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ANALYTIC ASPECTS OF EVOLUTION ALGEBRAS

P. MELLON[1] and M. VICTORIA VELASCO[2*]

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Abstract. We prove that every evolution algebra A is a normed algebra, for an l_1-norm defined in terms of a fixed natural basis. We further show that a normed evolution algebra A is a Banach algebra if and only if $A = A_1 \oplus A_0$, where A_1 is finite-dimensional and A_0 is a zero-product algebra. In particular, every nondegenerate Banach evolution algebra must be finite-dimensional and the completion of a normed evolution algebra is therefore not, in general, an evolution algebra. We establish a sufficient condition for continuity of the evolution operator L_B of A with respect to a natural basis B, and we show that L_B need not be continuous. Moreover, if A is finite-dimensional and $B = \{e_1, \ldots, e_n\}$, then L_B is given by L_e, where $e = \sum_i e_i$ and L_a is the multiplication operator $L_a(b) = ab$, for $b \in A$. We establish necessary and sufficient conditions for convergence of $(L_a^n(b))_n$, for all $b \in A$, in terms of the multiplicative spectrum $\sigma_m(a)$ of a. Namely, $(L_a^n(b))_n$ converges, for all $b \in A$, if and only if $\sigma_m(a) \subseteq \Delta \cup \{1\}$ and $\nu(1, a) \leq 1$, where $\nu(1, a)$ denotes the index of 1 in the spectrum of L_a.

1. Introduction

The use of algebraic techniques to study genetic inheritance dates from 1856 with Mendel [18], leading to subsequent work by various authors over the next four decades (see [9]–[12], [22]), and culminating in the algebraic formulation of Mendel's laws in terms of nonassociative algebras (see [9], [10]). Since then many algebras, generally referred to as *genetic algebras* (Mendelian, gametic, and zygotic algebras, to name but a few), have provided a mathematical framework for

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*Corresponding author.
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studying various types of inheritance. On the other hand, certain genetic phenomena such as, for example, the case of incomplete dominance, systems of multiple alleles, and asexual inheritance, do not follow Mendel's laws and so evolution algebras were introduced by Tian and Vojtěchovský [25] in 2006, partly as an attempt to study such non-Mendelian behavior. Evolution algebras are highly nonassociative in general (they are not even power-associative), although they are commutative. (For a recent study of evolution algebras in infinite dimensions, see [1]. Other aspects of evolution algebras have been considered in [2]–[6], [8], [13], [14], [21], and [26].)

Recall that an algebra is a vector space A over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) provided with a bilinear map $A \times A \to A$, $(a, b) \to ab$, referred to as the *multiplication* of A (which, here, is not assumed to be either associative or commutative). When an algebra A is provided with a basis $B = \{e_i : i \in \Lambda\}$ such that $e_i e_j = 0$ if $i \neq j$, then we say that A is an *evolution algebra* and B is a *natural basis* of A. In Section 2, we study the existence or otherwise of algebra norms and complete algebra norms on an evolution algebra. Recall that A is a normed algebra if A has a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\|\|b\|$, for every $a, b \in A$, and A is a Banach algebra if it has a complete algebra norm. We prove that every evolution algebra A is a normed algebra, for an l_1-norm defined in terms of a fixed natural basis, and we also show that a normed evolution algebra A is a Banach algebra if and only if $A = A_1 \oplus A_0$, where A_1 is finite-dimensional and A_0 is a zero-product algebra. In particular, every nondegenerate Banach evolution algebra must be finite-dimensional and the completion of a normed evolution algebra is not, in general, itself an evolution algebra.

For evolution algebra A and basis $B = \{e_i : i \in \Lambda\}$ as above, the unique linear map $L_B : A \to A$ satisfying $L_B(e_i) = e_i^2$, for all $i \in \Lambda$, is known as the *evolution operator* on A associated to B. This is postulated in [24] as being central to the dynamics of A. In Section 3 we study the continuity of the evolution operator, giving a sufficient condition for its continuity and an example to show that it is not necessarily continuous.

