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TRANSLATION THEOREMS FOR THE FOURIER–FEYNMAN TRANSFORM ON THE PRODUCT FUNCTION SPACE $C^2_{a,b}[0,T]$

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Abstract. In this article, we establish the Cameron–Martin translation theorems for the analytic Fourier–Feynman transform of functionals on the product function space $C^2_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$ is induced by the generalized Brownian motion process associated with continuous functions $a(t)$ and $b(t)$ on the time interval $[0,T]$. The process used here is nonstationary in time and is subject to a drift $a(t)$. To study our translation theorem, we introduce a Fresnel-type class $\mathcal{F}^{a,b}_{A_1,A_2}$ of functionals on $C^2_{a,b}[0,T]$, which is a generalization of the Kallianpur and Bromley–Fresnel class $\mathcal{F}_{A_1,A_2}$. We then proceed to establish the translation theorems for the functionals in $\mathcal{F}^{a,b}_{A_1,A_2}$.

1. Introduction

Let H be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $\mathcal{M}(H)$ be the space of all complex-valued Borel measures on H. The Fourier transform of σ in $\mathcal{M}(H)$ is defined by

$$f(\sigma)(h') \equiv \widehat{\sigma}(h') = \int_H \exp\big\{i\langle h,h'\rangle\big\}\,d\sigma(h), \quad h' \in H. \tag{1.1}$$

The set of all functionals of the form (1.1) is denoted by $\mathcal{F}(H)$ and is called the *Fresnel class* of H. Let (B,H,ν) be an abstract Wiener space. It is known (see

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[20], [21]) that each functional of the form (1.1) can be extended to B uniquely by

$$\widehat{\sigma}^{\sim}(x) = \int_H \exp\{i(h,x)^{\sim}\}\, d\sigma(h), \quad x \in B, \tag{1.2}$$

where $(\cdot,\cdot)^{\sim}$ is the stochastic inner product between H and B given by

$$(h,x)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (x, e_j) & \text{if the limit exists,} \\ 0 & \text{otherwise,} \end{cases}$$

and where $\{e_n\}$ is a complete orthonormal system in H such that the e_n's are in B^*, the dual of B, and $(\cdot,\cdot)$ denotes the natural dual pairing (in the sense of the Riesz theorem) between B and B^*. The Fresnel class $\mathcal{F}(B)$ of B is the space of (equivalence classes of) all functionals of the form (1.2). (For an elementary introduction to the classes $\mathcal{F}(B)$ and $\mathcal{F}(H)$, see [19, Chapter 20].)

It is well known that both the analytic Wiener integral and the Feynman integral exist for all functionals in the Fresnel class $\mathcal{F}(B)$ (see Kallianpur and Bromley [20] (as mentioned in [7]) for a successful treatment of certain physical problems by means of a Feynman integral; e.g., the anharmonic oscillator of [1]—they introduced a class $\mathcal{F}_{A_1,A_2}$ larger than the Fresnel class $\mathcal{F}(B)$ and showed the existence of the analytic Feynman integral of functionals in $\mathcal{F}_{A_1,A_2}$). The Fresnel class $\mathcal{F}_{A_1,A_2}$ of B^2 is the space of (equivalence classes of) all functionals on B^2 of the form

$$F(x_1,x_2) = \int_H \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}h, x_j)^{\sim}\Big\}\, d\sigma(h),$$

where A_1 and A_2 are bounded, nonnegative, and self-adjoint operators on H and $\sigma \in \mathcal{M}(H)$.

Let A be a nonnegative self-adjoint operator on H, and let σ be any complex Borel measure on H. Then the functional

$$F(x) = \int_H \exp\{i(A^{1/2}h, x)^{\sim}\}\, d\sigma(h) \tag{1.3}$$

belongs to the Fresnel class $\mathcal{F}(B)$ on B because it can be rewritten as

$$\int_H \exp\{i(h,x)^{\sim}\}\, d\sigma_A(h)$$

for $\sigma_A = \sigma \circ (A^{1/2})^{-1}$. For the functional F given by (1.3), the analytic Feynman integral $\int_B^{\mathrm{anf}_1} F(x)\, d\nu(x)$ with parameter $q = 1$ (based on the connection with the Fresnel integral of F in $\mathcal{F}(H)$ by Albeverio and Høegh-Krohn [1], the most important value of the parameter q is $q = 1$) on B exists and is given by

$$\int_B^{\mathrm{anf}_1} F(x)\, d\nu(x) = \int_H \exp\Big\{-\frac{i}{2}\langle Ah, h\rangle\Big\}\, d\sigma(h). \tag{1.4}$$

If we choose A to be the identity operator on H, then (1.4) is equal to "the Fresnel integral $\mathcal{F}(f)$" of $f(\sigma)$ studied in [1]. The concept of the Fresnel integral is not based on the technique of analytic continuation but rather is derived from solutions of important problems in quantum mechanics and quantum field theory; to achieve this, Kallianpur and Bromley [20] suggested the general Fresnel class $\mathcal{F}_{A_1,A_2}$.

Let A be a bounded self-adjoint operator on H. Then we may write

$$A = A_+ - A_-,$$

where A_+ and A_- are each bounded, nonnegative, and self-adjoint. Take $A_1 = A_+$ and $A_2 = A_-$ in the definition of $\mathcal{F}_{A_1,A_2}$ above. For any F in $\mathcal{F}_{A_+,A_-}$, the analytic Feynman integral of F with parameter $(1,-1)$ is given by

$$\int_{B^2}^{\mathrm{anf}_{(1,-1)}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2) = \int_H \exp\Big\{-\frac{i}{2}\langle Ah,h\rangle\Big\}\,d\sigma(h). \tag{1.5}$$

Kallianpur and Bromley, using this idea, studied relationships between Albeverio and Høegh–Krohn's Fresnel integral with respect to a symmetric bilinear form Δ on H (see [1, Chapter 4]) and the analytic Feynman integral given by (1.5).

Let $(C_0[0,T], m_w)$ denote a 1-parameter Wiener space, where $C_0[0,T]$ is the space of all real-valued continuous functions x on the compact interval $[0,T]$ with $x(0) = 0$ and where m_w is the Gaussian measure on $C_0[0,T]$ with mean zero and covariance function $r(s,t) = \min\{s,t\}$. It is well known that there is no quasi-invariant measure on infinite-dimensional linear spaces. Thus, the Wiener measures m_w and the abstract Wiener measure ν are not quasi-invariant. Based on this circumstance, numerous constructions and applications of the Cameron–Martin translation theorem for integrals on infinite-dimensional spaces have been studied in various research fields, including mathematics and physics. Most of the results in the literature are concentrated on the Wiener space $C_0[0,T]$. Translation theorems for Wiener integrals were given by Cameron and Martin in [3] and by Cameron and Graves in [2]. Translation theorems for analytic Feynman integrals were given by Cameron and Storvick in [4] and [6], and translation theorems for analytic Feynman integrals on abstract Wiener and Hilbert spaces were given by Chung and Kang in [17].

On the other hand, the translation theorem for the function space integral and the *generalized analytic Fourier–Feynman transform (GFFT)* have been developed for the functionals on the very general function space $C_{a,b}[0,T]$ in [9], [10], and [13]. The function space $C_{a,b}[0,T]$, induced by the *generalized Brownian motion process (GBMP)*, was introduced by Yeh [24], [25] and used extensively in [8], [12]–[16], and [23] (for the precise definition of GBMP, see [24] and [25]).

The purpose of this article is to establish a more general translation theorem for the GFFT of functionals on the product function space $C^2_{a,b}[0,T]$. To do this, we first introduce the class $\mathcal{F}^{a,b}_{A_1,A_2}$ of functionals on $C^2_{a,b}[0,T]$, which is a generalization of the Kallianpur and Bromley–Fresnel class $\mathcal{F}_{A_1,A_2}$. We next illustrate the existence of the GFFT of functionals in $\mathcal{F}^{a,b}_{A_1,A_2}$. We then proceed to establish a general translation theorem on the product function space $C^2_{a,b}[0,T]$.

The Wiener process used in [2]–[7], [17], [18], [20], and [21] is stationary in time and free of drift, while the stochastic process used in this article, as well as in [8]–[16], [23], and [24], is nonstationary in time, subject to a drift $a(t)$, and can be used to explain the position of the Ornstein–Uhlenbeck process in an external force field (see [22]).

2. Definitions and preliminaries

Let $(\Omega, \mathcal{W}, P)$ be a probability measure space, and let $T > 0$ be a positive real number. A real-valued stochastic process Y on $(\Omega, \mathcal{W}, P)$ and the compact interval $[0,T]$ is called a *GBMP* if $Y(0,\omega) = 0$ almost everywhere and for $0 = t_1 < t_2 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega),\dots,Y(t_n,\omega))$ is normally distributed with density function

$$K_n(\vec{t},\vec{u}) = \Big(\prod_{j=1}^{n} 2\pi\big(b(t_j)-b(t_{j-1})\big)\Big)^{-1/2} \times \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{[(u_j-a(t_j))-(u_{j-1}-a(t_{j-1}))]^2}{b(t_j)-b(t_{j-1})}\Big\},$$

where $\vec{u} = (u_1,\dots,u_n)$, $u_0 = 0$, $\vec{t} = (t_1,\dots,t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [25, pp. 18–20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^{[0,T]}, \mathcal{B}^{[0,T]})$, where $\mathbb{R}^{[0,T]}$ is the space of all real-valued functions $x(t)$, $t \in [0,T]$, and $\mathcal{B}^{[0,T]}$ is the smallest σ-algebra of subsets of $\mathbb{R}^{[0,T]}$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^{[0,T]}$ are measurable. The triple $(\mathbb{R}^{[0,T]}, \mathcal{B}^{[0,T]}, \mu)$ is a probability measure space. This measure space is called the *function space* induced by the GBMP Y determined by $a(\cdot)$ and $b(\cdot)$.

