Banach J. Math. Anal. 13 (2019), no. 2, 275–292
https://doi.org/10.1215/17358787-2018-0030
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

DUAL TRUNCATED TOEPLITZ C^*-ALGEBRAS

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In memory of R. G. Douglas

Communicated by M. Frank

Abstract. We establish the short exact sequences associated with the algebras generated by dual truncated Toeplitz operators on the orthogonal complement of the model space K_u^2, and discuss spectral properties of dual truncated Toeplitz operators.

1. Introduction

As a result of the seminal paper of Sarason [22], much work in the study of truncated Toeplitz operators has been done over the past ten years (see [2], [3], [15]). In particular, the algebra of truncated Toeplitz operators is an active area of research (see [6], [23]). In [12], the first and third authors introduced the dual truncated Toeplitz operators for the first time, which are defined on a Hilbert space of harmonic functions that are closely related to truncated Toeplitz operators. The present article aims to study the algebras associated with dual truncated Toeplitz operators. The structure of these algebras can provide us with more tools for studying the invertibility, Fredholmness, and spectral theory of dual truncated Toeplitz operators.

Let H^2 be the classical Hardy space of open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $L^2 = L^2(\mathbb{T})$ be the usual Lebesgue space on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The space L^∞ is the collection of all essentially bounded Lebesgue

Received Sep. 13, 2018; Accepted Oct. 1, 2018.
First published online Jan. 26, 2019.
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2010 *Mathematics Subject Classification.* Primary 47B35; Secondary 31A05.
Keywords. C*-algebra, model space, dual truncated Toeplitz operator.

measurable functions on $\mathbb{T}$, and the space H^∞ consists of all the functions that are analytic and bounded on $\mathbb{D}$. To each nonconstant inner function u, we denote the model space

$$K_u^2 := H^2 \ominus uH^2.$$

Letting M_u and $M_{\bar{u}}$ denote the multiplication operators on L^2 induced by u and $\bar{u}$, P is the orthogonal projection from L^2 onto H^2, and $P_u = P - M_u P M_{\bar{u}}$ is the orthogonal projection of L^2 onto K_u^2. For f in L^2, the truncated Toeplitz operator A_f with the symbol f is densely defined on K_u^2, given by

$$A_f x = P_u(fx), \quad x \in K_u^2 \cap H^\infty.$$

Then we define the dual truncated Toeplitz operator D_f on the orthogonal complement of K_u^2 as

$$D_f y = (I - P_u)(fy), \quad y \in [K_u^2]^\perp \cap L^\infty.$$

Note that $[K_u^2]^\perp = uH^2 \oplus \overline{zH^2}$, and D_f is an operator defined on a Hilbert space of harmonic functions.

Recall that, for f and g in L^∞, the Toeplitz operator T_f with symbol f and dual Toeplitz operator S_g with symbol g are defined on H^2 and $[H^2]^\perp$, respectively, as follows:

$$T_f x = P(fx), \quad x \in H^2,$$
$$S_g y = (I - P)(gy), \quad y \in [H^2]^\perp.$$

In our situation, writing $B([K_u^2]^\perp)$ for the set of all bounded linear operators on $[K_u^2]^\perp$, let $\mathfrak{X}$ be a closed self-adjoint subalgebra of L^∞, and let

$$\mathfrak{D}_{\mathfrak{X}} = \operatorname{clos}\Big\{\sum_{i=1}^{n} \prod_{j=1}^{m} D_{\phi_{ij}} : \phi_{ij} \in \mathfrak{X}\Big\}$$

be the smallest norm-closed subalgebra of $B([K_u^2]^\perp)$ containing $\{D_\phi, \phi \in \mathfrak{X}\}$. Hence $\mathfrak{D}_{\mathfrak{X}}$ is a C^*-algebra generated by $\{D_\phi, \phi \in \mathfrak{X}\}$. Let us call the closed ideal of $\mathfrak{D}_{\mathfrak{X}}$, generated by all semicommutators

$$[D_\phi, D_\psi) \overset{\text{def}}{=} D_\phi D_\psi - D_{\phi\psi}, \quad \phi, \psi \in \mathfrak{X},$$

the *semicommutator ideal* $\mathfrak{S}\mathfrak{D}_{\mathfrak{X}}$; the commutator ideal $\mathfrak{C}\mathfrak{D}_{\mathfrak{X}}$ of $\mathfrak{D}_{\mathfrak{X}}$ is the closed ideal generated by elements of the form

$$[D_\phi, D_\psi] \overset{\text{def}}{=} D_\phi D_\psi - D_\psi D_\phi, \quad \phi, \psi \in \mathfrak{X}.$$

In this article, we consider two kinds of closed subalgebras of $B([K_u^2]^\perp)$: the algebra generated by all bounded dual truncated Toeplitz operators, and the algebra generated by dual truncated Toeplitz operators with continuous symbol.

In the late 1960s, Coburn [7], [8] studied the C^*-algebra generated by T_z on the Hardy space. For truncated Toeplitz operators, Garcia, Ross, and Wogen [16] obtained an analogue of Coburn's work. Here the symbol map on the Toeplitz algebra in the Hardy space is an important tool for studying the structure of Toeplitz algebras (see [1], [4], [13], [14]). Analogous to the symbol map in the classical Hardy space setting, in Lemma 2.3 we construct a symbol map on the dual truncated Toeplitz algebra.

On the other hand, in the case of harmonic function spaces, Guo and Zheng [17] investigated the C^*-algebra generated by Toeplitz operators on the harmonic Bergman space with continuous symbols and showed that the Toeplitz operators with monomial symbols are invertible. We found that the C^*-algebra generated by dual truncated Toeplitz operators on $[K_u^2]^\perp$ with continuous symbols is similar to the case of harmonic Bergman space. The invertibility of dual truncated Toeplitz operators with monomial symbols is complicated; for example, if $u(0) \neq 0, D_z$ is invertible, if $u(0) = 0, D_z$ is not invertible (see Example 4.4).

Let $C(\mathbb{T})$ denote the set of continuous complex-valued functions on $\mathbb{T}$. The set of all compact operators on $[K_u^2]^\perp$ will be denoted by $\mathcal{K}$.

This article is organized in the following way. In Sections 2 and 3, we discuss the structures of the dual truncated Toeplitz algebras $\mathfrak{D}_{L^\infty}$ and $\mathfrak{D}_{C(\mathbb{T})}$ and obtain two short exact sequences

$$0 \longrightarrow \mathfrak{S}\mathfrak{D}_{L^\infty} \longrightarrow \mathfrak{D}_{L^\infty} \longrightarrow L^\infty \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathfrak{D}_{C(\mathbb{T})} \longrightarrow C(\mathbb{T}) \longrightarrow 0.$$

We give a necessary and sufficient condition for the semicommutator of two dual truncated Toeplitz operators to be a compact or finite rank operator. In the final section, we discuss spectral properties of dual Toeplitz operators and prove a spectral inclusion theorem analogous to the spectral inclusion of Toeplitz operators on Hardy space. Moreover, we obtain the spectrum and essential spectrum of dual truncated Toeplitz operators with symbols in $K_{zu}^2 \cap H^\infty$ and QC, respectively.