In particular, if $\dim A < \infty$ and $B = \{e_1, \ldots, e_n\}$, then L_B is the multiplication map L_e, for $e = \sum_{i=1}^n e_i$ (of course, L_B is then automatically continuous). For $b \in A$ and $m \in \mathbb{N}$, the element $L_B^m(b)$ has biological meaning, and a typical question in this framework is to study possible accumulation points of $(L_B^m(b))_m$. Section 4 tackles this topic, and in light of results from Section 2, we assume that A is finite-dimensional and thus $L_B = L_e$. On the other hand, $\tilde{e} := \lambda e$ for $\lambda \in \mathbb{K}\backslash\{0\}$ is another evolution element (corresponding to basis $\widetilde{B} = \{\lambda e_1, \ldots, \lambda e_n\}$) with $L_{\tilde{e}}^m = \lambda^m L_e^m$. Clearly, then, $(L_{\tilde{e}}^m)_m$ may not converge even if $(L_e^m)_m$ does. In other words, the role of the evolution element (even assuming norm 1) is not central and we study instead convergence of $L_a^m(b)$, for arbitrary a, b in A. To this end, we employ the multiplicative spectrum $\sigma_m(a)$ of a, as introduced in [17]. Section 4 then proves that $(L_a^m(b))_m$ converges for all $b \in A$ if and only if

$$\sigma_m(a) \subseteq \Delta \cup \{1\} \qquad \text{and} \qquad \nu(1, a) \leq 1,$$

where $\nu(1, a)$ is the index of 1 as an eigenvalue of L_a, and Δ is the open unit disk in $\mathbb{C}$. Alternative formulations of this are given in Corollaries 4.18 and 4.19.

For example, $(L_a^m(b))_m$ converges for all $b \in A$ if and only if $L_a = P + S$, for linear maps $P, S \in L(A)$ satisfying $P = P^2$, $PS = SP = 0$ and $\rho(S) < 1$. Moreover, we show that if $(L_a^m(b))_m$ converges for all $b \in A$, then $P := \lim_m L_a^m$ is a projection onto the subspace $A_a = \ker(L_a - I)$ and $P = 0$ if and only if $\nu(1, a) = 0$. Theorem 4.22 and Corollary 4.23 examine cases where the dynamical system $L_a^m(b)$ displays recurrent states.

2. Evolution algebras as Banach algebras

While finite-dimensional evolution algebras were introduced in [25] and evolution algebras with a countable basis were studied in [24], the first general algebraic study of evolution algebras of arbitrary dimension was presented in [1]. As the definition there generalizes the earlier ones, we use it throughout this article.

Definition 2.1. An *evolution algebra* is an algebra A provided with a basis $B = \{e_i : i \in \Lambda\}$ such that $e_i e_j = 0$ for $i, j \in \Lambda$ with $i \neq j$, where Λ is an arbitrary (possibly uncountable) nonempty set of indices. Such a basis B is said to be a *natural basis* of A. The product of A is then determined by the equalities $e_i^2 = \sum_{k \in \Lambda} \omega_{ki} e_k$, for all $i \in \Lambda$, and, for fixed $k \in \Lambda$, we note that ω_{ki} is nonzero for only a finite number of indices.

The map : $\Lambda \times \Lambda \to \mathbb{K}$ such that $(i, j) \to \omega_{ij}$ encodes the algebra structure of A with respect to B. It is therefore useful to represent this map as a $\Lambda \times \Lambda$ matrix, which we denote by $M_A(B) = (\omega_{ij})_{i,j}$ and which we refer to as the *evolution matrix* of A with respect to B.

In this section, we are primarily interested in what happens when an evolution algebra A is endowed with an algebra norm (i.e., a norm making the product continuous). When A is provided with such a norm, we will say that A is a *normed evolution algebra*, and when that norm is also complete, we will say that A is a *Banach evolution algebra*.