Yeh [25] showed that the GBMP Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By [25, Theorem 14.2, p. 187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence, $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y, where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$. To make a long story short, the function space $C_{a,b}[0,T]$ can be considered as a space of continuous sample paths of the process Y. We then complete this function space to obtain $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$, where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0,T]$. We note that the coordinate process defined by $e_t(x) = x(t)$ on $C_{a,b}[0,T] \times [0,T]$ is also the GBMP determined by $a(t)$ and $b(t)$; that is, for each $t \in [0,T]$, $e_t(x) \sim N(a(t), b(t))$, and the process $\{e_t : 0 \le t \le T\}$ has nonstationary and independent increments.

Recall that the process $\{e_t : 0 \le t \le T\}$ on $C_{a,b}[0,T]$ is a continuous process. Thus the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ (considered in [2]–[6], [18]) if and only if $a(t) \equiv 0$ and $b(t) = t$ for all $t \in [0,T]$. Let $L^2_{a,b}[0,T]$ be the set of functions on $[0,T]$ which are Lebesgue-measurable and square-integrable with respect to the Lebesgue–Stieltjes measure on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; that is,

$$L^2_{a,b}[0,T] = \Big\{ v : \int_0^T v^2(s)\,db(s) < \infty \text{ and } \int_0^T v^2(s)\,d|a|(s) < \infty \Big\},$$

where $|a|(\cdot)$ is the total variation function of $a(\cdot)$. Then $L^2_{a,b}[0,T]$ is a separable Hilbert space with inner product defined by

$$(u,v)_{a,b} = \int_0^T u(s)v(s)\,dm_{|a|,b}(s) \equiv \int_0^T u(s)v(s)\,d\big[b(s)+|a|(s)\big],$$

where $m_{|a|,b}$ denotes the Lebesgue–Stieltjes measure induced by the increasing function $|a|(\cdot)+b(\cdot)$ on $[0,T]$. In particular, note that $\|u\|_{a,b} \equiv \sqrt{(u,u)_{a,b}} = 0$ if and only if $u(t)=0$ a.e. on $[0,T]$.

Remark 2.1. Recall that above, as well as in [12], [14], [15], and [23], we require that $a:[0,T]\to\mathbb{R}$ be an absolutely continuous function with $a(0)=0$ and with $\int_0^T |a'(t)|^2\,dt < \infty$. Now throughout this paper, we add the requirement that

$$\int_0^T \big|a'(t)\big|^2\,d|a|(t) < \infty. \tag{2.1}$$

Remark 2.2. Note that the function $a(t)=t^{2/3}$, $0\le t\le T$, does not satisfy condition (2.1) even though its derivative is an element of $L^2[0,T]$.

Remark 2.3. The function $a:[0,T]\to\mathbb{R}$ satisfies the requirements in Remark 2.1 if and only if the function a' is an element of $L^2_{a,b}[0,T]$.

The following subspace of $C_{a,b}[0,T]$ plays an important role throughout this article. Let

$$C'_{a,b}[0,T] = \Big\{ w\in C_{a,b}[0,T] : w(t) = \int_0^t z(s)\,db(s) \text{ for some } z\in L^2_{a,b}[0,T] \Big\}.$$

For $w\in C'_{a,b}[0,T]$, with $w(t)=\int_0^t z(s)db(s)$ for $t\in[0,T]$, let $D: C'_{a,b}[0,T]\to L^2_{a,b}[0,T]$ be defined by

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.2}$$

Then $C'_{a,b}\equiv C'_{a,b}[0,T]$ with inner product

$$(w_1,w_2)_{C'_{a,b}} = \int_0^T Dw_1(t)Dw_2(t)\,db(t) = \int_0^T z_1(t)z_2(t)\,db(t)$$

is a separable Hilbert space.

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the linear operator given by (2.2). In fact, the inverse operator $D^{-1}: L^2_{a,b}[0,T]\to C'_{a,b}[0,T]$ is given by

$$D^{-1}z = \int_0^t z(s)\,db(s).$$

It is easy to show that D^{-1} is a bounded operator since

$$\|D^{-1}z\|_{C'_{a,b}} = \Big\|\int_0^t z(s)\,db(s)\Big\|_{C'_{a,b}} = \Big(\int_0^T z^2(t)\,db(t)\Big)^{1/2} \le \Big(\int_0^T z^2(t)\,d\big[b(t)+|a|(t)\big]\Big)^{1/2} = \|z\|_{a,b}.$$

Applying the open-mapping theorem, we see that D is also bounded and there exist positive real numbers α and β such that $\alpha\|w\|_{C'_{a,b}} \le \|Dw\|_{a,b} \le \beta\|w\|_{C'_{a,b}}$ for all $w \in C'_{a,b}[0,T]$.

Remark 2.4. Our conditions on $b : [0,T] \to \mathbb{R}$ imply that $0 < \delta < b'(t) < M$ for some positive real numbers δ and M, and all $t \in [0,T]$.

The following lemma follows quite easily from Remarks 2.1, 2.3, and 2.4 above and the fact that $a(t) = \int_0^t \frac{a'(s)}{b'(s)} db(s)$ on $[0,T]$.

Lemma 2.5. *The function $a : [0,T] \to \mathbb{R}$ satisfies the conditions in Remark 2.1 if and only if a is an element of $C'_{a,b}[0,T]$.*

For each $w \in C'_{a,b}[0,T]$, the *Paley–Wiener–Zygmund (PWZ)* stochastic integral $(w,x)^\sim$ is given by the formula

$$(w,x)^\sim = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (w,g_j)_{C'_{a,b}} Dg_j(t)dx(t)$$

for μ-a.e. $x \in C_{a,b}[0,T]$, where $\{g_j\}_{j=1}^\infty$ is a complete orthonormal set of functions in $C'_{a,b}[0,T]$ such that, for each $j \in \mathbb{N}$, Dg_j is of bounded variation on $[0,T]$.

We will emphasize the following fundamental facts.

(i) The limit defining the PWZ stochastic integral $(w,x)^\sim$ is essentially independent of the choice of the complete orthonormal set $\{g_j\}_{j=1}^\infty$.

(ii) For each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w,x)^\sim$ exists for *scale-invariant almost everywhere (s-a.e.)* $x \in C_{a,b}[0,T]$. (For the precise definition of the concept of scale-invariant almost everywhere on the function space $C_{a,b}[0,T]$, see [8].)

(iii) If $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w,x)^\sim$ equals the Riemann–Stieltjes integral $\int_0^T z(t)\,dx(t)$.

(iv) For each $w \in C'_{a,b}[0,T]$, the random variable $x \mapsto (w,x)^\sim$ is Gaussian with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$.

(v) We have that $(w,\alpha x)^\sim = (\alpha w,x)^\sim = \alpha(w,x)^\sim$ for any real number α, $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$.

(vi) Note that for all $w_1, w_2 \in C'_{a,b}[0,T]$,

$$\int_{C_{a,b}[0,T]} (w_1,x)^\sim (w_2,x)^\sim\, d\mu(x) = (w_1,w_2)_{C'_{a,b}} + (w_1,a)_{C'_{a,b}}(w_2,a)_{C'_{a,b}}.$$

Hence, if $\{w_1,\dots,w_n\}$ is an orthonormal set in $C'_{a,b}[0,T]$, then the random variables $(w_i,x)^\sim$ are independent.

(vii) It follows from the definition of the PWZ stochastic integral and from Parseval's equality that if $w \in C'_{a,b}[0,T]$ and $x \in C'_{a,b}[0,T]$, then $(w,x)^{\sim}$ exists and we have $(w,x)^{\sim} = (w,x)_{C'_{a,b}}$.

Pierce and Skoug [23] used the inner product $(\cdot,\cdot)_{a,b}$ on $L^2_{a,b}[0,T]$ rather than the inner product $(\cdot,\cdot)_{C'_{a,b}}$ on $C'_{a,b}[0,T]$ to study the PWZ stochastic integral and the related integration formula on the function space $C_{a,b}[0,T]$. We denote the function space integral of a $\mathcal{W}(C_{a,b}[0,T])$-measurable functional F by

$$E[F] \equiv E_x\big[F(x)\big] = \int_{C_{a,b}[0,T]} F(x)\,d\mu(x)$$

whenever the integral exists.

3. The GFFT of functionals in a Banach algebra $\mathcal{F}^{a,b}_{A_1,A_2}$

Let $\mathcal{M}(C'_{a,b}[0,T])$ be the space of complex-valued, countably additive (and hence finite) Borel measures on $C'_{a,b}[0,T]$. The space $\mathcal{M}(C'_{a,b}[0,T])$ is a Banach algebra under the total variation norm and with convolution as multiplication. We define the Fresnel-type class $\mathcal{F}(C_{a,b}[0,T])$ of functionals on $C_{a,b}[0,T]$ as the space of all stochastic Fourier transforms of elements of $\mathcal{M}(C'_{a,b}[0,T])$; that is, $F \in \mathcal{F}(C_{a,b}[0,T])$ if and only if there exists a measure f in $\mathcal{M}(C'_{a,b}[0,T])$ such that

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\big\{i(w,x)^{\sim}\big\}\,df(w) \tag{3.1}$$

for $x \in C_{a,b}[0,T]$ s-a.e. More precisely, since we will identify functionals which coincide scale-invariant almost everywhere on $C_{a,b}[0,T]$, $\mathcal{F}(C_{a,b}[0,T])$ can be regarded as the space of all s-equivalence classes of functionals of the form (3.1).

The Fresnel-type class $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{C'_{a,b}[0,T]} d|f|(w).$$

In fact, the correspondence $f \mapsto F$ is injective, carries convolution into pointwise multiplication, and is a Banach algebra isomorphism where f and F are related by (3.1).