2. Dual truncated Toeplitz algebra $\mathfrak{D}_{L^\infty}$

For $f \in L^2$, define an operator V on L^2 by

$$Vf(w) = \overline{w}\overline{f(w)}.$$

It is easy to check that V is antiunitary. The operator V satisfies the following properties:

$$V = V^{-1}, \qquad VT_f = S_{\bar{f}}V. \tag{2.1}$$

The Hankel operator H_f with symbol f is densely defined by

$$H_f x = (I - P)(fx), \quad \text{for } x \in H^\infty,$$

and H_f^* is densely defined by

$$H_f^* y = P(\bar{f}y), \quad \text{for } y \in [H^2]^\perp \cap L^\infty.$$

Write M_f for the multiplication operator defined on L^2 by $M_f\phi = f\phi$. If M_f is expressed as an operator matrix with respect to the decomposition $L^2 = H^2 \oplus$

$\overline{zH^2}$, then the result is of the form

$$M_f = \begin{pmatrix} T_f & H^*_{\bar{f}} \\ H_f & S_f \end{pmatrix}.$$

Define the unitary operator

$$U : L^2 = H^2 \oplus \overline{zH^2} \longrightarrow [K_u^2]^{\perp} = uH^2 \oplus \overline{zH^2}$$

by

$$U = \begin{pmatrix} M_u & 0 \\ 0 & I_{\overline{zH^2}} \end{pmatrix},$$

where $I_{\overline{zH^2}}$ is the identity on $\overline{zH^2}$, and M_u is a unitary operator:

$$\begin{aligned} M_u : H^2 &\longrightarrow uH^2 \\ f &\longmapsto uf. \end{aligned}$$

Clearly, U^* maps $[K_u^2]^{\perp}$ to L^2 and equals

$$U^* = \begin{pmatrix} M_{\bar{u}} & 0 \\ 0 & I_{\overline{zH^2}} \end{pmatrix}.$$

The next lemma shows that D_ϕ is unitarily equivalent to an operator on L^2 and gives a matrix representation of D_ϕ. The representation is useful in this article and shows that the dual truncated Toeplitz operators on $[K_u^2]^{\perp}$ are closely related to the Toeplitz operators and Hankel operators on H^2.

Lemma 2.1 ([21, Lemma 2.2])**.** *On $L^2(\mathbb{T}) = H^2 \oplus \overline{zH^2}$,*

$$U^* D_\phi U = \begin{pmatrix} T_\phi & H^*_{u\bar{\phi}} \\ H_{u\phi} & S_\phi \end{pmatrix}. \tag{2.2}$$

In the following, for $T \in B([K_u^2]^{\perp})$, define

$$\widetilde{T} = U^* T U.$$

Since $\widetilde{T}$ and T are unitarily equivalent, $\|T\|_{[K_u^2]^{\perp}} = \|\widetilde{T}\|_{L^2}$. Thus, we will frequently omit all norm subscripts when the contextual meaning is clear. As the first application of Lemma 2.1, we have the following lemma.

Lemma 2.2 ([12, Property 2.1])**.** *Let $f \in L^2$. Then D_f is bounded on $[K_u^2]^{\perp}$ if and only if $f \in L^\infty$. If D_f is bounded, then $\|D_f\| = \|f\|_\infty$.*

Proof. By the definition of D_f, we have

$$\|D_f\| = \big\|(I - P_u)f\big\| \le \|f\|_\infty.$$

Using (2.2),

$$\|D_f\| = \|\widetilde{D}_f\| \ge \left\| \begin{pmatrix} T_f & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|T_f\| = \|f\|_\infty. \qquad \square$$

Let k_z denote the normalized reproducing kernel

$$\frac{\sqrt{1-|z|^2}}{1-w\bar{z}}$$

of H^2 at the point $z \in \mathbb{D}$. Given $f \in L^2$, the harmonic extension of f is given by

$$\begin{aligned}\widetilde{f}(r\xi) &= \int_0^{2\pi} f(e^{i\theta})\left|k_{r\xi}(e^{i\theta})\right|^2 \frac{d\theta}{2\pi}\\ &= \langle f k_{r\xi}, k_{r\xi}\rangle,\end{aligned}$$

where $|k_{r\xi}|^2$ is the Poisson kernel for $r\xi \in \mathbb{D}$. Then $\widetilde{f}$ is harmonic on $\mathbb{D}$. By Fatou's theorem,

$$\lim_{r\to 1^-} \widetilde{f}(r\xi) = f(\xi)$$

for almost all $\xi \in \mathbb{T}$.

For every operator L in $B([K_u^2]^\perp)$, define

$$\widehat{L}_r(\xi) = \langle Luk_{r\xi}, uk_{r\xi}\rangle.$$

If $\lim_{r\to 1^-} \widehat{L}_r(\xi)$ exists for almost all $\xi \in \mathbb{T}$, let

$$\widehat{L}(\xi) = \lim_{r\to 1^-} \widehat{L}_r(\xi),$$

as $|\langle Luk_{r\xi}, uk_{r\xi}\rangle| \le \|L\|$, and $\widehat{L}$ is a bounded function a.e. on $\mathbb{T}$.

Lemma 2.3. *Let $f, f_1, f_2, \ldots, f_n \in L^\infty$.*

(1) *The radial limit*

$$\lim_{r\to 1^-} \langle D_{f_1}\cdots D_{f_n} uk_{r\xi}, uk_{r\xi}\rangle = f_1(\xi)\cdots f_n(\xi)$$

for almost all $\xi \in \mathbb{T}$.

