Y_1$ is bounded and so

$$ST\colon (X_0, X_0) \to (Y_0, Y_1) \qquad \text{and} \qquad ST\colon (X_1, X_1) \to (Y_0, Y_1).$$

This implies that

$$e_m(ST\colon X_0 \to Y_0) \le e_m(T_0)\|S\|_{Y_0\to Y_0},$$
$$e_m(ST\colon X_0 \to Y_1) \le e_m(T_0)\|S\|_{Y_0\to Y_1}.$$

Then from Proposition 4.5, we obtain the following estimate:

$$e_{2m-1}(ST\colon X_0 \to Y) \le \phi_Y\big(\|S\|_{Y_0\to Y_0}, \|S\|_{Y_0\to Y_1}\big)e_m(T_0)).$$

Similarly, we get that

$$e_n(ST\colon X_1 \to Y_0) \le e_n(T_1)\|S\|_{Y_1\to Y_0},$$
$$e_n(ST\colon X_1 \to Y_1) \le e_n(T_1)\|S\|_{Y_1\to Y_1}$$

and so the second estimate follows as

$$e_{2n-1}(ST\colon X_1 \to Y) \le \phi_Y\big(\|S\|_{Y_1\to Y_0}, \|S\|_{Y_1\to Y_1}\big)e_n(T_1).$$

These estimates combined with Proposition 4.4 yield that there exists a constant $C > 0$ such that

$$\begin{aligned} &e_{2m+2n-3}(ST\colon X \to Y) \\ &\quad\le \psi_X\big(\phi_Y\big(\|S\|_{Y_0\to Y_0}, \|S\|_{Y_0\to Y_1}\big)e_m(T_0), \phi_Y\big(\|S\|_{Y_1\to Y_0}, \|S\|_{Y_1\to Y_1}\big)e_n(T_1)\big) \\ &\quad\le f(S)\overline{\psi}_X\big(e_m(T_0), e_n(T_1)\big), \end{aligned}$$

and this completes the proof. □

As an application, we have the following corollary.

Corollary 4.7. *Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be Banach couples, and let $\mathcal{F}$ be an interpolation functor of exponent θ with the characteristic function $\varphi(s,t) = s^{1-\theta}t^{\theta}$ for all $s, t > 0$. Then for any operators $T\colon \vec{X} \to \vec{Y}$ and $S\colon \vec{Y} \to \vec{Y}$ such that $S\colon Y_0 + Y_1 \to Y_0 \cap Y_1$ with $\|S\|_{\vec{Y}\to\vec{Y}} \le C$, we have*

$$e_{2m+2n-3}\big(ST\colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big) \le Cg(S)e_m(T_0)^{1-\theta}e_n(T_1)^{\theta},$$

where $g(S) = (\|S\|_{Y_0\to Y_1}\|S\|_{Y_1\to Y_0})^{(1-\theta)\theta}$.

In the following lemma we will use the inner entropy numbers. Following Pietsch [19, p. 169]), for each $n \in \mathbb{N}$ we denote by $f_n(T)$ the (dyadic) inner entropy number of an operator $T\colon X \to Y$ between Banach spaces, which is defined to be the supremum of all those $\varepsilon > 0$ such that there are $x_1, \dots, x_{2^{n-1}+1}$ in B_X with $\|Tx_i - Tx_j\|_Y \ge 2\varepsilon$ whenever i, j are distinct points in $\{1, \dots, 2^{n-1}+1\}$. Then the entropy and inner entropy numbers are related by

$$f_n(T) \le e_n(T) \le 2f_n(T), \quad n \in \mathbb{N}.$$

Lemma 4.8. *Let $\mathcal{F}$ be an interpolation functor with fundamental function φ, and let $\vec{Y} = (Y_0, Y_1) \in (\mathrm{H})$ be a Banach couple. Then there exists a constant $C > 1$ such that for each $m, n \in \mathbb{N}$, any $\varepsilon > 0$, and $T\colon (X_0, X_1) \to (Y_0, Y_1)$, there exists an operator $P\colon Y_0 + Y_1 \to Y_0 \cap Y_1$ such that*

$$\|T - PT\|_{\mathcal{F}(\vec{X})\to\mathcal{F}(\vec{Y})} \le C\varphi\big(\varepsilon + e_m(T\colon X_0 \to Y_0), \varepsilon + e_n(T\colon X_1 \to Y_1)\big).$$

Proof. For simplicity of notation, we put $T_j := T|_{X_j}$, for $j \in \{0,1\}$. Given $\varepsilon > 0$ and $m, n \in \mathbb{N}$, we can find sets $\{x_1^{(0)}, \dots, x_p^{(0)}\} \subset B_{X_0}$, $\{x_1^{(1)}, \dots, x_q^{(1)}\} \subset B_{X_1}$ with $p \le 2^{m-1}$ and $q \le 2^{n-1}$ such that for $\rho_0 = f_m(T_0) + \varepsilon/2$ and $\rho_1 = f_n(T_1) + \varepsilon/2$, we have

$$T(B_{X_0}) \subset \bigcup_{i=1}^{p}\{Tx_i^{(0)} + 2\rho_0 B_{Y_0}\},$$
$$T(B_{X_1}) \subset \bigcup_{k=1}^{q}\{Tx_k^{(1)} + 2\rho_1 B_{Y_1}\}.$$

Moreover, for any $x \in B_{X_0}$ (resp., $x \in B_{X_1}$), there exists $i \in \{1, \dots, p\}$ (resp., $k \in \{1, \dots, q\}$) such that

$$\|Tx - Tx_i^{(0)}\|_{Y_0} \le 2\rho_0 \quad \text{resp., } \|Tx - Tx_k^{(1)}\|_{Y_0} \le 2\rho_1.$$