Remark 3.1. The Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ contains several interesting functions which arise naturally in quantum mechanics. Let $\mathcal{M}(\mathbb{R})$ be the class of $\mathbb{C}$-valued countably additive measures on $\mathcal{B}(\mathbb{R})$, the Borel class of $\mathbb{R}$. For $\nu \in \mathcal{M}(\mathbb{R})$, the Fourier transform $\widehat{\nu}$ of ν is a complex-valued function defined on $\mathbb{R}$ by the formula

$$\widehat{\nu}(u) = \int_{\mathbb{R}} \exp\{iuv\}\,d\nu(v).$$

Let $\mathcal{G}$ be the set of all complex-valued functions on $[0,T]\times\mathbb{R}$ of the form $\theta(s,u) = \widehat{\sigma}_s(u)$, where $\{\sigma_s : 0 \le s \le T\}$ is a family from $\mathcal{M}(\mathbb{R})$ satisfying the following two conditions:

(i) for every $E \in \mathcal{B}(\mathbb{R})$, $\sigma_s(E)$ is Borel-measurable in s,

(ii) $\int_0^T \|\sigma_s\|\, db(s) < +\infty$.

Let $\theta \in \mathcal{G}$, and let H be given by

$$H(x) = \exp\Big\{\int_0^T \theta\big(t, x(t)\big)\, dt\Big\}$$

for $x \in C_{a,b}[0,T]$ s-a.e. It was shown in [11] that the function $\theta(t,u)$ is Borel-measurable and that $\theta(t,x(t))$, $\int_0^T \theta(t,x(t))\,dt$, and $H(x)$ are elements of $\mathcal{F}(C_{a,b}[0,T])$. This fact is relevant to quantum mechanics, where exponential functions play a prominent role. (For more details, see [11].)

Let A be a nonnegative self-adjoint operator on $C'_{a,b}[0,T]$, and let f be any complex measure on $C'_{a,b}[0,T]$. Then the functional

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\big\{i(A^{1/2}w, x)^{\sim}\big\}\, df(w) \tag{3.2}$$

belongs to $\mathcal{F}(C_{a,b}[0,T])$ because it can be rewritten as

$$\int_{C'_{a,b}[0,T]} \exp\big\{i(w, x)^{\sim}\big\}\, df_A(w)$$

for $f_A = f \circ (A^{1/2})^{-1}$. Let A be self-adjoint but not nonnegative. Then A has the form

$$A = A^+ - A^-,$$

where both A^+ and A^- are bounded, nonnegative self-adjoint operators.

In this section, we will extend the ideas of [20] to obtain expressions of the generalized analytic Feynman integral and the GFFT of functionals of the form (3.2) when A is no longer required to be nonnegative. To do this, we will introduce definitions and notation analogous to those in [15] and [12].

Let $\mathcal{W}(C^2_{a,b}[0,T])$ denote the class of all Wiener measurable subsets of the product function space $C_{a,b}[0,T] \times C_{a,b}[0,T] \equiv C^2_{a,b}[0,T]$. A subset B of $C^2_{a,b}[0,T]$ is said to be *scale-invariant measurable* provided $\{(\rho_1 x_1, \rho_2 x_2) : (x_1,x_2) \in B\}$ is $\mathcal{W}(C^2_{a,b}[0,T])$-measurable for every $\rho_1 > 0$ and $\rho_2 > 0$, and a scale-invariant measurable subset N of $C^2_{a,b}[0,T]$ is considered scale-invariant null provided that $(\mu\times\mu)(\{(\rho_1 x_1, \rho_2 x_2) : (x_1,x_2) \in N\}) = 0$ for every $\rho_1 > 0$ and $\rho_2 > 0$. A property that holds except on a scale-invariant null set is considered to hold scale-invariant almost everywhere (s-a.e.) on $C^2_{a,b}[0,T]$. A functional F on $C^2_{a,b}[0,T]$ is considered scale-invariant measurable provided that F is defined on a scale-invariant measurable set and $F(\rho_1\cdot, \rho_2\cdot)$ is $\mathcal{W}(C^2_{a,b}[0,T])$-measurable for every $\rho_1 > 0$ and $\rho_2 > 0$. If two functionals F and G defined on $C^2_{a,b}[0,T]$ are equal scale-invariant almost everywhere, then we write $F \approx G$. (For more details, see [7], [20].)

We denote the product function space integral of a $\mathcal{W}(C^2_{a,b}[0,T])$-measurable functional F by

$$E[F] \equiv E_{\vec{x}}\big[F(x_1,x_2)\big] = \int_{C^2_{a,b}[0,T]} F(x_1,x_2)\, d(\mu\times\mu)(x_1,x_2)$$

whenever the integral exists. Throughout this article, let $\mathbb{C}$, $\mathbb{C}_+$, and $\widetilde{\mathbb{C}}_+$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively. Furthermore, for all $\lambda \in \widetilde{\mathbb{C}}_+$, $\lambda^{-1/2}$ (or $\lambda^{1/2}$) is always chosen to have positive real part. We also assume that every functional F on $C_{a,b}^2[0,T]$ we consider is scale-invariant almost everywhere defined and scale-invariant measurable.

Definition 3.2. Let $\mathbb{C}_+^2 = \{\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \mathrm{Re}(\lambda_j) > 0 \text{ for } j = 1,2\}$, and let $\widetilde{\mathbb{C}}_+^2 = \{\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_j \neq 0 \text{ and } \mathrm{Re}(\lambda_j) \geq 0 \text{ for } j = 1,2\}$. Let $F : C_{a,b}^2[0,T] \to \mathbb{C}$ be a scale-invariant measurable functional such that the function space integral

$$J(\lambda_1, \lambda_2) = \int_{C_{a,b}^2[0,T]} F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2)\, d(\mu \times \mu)(x_1, x_2)$$

exists and is finite for each $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists a function $J^*(\lambda_1, \lambda_2)$ analytic in $\mathbb{C}_+^2$ such that $J^*(\lambda_1, \lambda_2) = J(\lambda_1, \lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$, then $J^*(\lambda_1, \lambda_2)$ is defined to be the *analytic function space integral* of F over $C_{a,b}^2[0,T]$ with parameter $\vec{\lambda} = (\lambda_1, \lambda_2)$, and for $\vec{\lambda} \in \mathbb{C}_+^2$ we write

$$E^{\mathrm{an}_{\vec{\lambda}}}[F] \equiv E_{\vec{x}}^{\mathrm{an}_{\vec{\lambda}}}\big[F(x_1, x_2)\big] \equiv E_{x_1,x_2}^{\mathrm{an}_{(\lambda_1,\lambda_2)}}\big[F(x_1, x_2)\big] = J^*(\lambda_1, \lambda_2). \tag{3.3}$$

Let q_1 and q_2 be nonzero real numbers. Let F be a functional such that $E^{\mathrm{an}_{\vec{\lambda}}}[F]$ exists for all $\vec{\lambda} \in \mathbb{C}_+^2$. If the following limit exists, we call it the *generalized analytic Feynman integral* of F with parameter $\vec{q} = (q_1, q_2)$, and we write

$$E^{\mathrm{anf}_{\vec{q}}}[F] \equiv E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\big[F(x_1, x_2)\big] \equiv E_{x_1,x_2}^{\mathrm{anf}_{(q_1,q_2)}}\big[F(x_1, x_2)\big] = \lim_{\vec{\lambda} \to -i\vec{q}} E^{\mathrm{an}_{\vec{\lambda}}}[F], \tag{3.4}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2) \to -i\vec{q} = (-iq_1, -iq_2)$ through values in $\mathbb{C}_+^2$.

Definition 3.3. Let q_1 and q_2 be nonzero real numbers, and let F be a scale-invariant measurable functional on $C_{a,b}^2[0,T]$ such that, for $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}_+^2$ and $(y_1, y_2) \in C_{a,b}^2[0,T]$, the following analytic function space integral exists:

$$T_{\vec{\lambda}}(F)(y_1, y_2) \equiv T_{(\lambda_1,\lambda_2)}(F)(y_1, y_2) = E_{\vec{x}}^{\mathrm{an}_{\vec{\lambda}}}\big[F(y_1 + x_1, y_2 + x_2)\big].$$

For $p \in (1,2]$, we define the L_p analytic GFFT, $T_{\vec{q}}^{(p)}(F)$ of F, by the formula

$$T_{\vec{q}}^{(p)}(F)(y_1, y_2) \equiv T_{(q_1,q_2)}^{(p)}(F)(y_1, y_2) = \lim_{\substack{\vec{\lambda} \to -i\vec{q} \\ \vec{\lambda} \in \mathbb{C}_+^2}} T_{\vec{\lambda}}(F)(y_1, y_2)$$

if it exists; that is, for each $\rho_1 > 0$ and $\rho_2 > 0$,

$$\lim_{\substack{\vec{\lambda} \to -i\vec{q} \\ \vec{\lambda} \in \mathbb{C}_+^2}} \int_{C_{a,b}^2[0,T]} \big|T_{\vec{\lambda}}(F)(\rho_1 y_1, \rho_2 y_2) - T_{\vec{q}}^{(p)}(F)(\rho_1 y_1, \rho_2 y_2)\big|^{p'}\, d(\mu \times \mu)(y_1, y_2) = 0, \tag{3.5}$$

where $1/p+1/p'=1$. We also define the L_1 analytic GFFT, $T_{\vec{q}}^{(1)}(F)$ of F, by the formula

$$\begin{aligned} T_{\vec{q}}^{(1)}(F)(y_1,y_2) &\equiv T_{(q_1,q_2)}^{(1)}(F)(y_1,y_2) \\ &= \lim_{\substack{\vec{\lambda}\to -i\vec{q} \\ \vec{\lambda}\in\mathbb{C}_+^2}} T_{\vec{\lambda}}(F)(y_1,y_2) = E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\big[F(y_1+x_1,y_2+x_2)\big] \end{aligned} \tag{3.6}$$

for $\vec{y}=(y_1,y_2)\in C_{a,b}^2[0,T]$ s-a.e., whenever this limit exists.