(2) *Assume that $T_n, T \in B([K_u^2]^\perp)$, $n \in \mathbb{Z}^+$,*

$$\lim_{n\longrightarrow\infty} \|T_n - T\| = 0,$$

and that $\lim_{r\to 1^-} (\hat{T}_n)_r(\xi)$ exists for almost all $\xi \in \mathbb{T}$. Then

$$\lim_{r\to 1^-} \widehat{T}_r(\xi) = \lim_{n\to\infty} \widehat{T}_n(\xi).$$

(3) *We have $D_{f_1}D_{f_2}\cdots D_{f_n} - D_{f_1 f_2\cdots f_n} \in \mathfrak{S}\mathfrak{D}_{L^\infty}$.*

(4) *If $A \in \mathfrak{S}\mathfrak{D}_{L^\infty}$, then*

$$\lim_{r\to 1^-} \langle Auk_{r\xi}, uk_{r\xi}\rangle = 0.$$

(5) *The uniform limit of a dual truncated Toeplitz operator is also a dual truncated Toeplitz operator.*

Proof. (1) We will prove this lemma by induction on n. For $n = 1$, we have

$$\begin{aligned}\lim_{r\to 1^-}\langle D_{f_1}uk_{r\xi}, uk_{r\xi}\rangle &= \lim_{r\to 1^-}\langle f_1 uk_{r\xi}, uk_{r\xi}\rangle \\ &= \lim_{r\to 1^-}\langle f_1 k_{r\xi}, k_{r\xi}\rangle \\ &= \lim_{r\to 1^-}\int_0^{2\pi} f_1(e^{i\theta})\big|k_{r\xi}(e^{i\theta})\big|^2\,\frac{d\theta}{2\pi} = f_1(\xi)\end{aligned}$$

for almost all $\xi \in \mathbb{T}$. Let $n \geq 2$. Assume that the result is true up to $m-1$. Observe that

$$\begin{aligned}\langle D_{f_1}\cdots D_{f_{m-1}}D_{f_m}uk_{r\xi}, uk_{r\xi}\rangle = {}&\langle D_{f_1}\cdots D_{f_{m-1}}D_{f_m - f_m(\xi)}uk_{r\xi}, uk_{r\xi}\rangle \\ &+ f_m(\xi)\langle D_{f_1}\cdots D_{f_{m-1}}uk_{r\xi}, uk_{r\xi}\rangle.\end{aligned}$$

Also

$$\begin{aligned}&\big|\langle D_{f_1}\cdots D_{f_{m-1}}D_{f_m - f_m(\xi)}uk_{r\xi}, uk_{r\xi}\rangle\big| \\ &\quad\leq \|D_{f_1}\cdots D_{f_{m-1}}\|\|D_{f_m - f_m(\xi)}uk_{r\xi}\| \\ &\quad\leq \|D_{f_1}\cdots D_{f_{m-1}}\|\big\|\big(f_m - f_m(\xi)\big)k_{r\xi}\big\| \\ &\quad= \|D_{f_1}\cdots D_{f_{m-1}}\|\Big(\int_0^{2\pi}\big|f_m(e^{i\theta}) - f_m(\xi)\big|^2\big|k_{r\xi}(e^{i\theta})\big|^2\,\frac{d\theta}{2\pi}\Big)^{\frac{1}{2}} \\ &\quad\longrightarrow 0, \quad \text{a.e. } \big(|r|\to 1\big).\end{aligned}$$

By induction hypothesis, the result holds.

(2) If T_n is a sequence in $B([K_u^2]^\perp)$ that converges uniformly to T, then

$$\lim_{n\to\infty}\|T_n - T\| = 0 \tag{2.3}$$

and

$$\lim_{r\to 1^-}(\widehat{T}_n)_r(\xi) = \widehat{T}_n(\xi) \tag{2.4}$$

for almost all $\xi \in \mathbb{T}$. Hence

$$\big|(\widehat{T}_n)_r(\xi) - (\widehat{T}_m)_r(\xi)\big| = \big|\big\langle (T_n - T_m)uk_{r\xi}, uk_{r\xi}\big\rangle\big| \leq \|T_n - T_m\|.$$

Taking limits as r approaches 1^- yields

$$\big|\widehat{T}_n(\xi) - \widehat{T}_m(\xi)\big| \leq \|T_n - T_m\|.$$

Therefore, $\widehat{T}_n(\xi)$ is a Cauchy sequence. Let

$$\widehat{E}(\xi) \stackrel{\text{def}}{=} \lim_{n\to\infty}\widehat{T}_n(\xi). \tag{2.5}$$

For almost all $\xi \in \mathbb{T}$, we have

$$\begin{aligned}\big|\widehat{T}_r(\xi) - \widehat{E}(\xi)\big| &= \big|\widehat{T}_r(\xi) - (\widehat{T}_n)_r(\xi) + (\widehat{T}_n)_r(\xi) - \widehat{T}_n(\xi) + \widehat{T}_n(\xi) - \widehat{E}(\xi)\big| \\ &\leq \big|\widehat{T}_r(\xi) - (\widehat{T}_n)_r(\xi)\big| + \big|(\widehat{T}_n)_r(\xi) - \widehat{T}_n(\xi)\big| + \big|\widehat{T}_n(\xi) - \widehat{E}(\xi)\big| \\ &\leq \|T_n - T\| + \big|(\widehat{T}_n)_r(\xi) - \widehat{T}_n(\xi)\big| + \big|\widehat{T}_n(\xi) - \widehat{E}(\xi)\big|.\end{aligned}$$

According to (2.3), (2.4), and (2.5), it follows that

$$\lim_{r\to 1^{-1}} \widehat{T_r}(\xi) = \lim_{n\to\infty} \widehat{T_n}(\xi).$$

(3) For $n = 2$, we have

$$D_{f_1}D_{f_2} - D_{f_1f_2} \in \mathfrak{S}\mathfrak{D}_{L^\infty}.$$

Let $n \geq 2$. Assume that the result is true up to $m-1$. Observe that

$$\begin{aligned} D_{f_1}D_{f_2}\cdots D_{f_m} - D_{f_1f_2\cdots f_m} &= D_{f_1}D_{f_2}\cdots D_{f_m} - D_{f_1}D_{f_2\cdots f_m} \\ &\quad + D_{f_1}D_{f_2\cdots f_m} - D_{f_1f_2\cdots f_m} \\ &= D_{f_1}(D_{f_2}\cdots D_{f_m} - D_{f_2\cdots f_m}) \\ &\quad + D_{f_1}D_{f_2\cdots f_m} - D_{f_1f_2\cdots f_m}. \end{aligned}$$

By induction hypothesis, the result holds.

(4) By the definition of $\mathfrak{S}\mathfrak{D}_{L^\infty}$,

$$\mathcal{A} = \operatorname{span}\bigl\{D_{f_1}D_{f_2}\cdots D_{f_n}[D_f, D_g)D_{g_1}\cdots D_{g_m} : g, g_1, g_2, \ldots, g_m \in L^\infty\bigr\}$$

is a self-adjoint dense (unclosed) subset of $\mathfrak{S}\mathfrak{D}_{L^\infty}$. Since

$$\begin{aligned} &D_{f_1}D_{f_2}\cdots D_{f_n}[D_f, D_g)D_{g_1}\cdots D_{g_m} \\ &\quad = D_{f_1}D_{f_2}\cdots D_{f_n}D_fD_gD_{g_1}\cdots D_{g_m} \\ &\qquad - D_{f_1}D_{f_2}\cdots D_{f_n}D_{fg}D_{g_1}\cdots D_{g_m}, \end{aligned}$$