Of course, all finite-dimensional normed evolution algebras are automatically Banach evolution algebras since all norms are then complete. In what follows, we show that the concept of an infinite-dimensional Banach evolution algebra is not as straightforward as one might expect. In fact, an immediate consequence of the Baire category theorem is that an infinite-dimensional Banach space cannot have a countable basis, and hence an infinite-dimensional Banach evolution algebra cannot have a countable natural basis. In particular, this means that infinite-dimensional evolution algebras with countable basis in the sense of [24, Definition 3] are never Banach algebras.

We first show that every evolution algebra is a normed evolution algebra.

Definition 2.2. If $B = \{e_i : i \in \Lambda\}$ is a natural basis of an evolution algebra A, then the l_1-*norm with respect to* B is the norm $\|\cdot\|_1$ defined as

$$\|a\|_1 = \sum_{i \in \Lambda_a} |\alpha_i|$$

whenever $a = \sum_{i \in \Lambda} \alpha_i e_i = \sum_{i \in \Lambda_a} \alpha_i e_i$, and $\Lambda_a := \{i \in \Lambda : \alpha_i \neq 0\}$ is a finite subset of Λ.

Proposition 2.3. *Let A be an evolution algebra, let $B = \{e_i : i \in \Lambda\}$ be a natural basis, and let $\|\cdot\|_1$ be the l_1-norm with respect to B. Then $\|\cdot\|_1$ is an algebra norm on A if and only if $\|e_i^2\|_1 \leq 1$, for every $i \in \Lambda$.*

Proof. If $\|\cdot\|_1$ is an algebra norm on A, then $\|e_i^2\|_1 \leq \|e_i\|_1^2 = 1$. Conversely, if $\|e_i^2\|_1 \leq 1$ for every $i \in \Lambda$, then for $a = \sum_{i\in\Lambda_a} \alpha_i e_i$ and $b = \sum_{i\in\Lambda_b i} \beta_i e_i$, we have

$$\|ab\|_1 = \Big\| \sum_{i\in\Lambda_a\cap\Lambda_b} \alpha_i\beta_i e_i^2 \Big\|_1 \leq \sum |\alpha_i\beta_i| \leq \Big(\sum_{i\in\Lambda_a} |\alpha_i|\Big)\Big(\sum_{i\in\Lambda_b} |\beta_i|\Big) = \|a\|_1\|b\|_1,$$

namely, $\|\cdot\|_1$ is an algebra norm on A. □

This contrasts with [24, Section 3.3.1], where algebra norms are not considered. Proposition 2.3 also motivates the following.

Definition 2.4. Let A be an evolution algebra, and let $B = \{e_i : i \in \Lambda\}$ be a natural basis. We say that B is a *normalized natural basis* if $\|e_i^2\|_1 = 1$ for every $i \in \Lambda$ such that $e_i^2 \neq 0$.

It is easy to check that every evolution algebra A has a normalized natural basis. In fact, given a natural basis $B = \{u_i : i \in \Lambda\}$ of A, for $i \in \Lambda$, define $e_i := \frac{1}{\sqrt{\|u_i^2\|_1}} u_i$ if $u_i^2 \neq 0$ and $e_i = u_i$ otherwise. Then $\{e_i : i \in \Lambda\}$ is a normalized natural basis which we call the *normalized natural basis derived from B*.

The following is now immediate from Proposition 2.3.

Corollary 2.5. *Every evolution algebra A is a normed evolution algebra; namely, if B is a normalized natural basis, then the l_1-norm with respect to B is an algebra norm on A.*

Definition 2.6. Let $\|\cdot\|$ be an algebra norm on an evolution algebra A, and let $B = \{e_i : i \in \Lambda\}$ be a natural basis. We say that B is *unital* if $\|e_i\| = 1$, for every $i \in \Lambda$.

We may assume without loss of generality that for a given algebra norm the natural basis B is unital. The following example shows that the completion of a normed evolution algebra is not, in general, itself an evolution algebra (for the same underlying product).