For $p\in[1,2]$, we note that $T_{\vec{q}}^{(p)}(F)$ is defined only scale-invariant almost everywhere. We also note that if $T_{\vec{q}}^{(p)}(F)$ exists and if $F\approx G$, then $T_{\vec{q}}^{(p)}(G)$ exists and $T_{\vec{q}}^{(p)}(G)\approx T_{\vec{q}}^{(p)}(F)$. Moreover, from equations (3.3), (3.4), and (3.6), it follows that for $q_1,q_2\in\mathbb{R}\setminus\{0\}$,

$$E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\big[F(x_1,x_2)\big] = T_{\vec{q}}^{(1)}(F)(0,0). \tag{3.7}$$

Next we give the definition of the generalized Fresnel-type class $\mathcal{F}_{A_1,A_2}^{a,b}$.

Definition 3.4. Let A_1 and A_2 be bounded, nonnegative self-adjoint operators on $C'_{a,b}[0,T]$. The generalized Fresnel-type class $\mathcal{F}_{A_1,A_2}^{a,b}$ of functionals on $C_{a,b}^2[0,T]$ is defined as the space of all functionals F on $C_{a,b}^2[0,T]$ of the form

$$F(x_1,x_2) = \int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}w,x_j)^{\sim}\Big\}\,df(w) \tag{3.8}$$

for some $f\in\mathcal{M}(C'_{a,b}[0,T])$. More precisely, since we identify functionals which coincide scale-invariant almost everywhere on $C_{a,b}^2[0,T]$, $\mathcal{F}_{A_1,A_2}^{a,b}$ can be regarded as the space of all s-equivalence classes of functionals of the form (3.8).

Remark 3.5. (1) In Definition 3.4 above, let A_1 be the identity operator on $C'_{a,b}[0,T]$, and let $A_2\equiv 0$. Then $\mathcal{F}_{A_1,A_2}^{a,b}$ is essentially the Fresnel-type class $\mathcal{F}(C_{a,b}[0,T])$ and for $p\in[1,2]$ and nonzero real numbers q_1 and q_2,

$$T_{(q_1,q_2)}^{(1)}(F)(y_1,y_2) = T_{q_1}^{(1)}(F_0)(y_1),$$

if it exists, where $F_0(x_1)=F(x_1,x_2)$ for all $(x_1,x_2)\in C_{a,b}^2[0,T]$ and $T_{q_1}^{(1)}(F_0)(y)$ means the L_p analytic GFFT on $C_{a,b}[0,T]$ (see [12], [15]). Of course, if we choose $a(t)\equiv 0$, $b(t)=t$, $A_1=I$ (identity operator), and $A_2=0$ (zero operator), then the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ and the generalized Fresnel-type class $\mathcal{F}_{A_1,A_2}^{a,b}$ reduces to the Fresnel class $\mathcal{F}(C_0[0,T])$. It is known (see [18]) that $\mathcal{F}(C_0[0,T])$ forms a Banach algebra over the complex field and that $\mathcal{F}(C_0[0,T])$, $\mathcal{F}(H)$, and $\mathcal{S}(L^2[0,T])$ are isometrically isomorphic, where $\mathcal{S}(L^2[0,T])$ is the Cameron–Storvick Banach algebra of analytic Feynman integrable functionals on $C_0[0,T]$ (see [5]).

(2) The map $f\mapsto F$ defined by (3.8) sets up an algebra isomorphism between $\mathcal{M}(C'_{a,b}[0,T])$ and $\mathcal{F}_{A_1,A_2}^{a,b}$ if $\mathrm{Ran}(A_1+A_2)$ is dense in $C'_{a,b}[0,T]$, where Ran indicates the range of an operator. In this case, $\mathcal{F}_{A_1,A_2}^{a,b}$ becomes a Banach algebra

under the norm $\|F\| = \|f\|$. (For more details, see [20].)

Let

$$k(q_0;\vec{A};w) = \exp\Big\{\sum_{j=1}^{2}(2q_0)^{-1/2}\|A_j^{1/2}\|_o\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}, \tag{3.9}$$

where $\|A_j^{1/2}\|_o$ means the operator norm of $A_j^{1/2}$ for $j \in \{1,2\}$. For the existence of the GFFT of F, we define a subclass $\mathcal{F}^{q_0}_{A_1,A_2}$ of $\mathcal{F}^{a,b}_{A_1,A_2}$ by $F \in \mathcal{F}^{q_0}_{A_1,A_2}$ if and only if

$$\int_{C'_{a,b}[0,T]} k(q_0;\vec{A};w)\,d|f|(w) < +\infty,$$

where f and F are related by (3.8) and k is given by (3.9).

Remark 3.6. Note that in the case in which $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ and $(w,a)_{C'_{a,b}} = 0$ for all $w \in C'_{a,b}[0,T] = C'_0[0,T]$. Hence for all $\vec{\lambda} \in \widetilde{\mathbb{C}}^2_+$, $|\psi(\vec{\lambda};\vec{A};w)| \le 1$ and for any positive real number q_0, $\mathcal{F}^{q_0}_{A_1,A_2} = \mathcal{F}_{A_1,A_2}$, the Kallianpur and Bromley class introduced in Section 1.

The following theorem is due to Choi, Skoug, and Chang [16].

Theorem 3.7. *Let q_0 be a positive real number, and let F be an element of $\mathcal{F}^{q_0}_{A_1,A_2}$. Then for each $p \in [1,2]$, the L_p analytic GFFT of F, $T^{(p)}_{\vec{q}}(F)$ exists for all nonzero real numbers q_1 and q_2 with $|q_j| > q_0$, $j \in \{1,2\}$, belongs to $\mathcal{F}^{q_0}_{A_1,A_2}$, and is given by the formula*

$$T^{(p)}_{\vec{q}}(F)(y_1,y_2) = \int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}w,y_j)^{\sim}\Big\}\,df^{\vec{A}}_{\vec{q}}(w) \tag{3.10}$$

for $(y_1,y_2) \in C^2_{a,b}[0,T]$ s-a.e., where $f^{\vec{A}}_{\vec{q}}$ is a complex measure on $\mathcal{B}(C_{a,b}[0,T])$, the Borel σ-algebra of $C'_{a,b}[0,T]$, given by

$$f^{\vec{A}}_{\vec{q}}(B) = \int_B \psi(-i\vec{q};\vec{A};w)\,df(w), \quad B \in \mathcal{B}\big(C'_{a,b}[0,T]\big) \tag{3.11}$$

and where $\psi(-i\vec{q};\vec{A};w)$ is given by

$$\psi(-i\vec{q};\vec{A};w) = \exp\Big\{\sum_{j=1}^{2}\Big[-\frac{i(A_jw,w)_{C'_{a,b}}}{2q_j} + i(-iq_j)^{-1/2}(A_j^{1/2}w,a)_{C'_{a,b}}\Big]\Big\}. \tag{3.12}$$

The following theorem follows from (3.7) and (3.10).

Theorem 3.8. *Let q_0 and F be as in Theorem 3.7. Then for all real numbers q_1 and q_2 with $|q_j| > q_0$, $j \in \{1,2\}$, the generalized analytic Feynman integral $E^{\mathrm{anf}_{\vec{q}}}[F]$ of F exists and is given by the formula*

$$E^{\mathrm{anf}_{\vec{q}}}[F] = \int_{C'_{a,b}[0,T]} \psi(-i\vec{q};\vec{A};w)\,df(w),$$

where $\psi(-i\vec{q};\vec{A};w)$ is given by (3.12).

For a positive real number q_0, let

$$\Gamma_{q_0} = \Big\{\vec{\lambda} = (\lambda_1, \lambda_2) \in \widetilde{\mathbb{C}}^2_+ : \big|\mathrm{Im}(\lambda_j^{-1/2})\big| = \sqrt{\frac{|\lambda_j| - \mathrm{Re}(\lambda_j)}{2|\lambda_j|^2}} < \frac{1}{\sqrt{2q_0}}, j = 1, 2\Big\}.$$

Then it follows that for all $\vec{\lambda} = (\lambda_1, \lambda_2) \in \Gamma_{q_0}$,

$$\begin{aligned}\big|\psi(\vec{\lambda}; \vec{A}; w)\big| &\le \exp\Big\{\sum_{j=1}^{2}\big|\mathrm{Im}(\lambda_j^{-1/2})\big|\|A_j^{1/2}w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\} \\ &< \exp\Big\{\sum_{j=1}^{2}(2q_0)^{-1/2}\|A_j^{1/2}w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\} \\ &\le k(q_0; \vec{A}; w).\end{aligned}$$

We note that for all real q_j with $|q_j| > q_0$, $j \in \{1, 2\}$,

$$(-iq_j)^{-1/2} = \frac{1}{\sqrt{|2q_j|}} + \mathrm{sign}(q_j)\frac{i}{\sqrt{|2q_j|}}$$

and so $(-iq_1, -iq_2) \in \Gamma_{q_0}$. In fact, Γ_{q_0} is a connected open neighborhood of $-iq$ in $\tilde{\mathbb{C}}_+$. From these we can obtain the existence of the L_p analytic GFFT $T_{\vec{q}}^{(p)}(F)$ and the generalized analytic Feynman integral $E^{\mathrm{anf}_{\vec{q}}}[F]$ of F. (For more details, see [16].)

4. Translation theorems for the GFFT of functionals in $\mathcal{F}^{a,b}_{A_1,A_2}$

Cameron and Storvick [6] derived a translation theorem for the analytic Feynman integral of functionals in the Banach algebra $\mathcal{S}(L^2[0,T])$ on classical Wiener space, and Chang and Chung [13] derived a translation theorem for the function space integral of functionals on $C_{a,b}[0,T]$. Chang and Chung's translation theorem in [13], using the notation of this paper, states that if $x_0 \in C'_{a,b}[0,T]$ and if G is a μ-integrable functional on $C_{a,b}[0,T]$, then

$$E\big[G(x + x_0)\big] = \exp\Big\{-\frac{1}{2}\|x_0\|^2_{C'_{a,b}} - (x_0, a)_{C'_{a,b}}\Big\}E\big[G(x)\exp\{(x_0, x)^{\sim}\}\big]. \quad (4.1)$$

4.1. Special case. In [6], Cameron and Storvick established a translation theorem for the analytic Feynman integral of functionals in the Banach algebra $\mathcal{S}(L^2[0,T])$ using the concept of the Radon–Nikodym derivative and a direct calculation. In our next theorem, using the techniques similar to those used in [6], we derive a translation theorem for the GFFT of functionals F in the class $\mathcal{F}^{q_0}_{A_1,A_2}$.