by Lemma 2.3(1), for $T \in \mathcal{A}$, we have

$$\lim_{r\to 1^-} \langle Tuk_{r\xi}, uk_{r\xi}\rangle = 0.$$

For any $A \in \mathfrak{S}\mathfrak{D}_{L^\infty}$, there exists $\{A_n : n \in \mathbb{Z}^+\} \subset \mathcal{A}$ such that

$$\lim_{n\longrightarrow\infty} \|A_n - A\| = 0.$$

Note that

$$\begin{aligned} \bigl|\langle Auk_{r\xi}, uk_{r\xi}\rangle\bigr| &= \bigl|\bigl\langle (A - A_n + A_n)uk_{r\xi}, uk_{r\xi}\bigr\rangle\bigr| \\ &\leq \bigl|\bigl\langle (A - A_n)uk_{r\xi}, uk_{r\xi}\bigr\rangle\bigr| + \bigl|\langle A_nuk_{r\xi}, uk_{r\xi}\rangle\bigr| \\ &\leq \|A_n - A\| + \bigl|\langle A_nuk_{r\xi}, uk_{r\xi}\rangle\bigr|, \end{aligned}$$

and thus

$$\lim_{r\to 1^-} \langle Auk_{r\xi}, uk_{r\xi}\rangle = 0.$$

Furthermore, if $T \in \mathfrak{D}_{L^\infty}$, then the Cauchy–Schwarz inequality yields

$$\begin{aligned} \bigl|\langle TAuk_{r\xi}, uk_{r\xi}\rangle\bigr| &\leq \|T\|\|Auk_{r\xi}\| \\ &= \|T\|\langle Auk_{r\xi}, Auk_{r\xi}\rangle^{\frac{1}{2}} \\ &= \|T\|\langle A^*Auk_{r\xi}, uk_{r\xi}\rangle^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned}\left|\langle ATuk_{r\xi}, uk_{r\xi}\rangle\right| &= \left|\langle uk_{r\xi}, T^*A^*uk_{r\xi}\rangle\right| \\ &\le \|T^*\|\|A^*uk_{r\xi}\| \\ &= \|T\|\langle A^*uk_{r\xi}, A^*uk_{r\xi}\rangle^{\frac{1}{2}} \\ &= \|T\|\langle AA^*uk_{r\xi}, uk_{r\xi}\rangle^{\frac{1}{2}}.\end{aligned}$$

Since A^*A and AA^* are in $\mathfrak{SD}_{L^\infty}$, it follows that

$$\lim_{r\to 1^-}\langle TAuk_{r\xi}, uk_{r\xi}\rangle = 0,$$
$$\lim_{r\to 1^-}\langle ATuk_{r\xi}, uk_{r\xi}\rangle = 0.$$

(5) If $\lim_{n\to\infty}\|D_{f_n} - T\| = 0$, then $\widetilde{T}$ is a bounded operator on L^2. Under the decomposition $L^2(\mathbb{T}) = H^2 \oplus \overline{zH^2}$, write $\widetilde{T} = \left(\begin{smallmatrix} A & B \\ H & C\end{smallmatrix}\right)$. We have

$$\begin{aligned}\|D_{f_n} - T\| &= \|\widetilde{D}_{f_n} - \widetilde{T}\| \\ &\ge \left\|P(\widetilde{D}_{f_n} - \widetilde{T})P\right\| \\ &= \|T_{f_n} - A\|.\end{aligned}$$

Since the uniform limit of a Toeplitz operator on H^2 is also a Toeplitz operator, A is a Toeplitz operator on H^2, by [25, Theorem 3.2]. Let

$$A = T_\psi,$$

where $\psi = A1 - A1(0) + \overline{A^*1}$. By Lemma 2.2, we have

$$\|T_{f_n} - T_\psi\| = \|f_n - \psi\|_\infty = \|D_{f_n} - D_\psi\|.$$

Hence $T = D_\psi$. □

Remark 2.4 (of Lemma 2.3(2)). Let T'_n be another sequence in $B([K_u^2]^\perp)$ that converges uniformly to T, and let

$$\lim_{r\to 1^-}(\widehat{T'}_n)_r(\xi) = \widehat{T'}_n(\xi)$$

for almost all $\xi \in \mathbb{T}$. For a positive integer k, there exist T_{n_k}, T'_{n_k} such that

$$\|T_{n_k} - T\| \le \frac{1}{2k},$$
$$\|T'_{n_k} - T\| \le \frac{1}{2k}.$$

Hence

$$\|T_{n_k} - T'_{n_k}\| \le \|T_{n_k} - T\| + \|T'_{n_k} - T\| \le \frac{1}{k},$$
$$\left|\widehat{T'}_{n_k}(\xi) - \widehat{T}_{n_k}(\xi)\right| \le \frac{1}{k}.$$

That means that

$$\lim_{k\to\infty}\widehat{T'}_{n_k}(\xi) = \lim_{k\to\infty}\widehat{T}_{n_k}(\xi)$$

and

$$\lim_{n\to\infty}\widehat{T'}_n(\xi)=\lim_{n\to\infty}\widehat{T}_n(\xi)=\widehat{E}(\xi).$$

Then for any $T\in\mathfrak{D}_{L^\infty}$, there exist

$$\{T_n\}_{n\geq 0}\subset\left\{\sum_{i=1}^{n}\prod_{j=1}^{m}D_{\phi_{ij}}:\phi_{ij}\in L^\infty\right\}$$

such that

$$\lim_{n\longrightarrow\infty}\|T_n-T\|=0.$$

We define the mapping

$$\begin{aligned}\rho:\mathfrak{D}_{L^\infty}&\longrightarrow L^\infty,\\ T&\longmapsto\rho(T),\end{aligned}$$

where

$$\rho(T)(\xi)=\lim_{r\to 1^-}\langle Tuk_{r\xi},uk_{r\xi}\rangle=\lim_{n\to\infty}\widehat{T}_n(\xi)$$

for almost all $\xi\in\mathbb{T}$. We call ρ the *symbol map* on the dual truncated Toeplitz algebra $\mathfrak{D}_{L^\infty}$. We have the following analogue of the result by Stroethoff and Zheng [26, Theorem 8.4] of the dual Toeplitz algebra on the orthogonal complement of the Bergman space.