Example 2.7. Let c_{00} be the space of infinite sequences of finite support endowed with the product given by $e_n^2 = e_n$ and $e_n e_m = 0$ if $n \neq m$, for the standard (natural) basis $B = \{e_n : n \in \mathbb{N}\}$. Proposition 2.3 above implies that the l_1-norm is an algebra norm on c_{00} since $\|e_n^2\|_1 = \|e_n\|_1 = 1$. The completion of c_{00} with respect to this norm is the Banach space l_1. Suppose now that l_1 is an evolution algebra with natural basis given by $\overline{B} = \{u_i : i \in \Lambda\}$. From earlier, we know that Λ must be uncountable. For every $j \in \mathbb{N}$ there exist $m \in \mathbb{N}$ (depending on j), elements $u_{j_1}, \ldots, u_{j_m} \in \overline{B}$, and scalars $\gamma_1, \ldots, \gamma_m$ such that

$$e_j = \gamma_1 u_{j_1} + \cdots + \gamma_m u_{j_m}.$$

Then $\overline{B}_{00} := \bigcup_{j\in\mathbb{N}} \{u_{j_1}, \ldots, u_{j_m}\}$ is a countable subset of $\overline{B}$. Because Λ is not countable, there exists $u_{i_0} \in \overline{B}\backslash\overline{B}_{00}$ and it follows that $e_j u_{i_0} = 0$, for every $j \in \mathbb{N}$.

Fix $k_0 \in \mathbb{N}$. If $u_{i_0} = \sum_{k\in\mathbb{N}} \gamma_k e_k$ with $\sum_{k\in\mathbb{N}} |\gamma_k| < \infty$, then

$$0 = e_{k_0} u_{i_0} = e_{k_0} \sum_{k\in\mathbb{N}} \gamma_k e_k = \gamma_{k_0} e_{k_0}^2 = \gamma_{k_0} e_{k_0}.$$

In other words, $\gamma_{k_0} = 0$ and therefore $u_{i_0} = 0$. Since this is impossible, it follows that l_1 has no natural basis and is therefore not an evolution algebra.

We show next that nondegenerate infinite-dimensional Banach evolution algebras do not exist.

Lemma 2.8. *Let A be a Banach evolution algebra with norm $\|\cdot\|$ and natural basis $B = \{e_i : i \in \Lambda\}$. Then the set $\Lambda_B := \{i \in \Lambda : e_i^2 \neq 0\}$ is finite.*

Proof. We may assume without loss of generality that B is a unital natural basis so that if $\|\cdot\|_1$ denotes the corresponding l_1-norm associated to B as above, then $\|a\| \leq \|a\|_1$, for all $a \in A$. Suppose now that Λ_B is infinite. It is well known (via the axiom of choice and axiom of countable choice) that every infinite set has a countably infinite subset, so let $\{e_i : i \in \mathbb{N}\} \subseteq \Lambda_B$. Choose nonzero scalars α_n such that $\sum_{n\in\mathbb{N}} |\alpha_n| < \infty$. Let $u_n := \sum_{k=1}^{n} \alpha_k e_k$. Then $(u_n)_n$ is a $\|\cdot\|_1$-Cauchy sequence and hence, since B is unital, it is therefore also $\|\cdot\|$-Cauchy and consequently $\|\cdot\|$-convergent, so that the $\|\cdot\|$-limit $u = \lim_n u_n$ exists in A. On the other hand, since B is a basis

$$u = \beta_1 e_{\gamma_1} + \cdots + \beta_k e_{\gamma_k}, \tag{2.1}$$

for some $k \in \mathbb{N}$, nonzero scalars $\beta_1, \ldots, \beta_k$, and indices $\gamma_1, \ldots, \gamma_k \in \Lambda$. Fix now $j \in \mathbb{N}$ such that $e_j \notin \{e_{\gamma_1}, \ldots, e_{\gamma_k}\}$. Since $\lim_n \|u - u_n\| = 0$ and the product is $\|\cdot\|$-continuous, we have

$$\begin{aligned} 0 &= \lim_n \big\|e_j(u - u_n)\big\| \\ &= \lim_n \big\|e_j(\beta_1 e_{\gamma_1} + \cdots + \beta_k e_{\gamma_k} - u_n)\big\| \\ &= \lim_n \big\|e_j(u_n)\big\| = \|\alpha_j e_j^2\| = |\alpha_j| \|e_j^2\|. \end{aligned}$$