Theorem 4.1. *Let q_0 and F be as in Theorem 3.7. Let g be a function in $C'_{a,b}[0,T]$. Then for all $p \in [1,2]$ and all real numbers q_1 and q_2 with $|q_j| > q_0$,*

$j \in \{1,2\}$,

$$\begin{aligned} T_{\vec{q}}^{(p)}(F)\Big(y_1 + \frac{1}{q_1}A_1^{1/2}g, y_2 + \frac{1}{q_2}A_2^{1/2}g\Big) \\ = \exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{2q_j}(A_j g, g)_{C'_{a,b}} + i(-iq_j)^{-1/2}(A_j^{1/2}g, a)_{C'_{a,b}}\Big]\Big\} \\ \times \exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}g, y_j)^{\sim}\Big\}T_{\vec{q}}^{(p)}(FR_{(g,g)}^{a,b})(y_1, y_2) \end{aligned} \tag{4.2}$$

for $(y_1, y_2) \in C_{a,b}^2[0,T]$ *s-a.e., where*

$$R_{(g,g)}(x_1, x_2) = \exp\Big\{\sum_{j=1}^{2}\big[-i(A_j^{1/2}g, x_j)^{\sim}\big]\Big\}. \tag{4.3}$$

Proof. First, using (3.8), it follows that for $(y_1, y_2) \in C_{a,b}^2[0,T]$ s-a.e.,

$$\begin{aligned} F(y_1,y_2)R_{(g,g)}(y_1,y_2) &= F(y_1,y_2)\exp\Big\{\sum_{j=1}^{2}\big[-i(A_j^{1/2}g, y_j)^{\sim}\big]\Big\} \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{i\sum_{j=1}^{2}\big(A_j^{1/2}(w-g), y_j\big)^{\sim}\Big\}\, df(w) \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}z, y_j)^{\sim}\Big\}\, df^g(z), \end{aligned}$$

where f^g is the complex measure in $\mathcal{M}(C'_{a,b}[0,T])$ such that $f^g(B) = f(B+g)$ for $B \in \mathcal{B}(C'_{a,b}[0,T])$. Using the Minkowski inequality, we observe that

$$\begin{aligned} &\int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^{2}(2q_0)^{-1/2}\|A_j^{1/2}\|_o\|z\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\, d|f^g|(z) \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^{2}(2q_0)^{-1/2}\|A_j^{1/2}\|_o\|w-g\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\, d|f|(w) \\ &\le \exp\Big\{\sum_{j=1}^{2}(2q_0)^{-1/2}\|A_j^{1/2}\|_o\|g\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\} \\ &\quad \times \int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^{2}(2q_0)^{-1/2}\|A_j^{1/2}\|_o\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\, d|f|(w) \\ &\le \exp\Big\{\sum_{j=1}^{2}(2q_0)^{-1/2}\|A_j^{1/2}\|_o\|g\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\}\int_{C'_{a,b}[0,T]} k(q_0;\vec{A};w)\, d|f|(w) \\ &< +\infty \end{aligned}$$

and so that the functional $F(y_1,y_2)R_{(g,g)}(y_1,y_2)$ is an element of $\mathcal{F}^{q_0}_{A_1,A_2}$. Thus, by Theorem 3.7, the L_p analytic GFFT $T^{(p)}_{\vec{q}}(FR_{(g,g)})$ of $FR_{(g,g)}$ exists. Next, using (3.10) together with (3.11) with f replaced by f^g, and (3.12), it follows that

$$\begin{aligned}
&T^{(p)}_{\vec{q}}(FR_{(g,g)})(y_1,y_2)\\
&\quad=\int_{C'_{a,b}[0,T]}\exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}z,y_j)^{\sim}\Big\}\psi(-i\vec{q};\vec{A};z)\,df^g(z)\\
&\quad=\int_{C'_{a,b}[0,T]}\exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}z,y_j)^{\sim}\Big\}\\
&\qquad\times\exp\Big\{\sum_{j=1}^{2}\Big[-\frac{i(A_jz,z)_{C'_{a,b}}}{2q_j}+i(-iq_j)^{-1/2}(A_j^{1/2}z,a)_{C'_{a,b}}\Big]\Big\}\,df^g(z)\\
&\quad=\exp\Big\{\sum_{j=1}^{2}\Big[-i(A_j^{1/2}g,y_j)^{\sim}-\frac{i}{2q_j}(A_jg,g)_{C'_{a,b}}-i(-iq_j)^{-1/2}(A_j^{1/2}g,a)_{C'_{a,b}}\Big]\Big\}\\
&\qquad\times\int_{C'_{a,b}[0,T]}\exp\Big\{\sum_{j=1}^{2} i(A_j^{1/2}z,y_j+q_j^{-1}A_j^{1/2}g)^{\sim}\Big\}\psi(-i\vec{q};\vec{A};z)\,df(z)\\
&\quad=\exp\Big\{\sum_{j=1}^{2}\Big[-i(A_j^{1/2}g,y_j)^{\sim}-\frac{i}{2q_j}(A_jg,g)_{C'_{a,b}}-i(-iq_j)^{-1/2}(A_j^{1/2}g,a)_{C'_{a,b}}\Big]\Big\}\\
&\qquad\times T^{(p)}_{\vec{q}}(F)(y_1+q_1^{-1}A_1^{1/2}g,y_2+q_2^{-1}A_2^{1/2}g).
\end{aligned}$$

This yields (4.2). □

By Remark 3.5(1) and Theorem 4.1 above, we have the following corollary.

Corollary 4.2. *Let $\mathcal{F}(C_{a,b}[0,T])$ be the Fresnel-type class of functionals F given by (3.1). Given a positive real number q_0, let $F\in\mathcal{F}(C_{a,b}[0,T])$ satisfy the condition*

$$\int_{C'_{a,b}[0,T]}\exp\big\{(2q_0)^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\big\}\,d|f|(w)<+\infty,$$

where f and F are related by (3.1). Then for each $g\in C'_{a,b}[0,T]$ and any real number q with $|q|>q_0$, it follows that

$$T^{(p)}_q(F)(y+g)=\exp\Big\{i(g,y)^{\sim}+\frac{iq}{2}\|g\|^2_{C'_{a,b}}-(-iq)^{1/2}(g,a)_{C'_{a,b}}\Big\}T^{(p)}_q(FR_g)(y)\quad(4.4)$$

for $y\in C_{a,b}[0,T]$ s-a.e., where $T^{(p)}_q(F)$ denotes the L_p analytic GFFT on $C_{a,b}[0,T]$ (see Remark 3.5) and

$$R_g(x)=\exp\big\{-i(g,x)^{\sim}\big\}.$$

Proof. Simply choose $q_1=q$, $A_1=I$ (identity operator), and $A_2=0$ (zero operator), and replace g with qg in (4.2). □

Of course, if we choose $a(t) \equiv 0$, $b(t) = t$, then the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ and the generalized Fresnel-type class $\mathcal{F}^{a,b}_{A_1,A_2}$ reduces to the Kallianpur and Bromley–Fresnel class $\mathcal{F}_{A_1,A_2}$, where both A_1 and A_2 are bounded, nonnegative, self-adjoint operators on

$$\begin{aligned} C_0'[0,T] &= \Big\{ w \in C_0[0,T] : w(t) = \int_0^t z(s)\,ds \text{ for some } z \in L_2[0,T] \Big\} \\ &= \big\{ w \in C_0[0,T] : w \text{ is absolutely continuous on } [0,T] \\ &\qquad \text{with } w' \in L_2[0,T] \big\}. \end{aligned}$$

In this case, as commented in Remark 3.6, it follows that $\mathcal{F}^{q_0}_{A_1,A_2} = \mathcal{F}_{A_1,A_2}$ for all real $q_0 > 0$. Thus we obtain the following corollary.

Corollary 4.3. *Let $\mathcal{F}_{A_1,A_2}$ be the Kallianpur and Bromley–Fresnel class. Then for each functional $F \in \mathcal{F}_{A_1,A_2}$, each $g \in C_0'[0,T]$, and any nonzero real numbers q_1 and q_2, it follows that*

$$\begin{aligned} &T^{(p)}_{\vec{q}}(F)\Big(y_1 + \frac{1}{q_1}A_1^{1/2}g, y_2 + \frac{1}{q_2}A_2^{1/2}g\Big) \\ &\quad = \exp\Big\{\sum_{j=1}^2 i(A_j^{1/2}g, y_j)^\sim + \sum_{j=1}^2 \frac{i}{2q_j}(A_j g, g)_{C_0'}\Big\} T^{(p)}_{\vec{q}}(FR_{(g,g)})(y_1,y_2) \end{aligned} \tag{4.5}$$

for $(y_1,y_2) \in C_0^2[0,T]$ s-a.e., where $R_{(g,g)}(x_1,x_2)$ is given by the right-hand side of (4.3) for $(x_1,x_2) \in C_0^2[0,T]$.

4.2. General case. In the left-hand side of (4.2), the translated functions $A_1^{1/2}g$ and $A_2^{1/2}g$ both depend on the same function g. We will present a more general translation theorem for functionals in $\mathcal{F}^{a,b}_{A_1,A_2}$. In our next theorem we establish the general translation theorem for the L_1 analytic GFFT.