Theorem 2.5. *The sequence*

$$0\longrightarrow\mathfrak{S}\mathfrak{D}_{L^\infty}\longrightarrow\mathfrak{D}_{L^\infty}\longrightarrow L^\infty\longrightarrow 0$$

*is a short exact sequence; that is, the quotient algebra $\mathfrak{D}_{L^\infty}/\mathfrak{S}\mathfrak{D}_{L^\infty}$ is *-isometrically isomorphic to L^∞.*

Proof. By the definition of ρ, we have

$$\|\rho(T)\|_\infty\leq\|T\|. \tag{2.6}$$

Furthermore,

$$\begin{aligned}\rho(T^*)(\xi)&=\lim_{r\to 1^{-1}}\langle T^*uk_{r\xi},uk_{r\xi}\rangle\\ &=\lim_{r\to 1^{-1}}\langle uk_{r\xi},Tuk_{r\xi}\rangle\\ &=\overline{\rho(T)(\xi)}.\end{aligned}$$

Since linear combinations of operators of the form $D_{f_1}D_{f_2}\cdots D_{f_n}$ span a dense subset of $\mathfrak{D}_{L^\infty}$,

$$\prod_{j=1}^{n}D_{f_j}=D_{\prod_{j=1}^{n}f_j}+\prod_{j=1}^{n}D_{f_j}-D_{\prod_{j=1}^{n}f_j},$$

and by Lemma 2.3(3), it follows that operators of the form

$$D=D_\phi+A,\quad\phi\in L^\infty,A\in\mathfrak{S}\mathfrak{D}_{L^\infty} \tag{2.7}$$

form a dense subset of $\mathfrak{D}_{L^\infty}$. If $D_g\in\mathfrak{S}\mathfrak{D}_{L^\infty}$, then by Lemma 2.3(1), (4), we have that $g=0$ a.e. on $\mathbb{T}$. According to Lemma 2.3(5), since $\mathfrak{S}\mathfrak{D}_{L^\infty}$ is closed, every operator in $\mathfrak{D}_{L^\infty}$ is of the form (2.7). In fact, the mapping ρ has a more precise form

$$\rho : D_\phi + A \longmapsto \phi.$$

Its kernel is precisely the semicommutator ideal $\mathfrak{S}\mathfrak{D}_{L^\infty}$ of $\mathfrak{D}_{L^\infty}$. By Lemma 2.3(4), we have $\|\rho(D_\phi + A)\| = \|\phi\|_\infty$. For any $D_\phi + \mathfrak{S}\mathfrak{D}_{L^\infty} \in \mathfrak{D}_{L^\infty}/\mathfrak{S}\mathfrak{D}_{L^\infty}$, we define the mapping

$$\begin{aligned} \widetilde{\rho} : \mathfrak{D}_{L^\infty}/\mathfrak{S}\mathfrak{D}_{L^\infty} &\longrightarrow L^\infty, \\ D_\phi + \mathfrak{S}\mathfrak{D}_{L^\infty} &\longmapsto \phi. \end{aligned}$$

Then $\widetilde{\rho}$ is a bijection. For every $\varphi \in L^\infty$, we define the mapping

$$\begin{aligned} \sigma : L^\infty &\longrightarrow \mathfrak{D}_{L^\infty}, \\ \varphi &\longmapsto D_\varphi. \end{aligned}$$

The mapping σ is obviously linear and contractive. Observe that for $f, g \in L^\infty(\mathbb{T})$, we have

$$\sigma(f)\sigma(g) - \sigma(fg) = D_f D_g - D_{fg} \in \mathfrak{S}\mathfrak{D}_{L^\infty}.$$

So $\widetilde{\rho}$ is homomorphism. □

3. Dual truncated Toeplitz algebra $\mathfrak{D}_{C(\mathbb{T})}$

Let $C(\mathbb{T})$ denote the set of continuous complex-valued functions on $\mathbb{T}$.

Lemma 3.1 ([20, Theorem 5.5]). *Let $f \in L^\infty$. The Hankel operator H_f is compact if and only if $f \in H^\infty + C(\mathbb{T})$.*

Since $M_f M_g = M_{fg}$, we have

$$T_{fg} = T_f T_g + H^*_{\bar{f}} H_g, \tag{3.1}$$

$$H_{fg} = H_f T_g + S_f H_g. \tag{3.2}$$

We consider the compact semicommutator of the dual truncated Toeplitz operator. By the matrix representation (2.2), we have

$$U^* D_f D_g U = \begin{pmatrix} T_f T_g + H^*_{u\bar{f}} H_{gu} & T_f H^*_{u\bar{g}} + H^*_{u\bar{f}} S_g \\ H_{fu} T_g + S_f H_{gu} & H_{fu} H^*_{\bar{g}u} + S_f S_g \end{pmatrix}$$

and

$$U^* D_{fg} U = \begin{pmatrix} T_{fg} & H^*_{u\overline{fg}} \\ H_{ufg} & S_{fg} \end{pmatrix}.$$

Using (3.1) and (3.2), we have

$$\begin{aligned}
T_fT_g + H^*_{u\bar{f}}H_{gu} - T_{fg} &= T_fT_g + T_{\bar{u}fgu} - T_{\bar{u}f}T_{gu} - T_{fg}\\
&= T_fT_g - T_{\bar{u}f}T_{gu},\\
H_{fu}H^*_{\bar{g}u} + S_fS_g - S_{fg} &= V(H^*_{fu}H_{\bar{g}u} + T_{\bar{f}}T_{\bar{g}} - T_{\overline{fg}})V\\
&= V(T_{\bar{f}\bar{u}\bar{g}u} - T_{\bar{f}\bar{u}}T_{\bar{g}u} + T_{\bar{f}}T_{\bar{g}} - T_{\overline{fg}})V\\
&= V(T_{\bar{f}}T_{\bar{g}} - T_{\bar{f}\bar{u}}T_{\bar{g}u})V,\\
H_{fu}T_g + S_fH_{gu} - H_{ufg} &= H_{fu}T_g - H_fT_{ug},
\end{aligned}$$

and

$$\begin{aligned}
(T_fH^*_{u\bar{g}} + H^*_{u\bar{f}}S_g - H^*_{u\overline{fg}})^* &= H_{u\bar{g}}T_{\bar{f}} + S_{\bar{g}}H_{u\bar{f}} - H_{u\overline{fg}}\\
&= H_{u\bar{g}}T_{\bar{f}} - H_{\bar{g}}T_{u\bar{f}}.
\end{aligned}$$

Hence, we have the following theorem.

Theorem 3.2. *If $f, g \in L^\infty$, then $D_fD_g - D_{fg}$ is compact (finite rank) if and only if $T_fT_g - T_{\bar{u}f}T_{gu}, T_{\bar{f}}T_{\bar{g}} - T_{\bar{f}\bar{u}}T_{\bar{g}u}, H_{uf}T_g - H_fT_{ug}$, and $H_{u\bar{g}}T_{\bar{f}} - H_{\bar{g}}T_{u\bar{f}}$ are all compact (finite rank).*

Remark 3.3. We can find that the conditions of $T_fT_g - T_{\bar{u}f}T_{gu}$, $T_{\bar{f}}T_{\bar{g}} - T_{\bar{f}\bar{u}}T_{\bar{g}u}$, $H_{uf}T_g - H_fT_{ug}$, and $H_{u\bar{g}}T_{\bar{f}} - H_{\bar{g}}T_{u\bar{f}}$ are finite rank in [11, Theorem 3.4] and [10, Theorem 4.2].