Since $j \in \Lambda_B$, then $e_j^2 \neq 0$. In particular, then $\alpha_j = 0$. Since the scalars α_n were chosen to be nonzero, this contradiction proves that Λ_B must be finite. □

Theorem 2.9. *If $(A, \|\cdot\|)$ is a Banach evolution algebra, then $A = A_0 \oplus A_1$, where A_1 is a finite-dimensional evolution algebra and A_0 is a zero-product subalgebra.*

Proof. Let $B = \{e_i : i \in \Lambda\}$ be a natural basis. By Lemma 2.8, the set $\Lambda_B := \{i \in \Lambda : e_i^2 \neq 0\}$ is finite. For $i \in \Lambda$, if $e_i^2 = \sum_{k\in\Lambda} \omega_{ki} e_k$, let

$$\widehat{\Lambda}_i := \{k \in \Lambda : \omega_{ki} \neq 0\} \cup \{i\}.$$

Let $\Lambda_1 := \bigcup_{i\in\Lambda_B} \widehat{\Lambda}_i$ and $\Lambda_0 := \Lambda \backslash \Lambda_1$. Then for $A_0 = \mathrm{lin}\{e_i : i \in \Lambda_0\}$ and $A_1 = \mathrm{lin}\{e_i : i \in \Lambda_1\}$ we have $A = A_0 \oplus A_1$, where A_0 is (a possibly infinite-dimensional) zero-product subalgebra and A_1 is a finite-dimensional evolution subalgebra of A. □

This motivates the following, originally introduced in [25, p. 2].

Definition 2.10. We say that an evolution algebra A is *nondegenerate* if for some natural basis $B = \{e_i : i \in \Lambda\}$, then $e_i^2 \neq 0$ for every $i \in \Lambda$.

One sees easily that Definition 2.10 is independent of the choice of natural basis, for suppose that $B = \{e_i : i \in \Lambda\}$ and $\tilde{B} = \{u_i : i \in \Omega\}$ are two natural bases of A and suppose that $e_{i_0}^2 = 0$, for some $i_0 \in \Lambda$. Then $e_j e_{i_0} = 0$ for all $j \in \Lambda$ and hence $ae_{i_0} = 0$ for all $a \in A$. There is a finite subset $\Omega_0 \subset \Omega$ such that $e_{i_0} = \sum_{j\in\Omega_0} \alpha_j u_j$, with $\alpha_j \neq 0$ for $j \in \Omega_0$. For $k \in \Omega_0$, we then have $0 = u_k e_{i_0} = \alpha_k u_k^2$. In other words, $u_k^2 = 0$, for all $k \in \Omega_0$, giving the required independence. The independence can also be seen as a consequence of [1, Corollary 2.19], namely, an evolution algebra is nondegenerate if and only if $\operatorname{ann}(A) = 0$, where $\operatorname{ann}(A)$ denotes the annihilator of A. The following is now immediate.

Corollary 2.11. *Nondegenerate Banach evolution algebras are finite-dimensional. Consequently, the completion of a nondegenerate infinite-dimensional normed evolution algebra is not an evolution algebra.*

If A is a degenerate normed evolution algebra, then its completion $\widehat{A}$ is an evolution algebra only when $\widehat{A}$ is an algebra of the type described in Theorem 2.9, in which case A must also be of the same type. The above corollary answers in the negative a question raised in [24, p. 18] as to whether or not infinite-dimensional evolution algebras can be Banach algebras.

3. Continuity of the evolution operator

We continue to study the continuity of the evolution operator, defined as in [24].

Definition 3.1. Let A be an evolution algebra, and let $B = \{e_i : i \in \Lambda\}$ be a natural basis. The *evolution operator* of A associated to B is the unique linear map $L_B : A \to A$ such that $L(e_i) = e_i^2$.