Theorem 4.4. *Let q_0 and F be as in Theorem 3.7. Let g_1 and g_2 be functions in $C'_{a,b}[0,T]$. Then for all real numbers q_1 and q_2 with $|q_j| > q_0$, $j \in \{1,2\}$ and for $(y_1,y_2) \in C^2_{a,b}[0,T]$ s-a.e.,*

$$\begin{aligned} &T^{(1)}_{\vec{q}}(F)(y_1 + A_1^{1/2}g_1, y_2 + A_2^{1/2}g_2) \\ &\quad = \exp\Big\{\sum_{j=1}^2 \Big[\frac{iq_j}{2}(A_j g_j, g_j)_{C'_{a,b}} - (-iq_j)^{1/2}(A_j^{1/2}g_j, a)_{C'_{a,b}}\Big]\Big\} \\ &\quad\quad \times \exp\Big\{\sum_{j=1}^2 iq_j(A_j^{1/2}g_j, y_j)^\sim\Big\} T^{(1)}_{\vec{q}}(FR_{\vec{q},\vec{g}})(y_1,y_2), \end{aligned} \tag{4.6}$$

where

$$R_{\vec{q},\vec{g}}(x_1,x_2) \equiv R_{(q_1,q_2;g_1,g_2)}(x_1,x_2) = \exp\Big\{\sum_{j=1}^2 \big[-iq_j(A_j^{1/2}g_j, x_j)^\sim\big]\Big\}. \tag{4.7}$$

Proof. By Theorem 3.7, the L_1 analytic GFFT $T_{\vec{q}}^{(1)}(F)$ of F exists and is given by the right-hand side of (3.10). Thus we only need to verify the equality in (4.6).

For $\lambda_j > 0$, $j \in \{1,2\}$ and $w \in C'_{a,b}[0,T]$, let $G_j(w;\cdot)$ be a functional on $C_{a,b}[0,T]$ given by

$$G_j(w;x_j) = \exp\big\{i(A_j^{1/2}w, \lambda_j^{-1/2}x_j)^\sim\big\}, \tag{4.8}$$

and let

$$x_{0,1} = \lambda_1^{1/2}A_1^{1/2}g_1 \qquad \text{and} \qquad x_{0,2} = \lambda_2^{1/2}A_2^{1/2}g_2. \tag{4.9}$$

Then for $j \in \{1,2\}$,

$$\|x_{0,j}\|^2_{C'_{a,b}} = \lambda_j(A_jg_j,g_j)_{C'_{a,b}} \qquad \text{and} \qquad (x_{0,j},a)_{C'_{a,b}} = \lambda_j^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}}. \tag{4.10}$$

Using (3.8), Fubini's theorem, (4.9), (4.8), (4.1), and (4.10), we obtain that for $\lambda_1 > 0$ and $\lambda_2 > 0$,

$$\begin{aligned}
&T_{\vec{\lambda}}(F)(y_1 + A_1^{1/2}g_1, y_2 + A_2^{1/2}g_2)\\
&\quad= \int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^2 i(A_j^{1/2}w,y_j)^\sim\Big\}\\
&\qquad\times \Big(\prod_{j=1}^2 E_{x_j}\big[\exp\big\{i(A_j^{1/2}w, \lambda_j^{-1/2}x_j + A_j^{1/2}g_j)^\sim\big\}\big]\Big)\,df(w)\\
&\quad= \int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^2 i(A_j^{1/2}w,y_j)^\sim\Big\}\Big(\prod_{j=1}^2 E_{x_j}\big[G_j(w;x_j + x_{0,j})\big]\Big)\,df(w)\\
&\quad= \int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^2 i(A_j^{1/2}w,y_j)^\sim\Big\}\\
&\qquad\times \exp\Big\{\sum_{j=1}^2\Big[-\frac{\lambda_j}{2}(A_jg_j,g_j)_{C'_{a,b}} - \lambda_j^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}}\Big]\Big\}\\
&\qquad\times \Big(\prod_{j=1}^2 E_{x_j}\big[\exp\big\{i(A_j^{1/2}w, \lambda_j^{-1/2}x_j)^\sim + \lambda_j^{1/2}(A_j^{1/2}g_j,x_j)^\sim\big\}\big]\Big)\,df(w)\\
&\quad= \exp\Big\{\sum_{j=1}^2\Big[-\frac{\lambda_j}{2}(A_jg_j,g_j)_{C'_{a,b}} - \lambda_j^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}}\Big]\Big\}\\
&\qquad\times E_{\vec{x}}\Big[\int_{C'_{a,b}[0,T]} \exp\Big\{\sum_{j=1}^2\big[i(A_j^{1/2}w,y_j)^\sim + i(A_j^{1/2}w,\lambda_j^{-1/2}x_j)^\sim\big]\Big\}\,df(w)\\
&\qquad\times \exp\Big\{\sum_{j=1}^2 \lambda_j^{1/2}(A_j^{1/2}g_j,x_j)^\sim\Big\}\Big]\\
&\quad= \exp\Big\{\sum_{j=1}^2\Big[-\frac{\lambda_j}{2}(A_jg_j,g_j)_{C'_{a,b}} - \lambda_j^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}} - \lambda_j(A_j^{1/2}g_j,y_j)^\sim\Big]\Big\}
\end{aligned}$$

$$\times E_{\vec{x}}\Big[F(y_1+\lambda_1^{-1/2}x_1, y_2+\lambda_2^{-1/2}x_2)$$
$$\times \exp\Big\{\sum_{j=1}^{2}\big[\lambda_j(A_j^{1/2}g_j, y_j)^{\sim}+\lambda_j^{1/2}(A_j^{1/2}g_j, x_j)^{\sim}\big]\Big\}\Big]. \tag{4.11}$$

Now, letting $(\lambda_1,\lambda_2)\to(-iq_1,-iq_2)$, it follows that

$$T_{\vec{q}}(F)(y_1+A_1^{1/2}g_1, y_2+A_2^{1/2}g_2)$$
$$=\exp\Big\{\sum_{j=1}^{2}\Big[\frac{iq_j}{2}(A_jg_j,g_j)_{C'_{a,b}}-(-iq_j)^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}}+iq_j(A_j^{1/2}g_j,y_j)^{\sim}\Big]\Big\}$$
$$\times E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\Big[F(y_1+x_1,y_2+x_2)\exp\Big\{\sum_{j=1}^{2}\big[-iq_j(A_j^{1/2}g_j,y_j)^{\sim}+(A_j^{1/2}g_j,x_j)^{\sim}\big]\Big\}\Big]$$
$$=\exp\Big\{\sum_{j=1}^{2}\Big[\frac{iq_j}{2}(A_jg_j,g_j)_{C'_{a,b}}-(-iq_j)^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}}+iq_j(A_j^{1/2}g_j,y_j)^{\sim}\Big]\Big\}$$
$$\times T_{\vec{q}}^{(1)}(FR_{\vec{q},\vec{g}})(y_1,y_2)(y_1,y_2),$$

and the theorem is proved. □

The following corollary follows from equations (4.6) and (3.7) above.

Corollary 4.5. *Let q_0, F, g_1, and g_2 be as in Theorem 4.4. Then for all real numbers q_1 and q_2 with $|q_j|>q_0$, $j\in\{1,2\}$,*

$$E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\big[F(x_1+A_1^{1/2}g_1, x_2+A_2^{1/2}g_2)\big]$$
$$=T_{\vec{q}}^{(1)}(F)(0,0)$$
$$=\exp\Big\{\sum_{j=1}^{2}\Big[\frac{iq_j}{2}(A_jg_j,g_j)_{C'_{a,b}}-(-iq_j)^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}}\Big]\Big\}$$
$$\times E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\big[F(x_1,x_2)R_{\vec{q},\vec{g}}(x_1,x_2)\big], \tag{4.12}$$

where $R_{\vec{q},\vec{g}}$ is given by (4.7).

By Remark 3.5(1) and Corollary 4.5 above, we have the following corollary.

Corollary 4.6. *Given a positive real number q_0, let $F\in\mathcal{F}(C_{a,b}[0,T])$ be given as Corollary 4.2. Then for each $g\in C'_{a,b}[0,T]$ and any real number q with $|q|>q_0$, it follows that*

$$E^{\mathrm{anf}_q}\big[F(\cdot+g)\big]=\exp\Big\{\frac{iq}{2}\|g\|^2_{C'_{a,b}}-(-iq)^{1/2}(g,a)_{C'_{a,b}}\Big\}E^{\mathrm{anf}_q}[FR_{q,g}], \tag{4.13}$$

where $E^{\mathrm{anf}_q}[F]$ means the generalized analytic Feynman integral of F on $C_{a,b}[0,T]$ (see [12], [15]), and

$$R_{q,g}(x)=\exp\{-iq(g,x)^{\sim}\}.$$

Remark 4.7. In Corollary 4.6, taking $a(t) \equiv 0$ and $b(t) = t$, the general function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. Also, we know that (4.13) becomes

$$\begin{aligned}&\int_{C_0[0,T]}^{\text{anf}_q} F(x+g_0)\,dm_w(x)\\&\quad= \exp\Big\{\frac{iq}{2}\|g_0'\|_2^2\Big\}\int_{C_0[0,T]}^{\text{anf}_q} F(x)\exp\Big\{-iq\int_0^T g_0'(t)\,dx(t)\Big\}\,dm_w(x),\end{aligned}$$

where $\|\cdot\|_2$ is the norm on $L^2[0,T]$. This result subsumes a similar known result obtained by Cameron and Storvick [6].

In our next theorem we establish a translation theorem for the L_p analytic GFFT with $p \in (1,2]$.

Theorem 4.8. *Let q_0 and F be as in Theorem 3.7. Let g_1 and g_2 be functions in $C'_{a,b}[0,T]$. Then for all $p \in [1,2]$, and all real numbers q_1 and q_2 with $|q_j| > q_0$, $j \in \{1,2\}$, the L_p analytic GFFT $T_{\vec{q}}^{(p)}(F)(y_1 + A_1^{1/2}g_1, y_2 + A_2^{1/2}g_2)$ is given by the right-hand side of (4.6) for $(y_1,y_2) \in C_{a,b}^2[0,T]$ s-a.e.*

Proof. By Theorem 3.7, the L_p analytic GFFT $T_{\vec{q}}^{(p)}(F)$ of F exists for each $p \in [1,2]$ and is given by the right-hand side of equation (3.10). Thus we only need to verify the equality in (4.6) with $T_{\vec{q}}^{(1)}(F)$ replaced with $T_{\vec{q}}^{(p)}(F)$, $1 < p \le 2$. But, in order to obtain (4.6) with $T_{\vec{q}}^{(p)}(F)$, $1 < p \le 2$, we have to use the concept of the scale-invariant limit (see (3.5) above) for the proof.