Corollary 3.4. *If $f \in L^\infty$ and $g \in QC = [H^\infty + C(\mathbb{T})] \cap \overline{[H^\infty + C(\mathbb{T})]}$, then $D_fD_g - D_{fg}$ is compact.*

Proof. By Lemma 3.1 and the fact that $H^\infty + C(\mathbb{T})$ is a closed subalgebra of L^∞, we have g, gu, $\bar{g}$, and $\bar{g}u \in H^\infty + C(\mathbb{T})$; hence H_g, H_{gu}, $H_{\bar{g}}$, and $H_{\bar{g}u}$ are compact. Due to (3.1) and (3.2), it follows that

$$\begin{aligned}
T_fT_g - T_{\bar{u}f}T_{gu} &= T_fT_g - T_{fg} + T_{fg} - T_{\bar{u}f}T_{gu}\\
&= T_fT_g - T_{fg} + T_{\bar{u}fgu} - T_{\bar{u}f}T_{gu}\\
&= -H^*_{\bar{f}}H_g + H^*_{u\bar{f}}H_{ug},\\
T_{\bar{f}}T_{\bar{g}} - T_{\bar{f}\bar{u}}T_{\bar{g}u} &= T_{\bar{f}}T_{\bar{g}} - T_{\bar{f}\bar{g}} + T_{\bar{f}\bar{g}} - T_{\bar{f}\bar{u}}T_{\bar{g}u}\\
&= T_{\bar{f}}T_{\bar{g}} - T_{\bar{f}\bar{g}} + T_{\bar{u}\bar{f}\bar{g}u} - T_{\bar{f}\bar{u}}T_{\bar{g}u}\\
&= -H^*_fH_{\bar{g}} + H^*_{uf}H_{u\bar{g}},\\
H_{fu}T_g - H_fT_{ug} &= H_{fu}T_g - H_{ufg} + S_fH_{gu}\\
&= S_{fu}H_g + S_fH_{gu}.
\end{aligned}$$

According to Theorem 3.2, $D_fD_g - D_{fg}$ is compact. □

Next, we investigate the dual truncated Toeplitz operators with continuous symbol.

Lemma 3.5. *Let φ be in $C(\mathbb{T})$. Then*

$$\|D_\varphi\|_e = \|\varphi\|_\infty.$$

Proof. Since φ is in $C(\mathbb{T})$, $u\varphi$ and $u\overline{\varphi}$ are in $H^\infty + C(\mathbb{T})$. It follows from Lemma 3.1 that both $H^*_{u\bar\varphi}$ and $H_{u\varphi}$ are compact. From the matrix representation (2.2), we have that

$$\begin{aligned}
\|D_\varphi\|_e = \|\widetilde{D}_\varphi\|_e &= \inf\left\{\left\|\begin{pmatrix} T_\varphi & H^*_{u\bar\varphi} \\ H_{u\varphi} & S_\varphi \end{pmatrix} + K\right\|, K \text{ is a compact operator}\right\} \\
&= \inf\left\{\left\|\begin{pmatrix} T_\varphi & 0 \\ 0 & S_\varphi \end{pmatrix} + K\right\|, K \text{ is a compact operator}\right\} \\
&= \left\|\begin{pmatrix} T_\varphi & 0 \\ 0 & S_\varphi \end{pmatrix}\right\|_e \\
&= \left\|\begin{pmatrix} T_\varphi & 0 \\ 0 & S_\varphi \end{pmatrix}^* \begin{pmatrix} T_\varphi & 0 \\ 0 & S_\varphi \end{pmatrix}\right\|_e^{1/2} \\
&= \left\|\begin{pmatrix} T^*_\varphi T_\varphi & 0 \\ 0 & (S_\varphi)^* S_\varphi \end{pmatrix}\right\|_e^{1/2} \\
&= \left[r_e \begin{pmatrix} T^*_\varphi T_\varphi & 0 \\ 0 & (S_\varphi)^* S_\varphi \end{pmatrix}\right]^{1/2} \qquad (r_e \text{ is essential spectral radius}) \\
&= \max\left\{r_e(T^*_\varphi T_\varphi), r_e\big((S_\varphi)^* S_\varphi\big)\right\}^{1/2} \\
&= \max\left\{\|T_\varphi\|_e, \|S_\varphi\|_e\right\} \\
&= \max\left\{\|T_\varphi\|_e, \|VT_{\bar\varphi}V\|_e\right\} = \|\varphi\|_\infty.
\end{aligned}$$

The last equality follows from the fact that $\|T_\varphi\|_e = \|\varphi\|_\infty$ (see [19, Corollary 4.5.3]). □

Lemma 3.6. *We have that $\mathfrak{D}_{C(\mathbb{T})}$ is an irreducible C^*-algebra.*

Proof. Suppose that $\mathfrak{D}_{C(\mathbb{T})}$ is reducible. Then there exists a nontrivial orthogonal projection P_0 which commutes with each D_φ for all $\varphi \in C(\mathbb{T})$. We have that $\widetilde{P}_0 = U^* P_0 U$ is an orthogonal projection on L^2. Under the decomposition $L^2(\mathbb{T}) = H^2 \oplus \overline{zH^2}$, write

$$\widetilde{P}_0 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, \tag{3.3}$$

where P_1 is an orthogonal projection on H^2, and P_2 is an orthogonal projection on $\overline{zH^2}$. For each integer n, $z^n \in C(\mathbb{T})$,

$$\widetilde{D}_{z^n}\widetilde{P}_0 = \widetilde{P}_0\widetilde{D}_{z^n}.$$

By the matrix representation (2.2), we have

$$\widetilde{D}_z\widetilde{P}_0 = \begin{pmatrix} T_z P_1 & H^*_{u\bar z} P_2 \\ 0 & VT_{\bar z}VP_2 \end{pmatrix}$$

and

$$\widetilde{P}_0\widetilde{D}_z = \begin{pmatrix} P_1 T_z & P_1 H^*_{u\bar z} \\ 0 & P_2 VT_{\bar z}V \end{pmatrix}.$$

Thus

$$T_zP_1 = P_1T_z, \qquad VT_{\bar{z}}VP_2 = P_2VT_{\bar{z}}V.$$

By [18, Problem 147], P_1 is an analytic Toeplitz operator, and VP_2V is a coanalytic Toeplitz operator. Since $P_1^2 = P_1$, $P_2^2 = P_2$ and the only idempotent Toeplitz operators are 0 and 1 (see [5, Corollary 6]), it follows that P_1 is 0 or 1 and P_2 is 0 or 1. Thus, we distinguish four cases.