Let $\delta = \varepsilon/2\|T\|_{\vec{X}\to\vec{Y}}$. Then our hypothesis yields that there exist a universal constant $\lambda > 0$ and an operator $P\colon Y_0 + Y_1 \to Y_0 \cap Y_1$ with $\|P\|_{\vec{Y}\to\vec{Y}} \le \lambda$ satisfying

$$\big\|P(Tx_i^{(0)}) - Tx_i^{(0)}\big\|_{Y_0} \le \delta\|Tx_i^{(0)}\|_{Y_0} \le \varepsilon/2, \quad 1 \le i \le p,$$
$$\big\|P(Tx_k^{(1)}) - Tx_k^{(1)}\big\|_{Y_1} \le \delta\|Tx_i^{(1)}\|_{Y_1} \le \varepsilon/2, \quad 1 \le k \le q.$$

Combining the above estimates, we conclude that for a given $x \in B_{X_0}$, there exists $i \in \{1, \dots, p\}$ such that $Tx = Tx_i^{(0)} + 2\rho_0 y_0$ with $y_0 \in B_{Y_0}$. This implies (by $f_m(T_0) \le e_m(T_0)$) for $C = 2(1+\lambda)$ that

$$\begin{aligned}\big\|Tx - PT(x)\big\|_{Y_0} &\le \big\|P(Tx_i^{(0)}) - Tx_i^{(0)}\big\|_{Y_0} + 2\rho_0\big\|(P-I)y_0\big\|_{Y_0}\\ &\le \varepsilon/2 + 2(1+\lambda)\rho_0 \le C\big(\varepsilon + e_m(T_0)\big).\end{aligned}$$

This implies that

$$\|T - PT\|_{X_0 \to Y_0} \le C\big(\varepsilon + e_m(T_0)\big).$$

Similarly, we get that $\|T - PT\|_{X_1 \to Y_1} \le C(\varepsilon + e_n(T_1))$. Thus the required estimate follows from the definition of φ. □

We will make an application of Theorem 4.6 to Banach couples satisfying the approximation hypothesis used in [5]. A Banach couple $\vec{Y} = (Y_0, Y_1)$ satisfies the approximation condition (AP) if there is a sequence $\{P_n\}_{n=1}^{\infty}$ of linear operators from $Y_0 + Y_1$ into $Y_0 \cap Y_1$ and two other sequences $\{Q_n^+\}$ and $\{Q_n^-\}_{n=1}^{\infty}$ of linear operators from $Y_0 + Y_1$ into $Y_0 + Y_1$ such that

(I) they are uniformly bounded in $\vec{Y}$:

$$C := \sup_{n \in \mathbb{N}} \big\{\|P_n\|_{\vec{Y} \to \vec{Y}}, \|Q_n^+\|_{\vec{Y} \to \vec{Y}}, \|Q_n^-\|_{\vec{Y} \to \vec{Y}}\big\} < \infty;$$

(II) the identity operator I on $Y_0 + Y_1$ can be written as

$$I = P_n + Q_n^+ + Q_n^-, \quad n \in \mathbb{N};$$

(III) for each $n \in \mathbb{N}$, we have $Q^+ \colon Y_0 \to Y_1$ and $Q_n^- \colon Y_1 \to Y_0$ with

$$\lim_{n \to \infty} \|Q_n^+\|_{Y_0 \to Y_1} = \lim_{n \to \infty} \|Q_n^-\|_{Y_1 \to Y_0} = 0.$$

Corollary 4.9. *Let $\vec{X} = (X_0, X_1)$ be a Banach couple, and assume that $\vec{Y} = (Y_0, Y_1)$ satisfies the approximation condition* (AP). *If $\mathcal{F}$ is an interpolation functor of exponent θ $(0 < \theta < 1)$, then for any operator $T \colon \vec{X} \to \vec{Y}$,*

$$\begin{aligned}&e_{4n-3}\big(T \colon \mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})\big)\\ &\quad\le C\big(\|P_n\|_{Y_0 \to Y_1}\|P_n\|_{Y_1 \to Y_0}\big)^{(1-\theta)\theta} + e_n(T_0)^{1-\theta} e_n(T_1)^{\theta}\\ &\qquad + \|Q_n^- T\|_{X_0 \to Y_0}^{1-\theta}\|Q_n^- T\|_{X_1 \to Y_1}^{\theta} + \|Q_n^+ T\|_{X_0 \to Y_0}^{1-\theta}\|Q_n^+ T\|_{X_1 \to Y_1}^{\theta}.\end{aligned}$$

Proof. Combining $T - P_n T = (I - P_n)T = Q_n^- T + Q_n^+$ for each $n \in \mathbb{N}$ with our hypothesis that $\mathcal{F}$ is of exponent θ, we get that

$$\begin{aligned}&\|T - P_n T\|_{\mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})}\\ &\quad\le \|Q_n^- T\|_{\mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})} + \|Q_n^+ T\|_{\mathcal{F}(\vec{X}) \to \mathcal{F}(\vec{Y})}\\ &\quad\le \|Q_n^- T\|_{X_0 \to Y_0}^{1-\theta}\|Q_n^- T\|_{X_1 \to Y_1}^{\theta} + \|Q_n^+ T\|_{X_0 \to Y_0}^{1-\theta}\|Q_n^+ T\|_{X_1 \to Y_1}^{\theta}.\end{aligned}$$

To conclude, it is enough to apply Corollary 4.7. □

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