Let

$$\Phi_1(x_1,x_2) = F(x_1,x_2)\exp\Big\{\sum_{j=1}^2\big[\lambda_j(A_j^{1/2}g_j,x_j)^{\sim}\big]\Big\}.$$

Then for $\lambda_1 > 0$ and $\lambda_2 > 0$, the last expression of (4.11) is represented by

$$\begin{aligned}&\exp\Big\{\sum_{j=1}^2\Big[-\frac{\lambda_j}{2}(A_jg_j,g_j)_{C'_{a,b}} - \lambda_j^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}} - \lambda_j(A_j^{1/2}g_j,y_j)^{\sim}\Big]\Big\}\\&\quad\times E_{\vec{x}}\big[\Phi_1(y_1+\lambda_1^{-1/2}x_1, y_2+\lambda_2^{-1/2}x_2)\big];\end{aligned}$$

that is, it follows that

$$\begin{aligned}&T_{\vec{\lambda}}(F)(y_1+A_1^{1/2}g_1, y_2+A_2^{1/2}g_2)\\&\quad= \exp\Big\{\sum_{j=1}^2\Big[-\frac{\lambda_j}{2}(A_jg_j,g_j)_{C'_{a,b}} - \lambda_j^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}} - \lambda_j(A_j^{1/2}g_j,y_j)^{\sim}\Big]\Big\}\\&\qquad\times E_{\vec{x}}\big[\Phi_1(y_1+\lambda_1^{-1/2}x_1, y_2+\lambda_2^{-1/2}x_2)\big].\end{aligned}$$

On the other hand, for $\lambda_1 > 0$ and $\lambda_2 > 0$, we see that

$$\begin{aligned}
&T_{\vec{\lambda}}(FR_{\vec{q},\vec{g}})(y_1, y_2)\\
&\quad= E_{\vec{x}}\big[(FR_{\vec{q},\vec{g}})(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)\big]\\
&\quad= E_{\vec{x}}\Big[F(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)\\
&\qquad\times \exp\Big\{\sum_{j=1}^{2}\big[-iq_j(A_j^{1/2}g_j, y_j + \lambda_j^{-1/2}x_j)^{\sim}\big]\Big\}\Big]\\
&\quad= E_{\vec{x}}\Big[F(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)\\
&\qquad\times \exp\Big\{\sum_{j=1}^{2}\big[-iq_j(A_j^{1/2}g_j, y_j)^{\sim} - iq_j\lambda_j^{-1/2}(A_j^{1/2}g_j, x_j)^{\sim}\big]\Big\}\Big].
\end{aligned}$$

Next, using Hölder's inequality with $\lambda_1 > 0$ and $\lambda_2 > 0$, it follows that

$$\begin{aligned}
&E_{\vec{x}}\big[\big|(FR_{\vec{q},\vec{g}})(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2) - \Phi_1(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)\big|\big]\\
&\quad= E_{\vec{x}}\Big[\big|F(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)\big|\\
&\qquad\times \Big|1 - \exp\Big\{\sum_{j=1}^{2}\big[(iq_j + \lambda_j)(A_j^{1/2}g_j, y_j)^{\sim}\\
&\qquad+ (iq_j\lambda_j^{-1/2} + \lambda_j^{1/2})(A_j^{1/2}g_j, x_j)^{\sim}\big]\Big\}\Big|\Big]\\
&\quad\le \big(E_{\vec{x}}\big[\big|F(y_1 + \lambda_1^{-1/2}x_1, y_2 + \lambda_2^{-1/2}x_2)\big|^p\big]\big)^{1/p}\\
&\qquad\times \Big(E_{\vec{x}}\Big[\Big|1 - \exp\Big\{\sum_{j=1}^{2}\big[(iq_j + \lambda_j)(A_j^{1/2}g_j, y_j)^{\sim}\\
&\qquad+ (iq_j\lambda_j^{-1/2} + \lambda_j^{1/2})(A_j^{1/2}g_j, x_j)^{\sim}\big]\Big\}\Big|^{p'}\Big]\Big)^{1/p'}.
\end{aligned}$$

Note that each factor in the last expression has a limit as $\vec{\lambda} = (\lambda_1, \lambda_2) \to -i\vec{q} = (-iq_1, -iq_2)$ in $\mathbb{C}_+^2$, and that

$$\begin{aligned}
&\Big(E_{\vec{x}}\Big[\Big|1 - \exp\Big\{\sum_{j=1}^{2}\big[(iq_j + \lambda_j)(A_j^{1/2}g_j, y_j)^{\sim}\\
&\qquad+ (iq_j\lambda_j^{-1/2} + \lambda_j^{1/2})(A_j^{1/2}g_j, x_j)^{\sim}\big]\Big\}\Big|^{p'}\Big]\Big)^{1/p'} \to 0
\end{aligned}$$

as $\vec{\lambda} = (\lambda_1, \lambda_2) \to -i\vec{q} = (-iq_1, -iq_2)$ in $\mathbb{C}_+^2$. Hence we have

$$\begin{aligned}
&T_{\vec{q}}^{(p)}(F)(y_1 + A_1^{1/2}g_1, y_2 + A_2^{1/2}g_2)\\
&\quad= \lim_{\substack{\vec{\lambda}\to -i\vec{q}\\ \vec{\lambda}\in\mathbb{C}_+^2}} T_{\vec{\lambda}}(F)(y_1 + A_1^{1/2}g_1, y_2 + A_2^{1/2}g_2)
\end{aligned}$$

$$= \lim_{\substack{\vec{\lambda}\to -i\vec{q}\\ \vec{\lambda}\in\mathbb{C}_+^2}} \exp\Big\{\sum_{j=1}^{2}\Big[-\frac{\lambda_j}{2}(A_j g_j, g_j)_{C'_{a,b}} - \lambda_j^{1/2}(A_j^{1/2} g_j, a)_{C'_{a,b}}$$

$$- \lambda_j (A_j^{1/2} g_j, y_j)^{\sim}\Big]\Big\} E_{\vec{x}}\big[\Phi_1(y_1+\lambda_1^{-1/2}x_1, y_2+\lambda_2^{-1/2}x_2)\big]$$

$$= \exp\Big\{\sum_{j=1}^{2}\Big[\frac{iq_j}{2}(A_j g_j, g_j)_{C'_{a,b}} - (-iq_j)^{1/2}(A_j^{1/2} g_j, a)_{C'_{a,b}} + iq_j (A_j^{1/2} g_j, y_j)^{\sim}\Big]\Big\}$$

$$\times \lim_{\substack{\vec{\lambda}\to -i\vec{q}\\ \vec{\lambda}\in\mathbb{C}_+^2}} E_{\vec{x}}\big[\Phi_1(y_1+\lambda_1^{-1/2}x_1, y_2+\lambda_2^{-1/2}x_2)\big]$$

$$= \exp\Big\{\sum_{j=1}^{2}\Big[\frac{iq_j}{2}(A_j g_j, g_j)_{C'_{a,b}} - (-iq_j)^{1/2}(A_j^{1/2} g_j, a)_{C'_{a,b}} + iq_j (A_j^{1/2} g_j, y_j)^{\sim}\Big]\Big\}$$

$$\times \lim_{\substack{\vec{\lambda}\to -i\vec{q}\\ \vec{\lambda}\in\mathbb{C}_+^2}} T_{\vec{\lambda}}(FR_{\vec{q},\vec{g}})(y_1, y_2)$$

$$= \exp\Big\{\sum_{j=1}^{2}\Big[\frac{iq_j}{2}(A_j g_j, g_j)_{C'_{a,b}} - (-iq_j)^{1/2}(A_j^{1/2} g_j, a)_{C'_{a,b}} + iq_j (A_j^{1/2} g_j, y_j)^{\sim}\Big]\Big\}$$

$$\times T_{\vec{q}}(FR_{\vec{q},\vec{g}})(y_1, y_2).$$

This completes the proof. □

Remark 4.9. Given any function g in $C'_{a,b}[0,T]$, setting $g_1 = q_1^{-1}g$ and $g_2 = q_2^{-1}g$ in (4.6) yields (4.2). Also, choosing $g_1 = g$, $A_1 = I$, and $A_2 = 0$ yields (4.4).

Corollary 4.10. *Let $\mathcal{F}_{A_1,A_2}$ be the Kallianpur and Bromley–Fresnel class. Then for all $p \in [1,2]$, every functional $F \in \mathcal{F}_{A_1,A_2}$, each $g \in C'_0[0,T]$, and any nonzero real number q_1 and q_2, it follows that*

$$T_{\vec{q}}^{(p)}(F)(y_1 + A_1^{1/2}g_1, y_2 + A_2^{1/2}g_2)$$
$$= \exp\Big\{\sum_{j=1}^{2} iq_j (A_j^{1/2} g_j, y_j)^{\sim} + \sum_{j=1}^{2}\frac{iq_j}{2}(A_j g_j, g_j)_{C'_0}\Big\} T_{\vec{q}}^{(1)}(FR_{\vec{q},\vec{g}})(y_1, y_2) \tag{4.14}$$

for $(y_1, y_2) \in C_0^2[0,T]$ s-a.e., and

$$E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\big[F(x_1 + A_1^{1/2}g_1, x_2 + A_2^{1/2}g_2)\big]$$
$$= \exp\Big\{\sum_{j=1}^{2}\frac{iq_j}{2}(A_j g_j, g_j)_{C'_0}\Big\} E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\big[F(x_1, x_2) R_{\vec{q},\vec{g}}(x_1, x_2)\big].$$

Remark 4.11. Given any function g in $C'_0[0,T]$, setting $g_1 = q_1^{-1}g$ and $g_2 = q_2^{-1}g$ in (4.14) yields (4.5).