Case 1. $\widetilde{P}_0 = I$;

Case 2. $\widetilde{P}_0 = O$;

Case 3.

$$\widetilde{P}_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix};$$

Case 4.

$$\widetilde{P}_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Cases 1 and 2 contradict the assumption of P_0. Case 3 is very similar to Case 4. We only need to consider Case 3. In Case 3, where H^2 is the reducing subspace for $\widetilde{D}_{z^n}$ and n is positive, we have

$$\widetilde{D}_{z^n} = \begin{pmatrix} T_{z^n} & H^*_{u\bar{z}^n} \\ 0 & VT_{\bar{z}^n}V \end{pmatrix}.$$

According to [9, Proposition 3.7], we have $H^*_{u\bar{z}^n} = 0$. Thus $u\bar{z}^n \in H^2$,

$$u \in \bigcap_{n=1}^{\infty} z^nH^2 = \{0\}.$$

This leads to a contradiction. □

Lemma 3.7 ([21, Theorem 2.5])**.** *Let $f, g \in L^\infty$. Assume that $D_fD_g = D_gD_f$. Then either*

(1) *both f and g are analytic; or*
(2) *both f and g are coanalytic; or*
(3) *a nontrivial linear combination of f and g is constant.*

Lemma 3.8. *The set of all compact operators on $[K_u^2]^\perp$ will be denoted by $\mathcal{K}$. Then*

(1) $\mathcal{K} \subset \mathfrak{D}_{C(\mathbb{T})}$,
(2) $\mathfrak{S}\mathfrak{D}_{C(\mathbb{T})} = \mathfrak{C}\mathfrak{D}_{C(\mathbb{T})} = \mathcal{K}$.

Proof. (1) If f and g are in $C(\mathbb{T})$, note that

$$D_fD_g - D_gD_f = D_fD_g - D_{fg} + D_{fg} - D_gD_f.$$

By Corollary 3.4, we have that $D_fD_g - D_gD_f$ is compact. Using Lemma 3.7 and $z, \bar{z} \in C(\mathbb{T})$, we have that $D_zD_{\bar{z}} - D_{\bar{z}}D_z$ is a nonzero compact operator. A theorem in [13, Theorem 5.39] states that the commutator ideal of every irreducible algebra contains the ideal of compact operators if it contains a nontrivial compact operator. By Lemma 3.6, we have that $\mathfrak{D}_{C(\mathbb{T})}$ is irreducible; therefore, $\mathcal{K} \subset \mathfrak{D}_{C(\mathbb{T})}$.

(2) By Corollary 3.4, if f and g are in $C(\mathbb{T})$, then both $[D_f, D_g)$ and $[D_f, D_g]$ are in $\mathfrak{D}_{C(\mathbb{T})}$. Since $\mathfrak{C}\mathfrak{D}_{C(\mathbb{T})}$ and $\mathfrak{S}\mathfrak{D}_{C(\mathbb{T})}$ are closed two-sided *-ideals of $\mathfrak{D}_{C(\mathbb{T})}$ and $\mathcal{K}$ contains no proper closed ideal, it follows that $\mathfrak{S}\mathfrak{D}_{C(\mathbb{T})} = \mathfrak{C}\mathfrak{D}_{C(\mathbb{T})} = \mathcal{K}$. □

Theorem 3.9. *The sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathfrak{D}_{C(\mathbb{T})} \longrightarrow C(\mathbb{T}) \longrightarrow 0$$

*is a short exact sequence; that is, the quotient algebra $\mathfrak{D}_{C(\mathbb{T})}/\mathcal{K}$ is *-isometrically isomorphic to $C(\mathbb{T})$.*

Proof. By (2.7) and Lemma 3.8, we have

$$\mathfrak{D}_{C(\mathbb{T})} = \big\{D_\phi + K : \phi \in C(\mathbb{T}), K \in \mathcal{K}\big\}.$$

Let us define the symbol map π given by

$$\pi(D_\varphi + \mathcal{K}) \to \varphi.$$

By Lemma 3.5, π is well defined. Thus π is *-isometrically isomorphic from $\mathfrak{D}_{C(\mathbb{T})}/\mathcal{K}$ to $C(\mathbb{T})$. □

Recall that the Toeplitz algebra $\mathfrak{T}_{C(\mathbb{T})}$ is the smallest closed subalgebra of $B(H^2)$ containing $\{T_\phi : \phi \in C(\mathbb{T})\}$. Using [13, Theorem 7.23], we have the following corollary.

Corollary 3.10. *There exists a *-homomorphism ζ from $\mathfrak{D}_{C(\mathbb{T})}/\mathcal{K}$ onto $\mathfrak{T}_{C(\mathbb{T})}/\mathcal{K}_{B(H^2)}$. Hence, if $f \in C(\mathbb{T})$, then $\sigma_e(D_f) = \sigma_e(T_f)$.*

Corollary 3.11. *There exists a *-homomorphism ζ from the quotient algebra $\mathfrak{D}_{L^\infty}/\mathcal{K}$ onto L^∞ such that the diagram*

$$\begin{array}{ccc} \mathfrak{D}_{L^\infty} & \xrightarrow{\ \pi\ } & \mathfrak{D}_{L^\infty}/\mathcal{K} \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\zeta} & \\ & L^\infty & \end{array}$$

commutes. If $\varphi \in L^\infty$ and D_φ is a Fredholm operator, then φ is invertible in L^∞.

4. Spectrum

Theorem 4.1. *If $\varphi \in L^\infty$, then*

$$\mathcal{R}(\varphi) \subset \sigma_e(D_\varphi) \subset \sigma(D_\varphi) \subset h\big(\mathcal{R}(\varphi)\big),$$

where $\mathcal{R}(\varphi)$ is the essential range of φ and $h(\mathcal{R}(\varphi))$ is the closed convex hull of $R(\varphi)$.

Proof. Let $\lambda \notin h(\mathcal{R}(\varphi))$. Since $h(\mathcal{R}(\varphi))$ is a compact subset of the complex plane, there is disk $B = B(a, r)$ such that $\mathcal{R}(\varphi) \subset B$ and $\lambda \notin B$. Thus

$$|\lambda - a| > \operatorname*{ess\,sup}_{\mathbb{T}} |\varphi - a| = \|\varphi - a\|_\infty = \|D_{\varphi - a}\|.$$

We deduce that $\lambda I - D_\varphi = (\lambda - a)I - D_{\varphi-a}$ is invertible, so $\lambda \notin \sigma(D_\varphi)$. Using Corollary 3.11 and $D_{\varphi-\lambda} = D_\varphi - \lambda$, we have $\mathcal{R}(\varphi) \subset \sigma_e(D_\varphi)$. □

Example 4.2. Using Theorem 4.1, we have $\sigma(D_{z^n+\overline{z^n}}) = \mathcal{R}(z^n + \overline{z^n}) = [-2, 2]$, where $n \in \mathbb{Z}_+$.