5. Examples

We finish this article with meaningful examples to which our translation theorems can be applied.

Example 5.1. Given two self-adjoint operators A_1 and A_2, as commented in Remark 3.5(2), the generalized Fresnel-type class $\mathcal{F}_{A_1,A_2}^{a,b}$ is a Banach algebra if $\mathrm{Ran}(A_1+A_2)$ is dense in $C'_{a,b}[0,T]$. In this case, using (4.12) with $F(x_1,x_2)\equiv 1$ (we can use (4.12), because $F\equiv 1\in\mathcal{F}_{A_1,A_2}^{q_0}$ for all positive real numbers q_0) and (4.7), it follows that, for all $q_1,q_2\in\mathbb{R}\setminus\{0\}$ and any functions g_1 and g_2 in $C'_{a,b}[0,T]$,

$$1 = \exp\Big\{\sum_{j=1}^{2}\Big[\frac{iq_j}{2}(A_jg_j,g_j)_{C'_{a,b}} - (-iq_j)^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}}\Big]\Big\} \\ \times E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\Big[\exp\Big\{\sum_{j=1}^{2}\big[-iq_j(A_j^{1/2}g_j,x_j)^{\sim}\big]\Big\}\Big].$$

Using this, we immediately obtain the generalized analytic Feynman integration formula on the product function space $C'_{a,b}[0,T]$ as follows: for any $\vec{q}=(q_1,q_2)$ with $q_1,q_2\in\mathbb{R}\setminus\{0\}$,

$$E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}\Big[\exp\Big\{\sum_{j=1}^{2}\big[-iq_j(A_j^{1/2}g_j,x_j)^{\sim}\big]\Big\}\Big] \\ = \exp\Big\{\sum_{j=1}^{2}\Big[-\frac{iq_j}{2}(A_jg_j,g_j)_{C'_{a,b}} + (-iq_j)^{1/2}(A_j^{1/2}g_j,a)_{C'_{a,b}}\Big]\Big\}. \tag{5.1}$$

In fact, (5.1) holds even if $\mathrm{Ran}(A_1+A_2)$ is not dense in $C'_{a,b}[0,T]$. But the direct calculation for the formula is very complicated.

Let $S: C'_{a,b}[0,T]\to C'_{a,b}[0,T]$ be the linear operator defined by

$$Sw(t)=\int_0^t w(s)\,db(s).$$

Then we see that the adjoint operator S^* of S is given by

$$S^*w(t) = w(T)b(t) - \int_0^t w(s)\,db(s) = \int_0^t \big[w(T)-w(s)\big]\,db(s),$$

and the linear operator $C=S^*S$ is given by

$$Cw(t)=\int_0^T \min\big\{b(s),b(t)\big\}w(s)\,db(s).$$

Furthermore, we see that C is a self-adjoint operator on $C'_{a,b}[0,T]$ and that

$$(w_1,Cw_2)_{C'_{a,b}} = (Sw_1,Sw_2)_{C'_{a,b}} = \int_0^T w_1(s)w_2(s)\,db(s)$$

for all $w_1,w_2\in C'_{a,b}[0,T]$. Hence C is a positive definite operator; that is, $(w,Cw)_{C'_{a,b}}\geq 0$ for all $w\in C'_{a,b}[0,T]$.

One can show that the orthonormal eigenfunctions $\{e_m\}$ of C are given by

$$e_m(t) = \frac{\sqrt{2b(T)}}{(m-\frac{1}{2})\pi} \sin\Big(\frac{(m-\frac{1}{2})\pi}{b(T)} b(t)\Big) \tag{5.2}$$

with corresponding eigenvalues $\{\beta_m\}$ given by

$$\beta_m = \Big(\frac{b(T)}{(m-\frac{1}{2})\pi}\Big)^2. \tag{5.3}$$

Furthermore, it can be shown that $\{e_m\}$ is a basis of $C'_{a,b}[0,T]$ and that C is a trace class operator and so S is a Hilbert–Schmidt operator on $C'_{a,b}[0,T]$. In fact, the trace of C is given by $\mathrm{Tr}C = \frac{1}{2}b^2(T) = \int_0^T b(t)\,db(t)$.

Define a self-adjoint operator on $C'_{a,b}[0,T]$ by

$$Aw = \sum_{m=1}^{\infty} \gamma_m (w, e_m)_{C'_{a,b}} e_m \tag{5.4}$$

where

$$\gamma_m = \begin{cases} \beta_m & m : \text{even}, \\ -\beta_m & m : \text{odd}. \end{cases}$$

Then

$$\begin{aligned} Aw &= \sum_{m=1}^{\infty} (-1)^m \beta_m (w, e_m)_{C'_{a,b}} e_m, \\ A_+^{1/2} w &= \sum_{m\,:\,\text{even}} \sqrt{\beta_m} (w, e_m)_{C'_{a,b}} e_m, \end{aligned} \tag{5.5}$$

and

$$A_-^{1/2} w = \sum_{m\,:\,\text{odd}} \sqrt{\beta_m} (w, e_m)_{C'_{a,b}} e_m. \tag{5.6}$$

In this case, we see that A_+ and A_- are the positive and the negative parts of A, respectively. One can show that $A_+^{1/2}$ and $A_-^{1/2}$ are trace class operators with $\mathrm{Tr}A_+^{1/2} = \frac{b^2(T)}{8}$ and $\mathrm{Tr}A_-^{1/2} = \frac{3b^2(T)}{8}$.

Example 5.2. Applying equations (5.1) through (5.6), it follows that for each $w \in C'_{a,b}[0,T]$,

$$\begin{aligned} &E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}} \big[\exp\{-i(A_+^{1/2} w, x_1)^\sim + i(A_-^{1/2} w, x_2)^\sim\}\big] \\ &= \exp\Big\{-\frac{i}{2} \sum_{m=1}^{\infty} (-1)^m \beta_m (w, e_m)^2_{C'_{a,b}} \\ &\quad + (-i)^{1/2} \sum_{m\,:\,\text{even}} \sqrt{\beta_m} (w, e_m)_{C'_{a,b}} (e_m, a)_{C'_{a,b}} \\ &\quad + (i)^{1/2} \sum_{m\,:\,\text{odd}} \sqrt{\beta_m} (w, e_m)_{C'_{a,b}} (e_m, a)_{C'_{a,b}} \Big\}. \end{aligned}$$

From this, it follows that if m is an even number, then

$$\begin{aligned}
&E_x^{\mathrm{anf}_1}\Big[\exp\Big\{i\int_0^T \cos\Big(\frac{(m-1/2)\pi}{b(T)}b(t)\Big)\,dx(t)\Big\}\Big]\\
&= E_x^{\mathrm{anf}_1}\Big[\exp\{i\Big(\frac{b(T)}{(m-1/2)\pi}\sin\Big(\frac{(m-1/2)\pi}{b(T)}b(t)\Big),x\Big)^{\sim}\Big]\\
&= E_x^{\mathrm{anf}_1}\Big[\exp\{-i\Big(A_+^{1/2}\Big(-\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m\Big),x\Big)^{\sim}\Big]\\
&= E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\Big[\exp\Big\{-i\Big(A_+^{1/2}\Big(-\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m\Big),x_1\Big)^{\sim}\\
&\quad + i\Big(A_-^{1/2}\Big(-\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m\Big),x_2\Big)^{\sim}\Big\}\Big]\\
&= \exp\Big\{-\frac{i}{2}\beta_m\Big(-\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m,e_m\Big)^2_{C'_{a,b}}\\
&\quad + (-i)^{1/2}\sqrt{\beta_m}\Big(-\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m,e_m\Big)_{C'_{a,b}}(e_m,a)_{C'_{a,b}}\Big\}\\
&= \exp\Big\{-\frac{i}{4}b(T)-(-i)^{1/2}\int_0^T \cos\Big(\frac{(m-1/2)\pi}{b(T)}b(t)\Big)\,da(t)\Big\},
\end{aligned}\tag{5.7}$$

and if m is an odd number, then

$$\begin{aligned}
&E_x^{\mathrm{anf}_{-1}}\Big[\exp\Big\{i\int_0^T \cos\Big(\frac{(m-1/2)\pi}{b(T)}b(t)\Big)\,dx(t)\Big\}\Big]\\
&= E_x^{\mathrm{anf}_{-1}}\Big[\exp\{i\Big(\frac{b(T)}{(m-1/2)\pi}\sin\Big(\frac{(m-1/2)\pi}{b(T)}b(t)\Big),x\Big)^{\sim}\Big]\\
&= E_x^{\mathrm{anf}_{-1}}\Big[\exp\{i\Big(A_-^{1/2}\Big(\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m\Big),x\Big)^{\sim}\Big]\\
&= E_{\vec{x}}^{\mathrm{anf}_{(1,-1)}}\Big[\exp\Big\{-i\Big(A_+^{1/2}\Big(\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m\Big),x_1\Big)^{\sim}\\
&\quad + i\Big(A_-^{1/2}\Big(\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m\Big),x_2\Big)^{\sim}\Big\}\Big]\\
&= \exp\Big\{\frac{i}{2}\beta_m\Big(\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m,e_m\Big)^2_{C'_{a,b}}\\
&\quad + (i)^{1/2}\sqrt{\beta_m}\Big(\frac{(m-1/2)\pi}{\sqrt{2b(T)}}e_m,e_m\Big)_{C'_{a,b}}(e_m,a)_{C'_{a,b}}\Big\}\\
&= \exp\Big\{\frac{i}{4}b(T)+(i)^{1/2}\int_0^T \cos\Big(\frac{(m-1/2)\pi}{b(T)}b(t)\Big)\,da(t)\Big\}.
\end{aligned}\tag{5.8}$$

In (5.7) and (5.8) above, $E_x^{\mathrm{anf}_q}[F(x)]$ indicates the generalized analytic Feynman integral of F on $C_{a,b}[0,T]$ with parameter q.

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