If $\varphi \in L^\infty$, then write $\varphi = \varphi_+ + \varphi_-$, where $\varphi_+ = P\varphi$ and $\varphi_- = (I-P)\varphi$. Assume that $\varphi_+, \overline{\varphi}_- \in K^2_{zu} \cap H^\infty$. Then $H_{u\varphi} = H_{u\varphi_-} = 0$ and $H_{u\bar\varphi} = H_{u\overline{\varphi_+}} = 0$. Therefore, D_φ is unitarily equivalent to a diagonal operator matrix, that is,

$$U^* D_\varphi U = \begin{pmatrix} T_\varphi & 0 \\ 0 & VT_{\bar\varphi}V \end{pmatrix}.$$

Since $\sigma(T_\varphi) = \sigma(VT_{\bar\varphi}V)$, $\sigma(D_\varphi) = \sigma(T_\varphi) \cup \sigma(VT_{\bar\varphi}V) = \sigma(T_\varphi)$. By [13, Theorem 7.21], we have Theorem 4.3(1). We provide an alternative proof as follows.

Theorem 4.3. *Let $\varphi \in L^\infty$.*

(1) *If $\varphi \in K^2_{zu} \cap H^\infty$, then $\sigma(D_\varphi) = \operatorname{clos}(\widetilde{\varphi}(\mathbb{D}))$, where $\widetilde{\varphi}$ is its harmonic extension of φ to $\mathbb{D}$.*
(2) *If $\varphi \in QC$, then $\sigma_e(D_\varphi) = \mathcal{R}(\varphi)$.*

Proof. (1) Recall that $k_\lambda = \frac{\sqrt{1-|\lambda|^2}}{1-w\bar\lambda}$ denotes the normalized Hardy reproducing kernel at λ. For $\lambda \in \mathbb{D}$, we have

$$\begin{aligned} D_{\bar\varphi} u k_\lambda &= (uP\bar u + I - P)\bar\varphi u k_\lambda \\ &= uP\bar\varphi k_\lambda + 0 \\ &= \overline{\widetilde{\varphi}(\lambda)} u k_\lambda; \end{aligned}$$

hence $\widetilde{\varphi}(\mathbb{D}) \subset \sigma(D_\varphi)$. For each nonzero constant λ and $\lambda \notin \operatorname{clos}(\widetilde{\varphi}(\mathbb{D}))$, we have $\lambda \perp uzH^2$. Then $\varphi - \lambda \in K^2_{zu}$. For some $\epsilon > 0$, we have $|\varphi(z) - \lambda| \geq \epsilon$, a.e. $z \in \mathbb{T}$. Then $\frac{1}{\varphi-\lambda}$ is in L^∞. Moreover,

$$U^* D_{\varphi-\lambda} U = \begin{pmatrix} T_{\varphi-\lambda} & 0 \\ 0 & S_{\varphi-\lambda} \end{pmatrix}.$$

Hence $D_{\varphi-\lambda} D_{\frac{1}{\varphi-\lambda}} = D_{\frac{1}{\varphi-\lambda}} D_{\varphi-\lambda} = I$, so $\lambda \notin \sigma(D_\varphi)$.

(2) By Theorem 4.1, we have $\mathcal{R}(\varphi) \subset \sigma_e(D_\varphi)$. If $\lambda \notin \mathcal{R}(\varphi)$, then for some $\epsilon > 0$ we have $|\varphi(z) - \lambda| \geq \epsilon$, a.e. $z \in \mathbb{T}$. Then $g = \frac{1}{\varphi-\lambda}$ is in L^∞. By Corollary 3.4,

$$D_{\varphi-\lambda} D_g = I + K_1, \qquad D_g D_{\varphi-\lambda} = I + K_2,$$

where K_1 and K_2 are compact. We have that $D_{\varphi-\lambda} + \mathcal{K}$ is invertible in the Calkin algebra, so $\lambda \notin \sigma_e(D_\varphi)$. □

Example 4.4. Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. If $u(0) = 0$, then $z \perp zuH^2$. By Theorem 4.3(1), we have $\sigma(D_z) = \overline{\mathbb{D}}$.

If $u(0) \neq 0$, note that

$$\widetilde{D}_z = \begin{pmatrix} T_z & H^*_{u\bar z} \\ 0 & S_z \end{pmatrix}.$$

For all $z^n \in L^2$, we have

$$\widetilde{D}_z \begin{pmatrix} z^n \\ 0 \end{pmatrix} = \begin{pmatrix} T_z z^n \\ 0 \end{pmatrix} = \begin{pmatrix} z^{n+1} \\ 0 \end{pmatrix}, \quad n \geq 0,$$
$$\widetilde{D}_z \begin{pmatrix} 0 \\ \bar{z} \end{pmatrix} = \begin{pmatrix} H^*_{u\bar{z}} \bar{z} \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{u}(0) \\ 0 \end{pmatrix},$$

and

$$\widetilde{D}_z \begin{pmatrix} 0 \\ \bar{z}^{m+2} \end{pmatrix} = \begin{pmatrix} 0 \\ S_z \bar{z}^{m+2} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{z}^{m+1} \end{pmatrix}, \quad m \geq 0.$$

This implies that, for all integers n,

$$\widetilde{D}_z z^n = w_n z^{n+1},$$

$$w_n = \begin{cases} 1 & \text{if } n \neq -1, \\ \bar{u}(0), \quad (0 < |u(0)| < 1) & \text{if } n = -1. \end{cases}$$

By [24, Theorem 2(a)], there exists an invertible operator A on L^2 such that

$$AM_z = \widetilde{D}_z A, \tag{4.1}$$

where M_z is the bilateral shift and

$$Az^n = \alpha_n z^n,$$

$$\alpha_n = \begin{cases} 1 & \text{if } n \geq 0, \\ \frac{1}{\bar{u}(0)} & \text{if } n \leq -1. \end{cases}$$

Under the decomposition $L^2 = H^2 \oplus \overline{zH^2}$,

$$A = \begin{pmatrix} I_{H^2} & 0 \\ 0 & \frac{1}{\bar{u}(0)} I_{\overline{zH^2}} \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} I_{H^2} & 0 \\ 0 & \bar{u}(0) I_{\overline{zH^2}} \end{pmatrix},$$

where I_{H^2} is the identity on H^2. Due to (4.1), $\widetilde{D}_z$ and M_z are similar. Note that M_z is the bilateral shift and $\sigma(M_z) = \mathbb{T}$ (see [13, Example 4.25]). Hence $\sigma(D_z) = \sigma(\widetilde{D}_z) = \mathbb{T}$.

Acknowledgments. The authors thank the referees for constructive comments. We are also grateful to Professor Dechao Zheng for various helpful suggestions.

Ding's work was partially supported by National Natural Science Foundation of China (NSFC) grant 11871122, by University Innovation Team Program of Chongqing grant CXTDX201601026, and by grant cstc2018jcyjAX0595. Qin and Sang's work was partially supported by NSFC grant 11501059.

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