Remark 3.2. If $\dim A < \infty$ and $B = \{e_1, \ldots, e_n\}$ is a natural basis of A, then for $a \in A$, $L_B(a) = ea$, where $e = \sum_{i=1}^n e_i$. In other words, L_B is the multiplication operator L_e. Of course, in infinite dimensions L_B is well defined even when $\sum_{i\in\Lambda} e_i$ is not.

Propositions 2.3 and 2.5 guarantee that A always has an algebra norm, namely, the l_1-norm with respect to a normalized natural basis. Moreover, we have the following.

Proposition 3.3. *Let A be an algebra provided with a norm $\|\cdot\|$. Then $\|\cdot\|$ is an algebra norm if and only if for every $a \in A$ the multiplication operator L_a is continuous with $\|L_a\| \leq \|a\|$.*

Proof. If $\|\cdot\|$ is an algebra norm, then $\|L_a(b)\| = \|ab\| \leq \|a\|\|b\|$, so L_a is continuous and $\|L_a\| \leq \|a\|$. Conversely, if $\|L_a\| \leq \|a\|$, then

$$\|ab\| = \big\|L_a(b)\big\| \leq \|L_a\|\|b\| \leq \|a\|\|b\|,$$

so that $\|\cdot\|$ is an algebra norm. □

We now show that the evolution operator is not necessarily continuous for every algebra norm in the infinite-dimensional case (of course, all norms are equivalent and every linear map is continuous in finite dimensions).

Proposition 3.4. *There exists a normed evolution algebra* $(A, \|\cdot\|)$ *with a natural basis such that* L_B *is not continuous.*

Proof. Let A be the space c_{00} of infinite sequences of finite support, as in Example 2.7 above. Let $B := \{e_n : n \in \mathbb{N}\}$, where $e_n := (\delta_{kn})_{k\in\mathbb{N}}$. For $n, m \in \mathbb{N}$, define $e_n^2 = ne_n$ and $e_n e_m = 0$, if $n \neq m$. Then A is an evolution algebra and B is a natural basis for A. Let $\gamma : \mathbb{N} \to \mathbb{N}$ be such that $\gamma(n) \geq n$, for every $n \in \mathbb{N}$. Let $F : A \to A$ be the unique linear operator such that $F(e_k) = \gamma(k)e_k$, for $k \in \mathbb{N}$. Define $\|a\| = \|F(a)\|_1$ for every $a \in A$. It is straightforward to check that this is a norm. In fact,

$$\begin{aligned}\|ab\| &= \big\|F(ab)\big\|_1 = \Big\|F\Big(\sum \alpha_n\beta_n e_n^2\Big)\Big\|_1 = \Big\|F\Big(\sum \alpha_n\beta_n n e_n\Big)\Big\|_1 \\ &= \Big\|\sum \alpha_n\beta_n n\gamma(n)e_n\Big\|_1 \leq \Big(\sum |\alpha_n|\gamma(n)\Big)\Big(\sum |\beta_n|\gamma(n)\Big) \\ &= \Big\|\sum \alpha_n F(e_n)\Big\|_1 \Big\|\sum \beta_n F(e_n)\Big\|_1 = \|a\|\|b\|.\end{aligned}$$

Obviously, $\|\cdot\|$ and $\|\cdot\|_1$ are not equivalent because $\|e_n\|_1 = 1$ while $\|e_n\| = \gamma(n) \to \infty$. We claim that $L_B : A \to A$ is not $\|\cdot\|$-continuous. For $k, n \in \mathbb{N}$, let α_k be such that $\alpha_k\gamma(k) = \frac{1}{k^2}$, and define $a_n := \sum_{k=1}^n \alpha_k e_k$. Then

$$\begin{aligned}\|a_n\| &= \big\|F(a_n)\big\|_1 = \Big\|F\Big(\sum_{k=1}^n \alpha_k e_k\Big)\Big\|_1 = \Big\|\sum_{k=1}^n \alpha_k\gamma(k)e_k\Big\|_1 \\ &= \sum_{k=1}^n \big|\alpha_k\gamma(k)\big| = \sum_{k=1}^n \frac{1}{k^2} < \sum_{k=1}^\infty \frac{1}{k^2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\big\|L_B(a_n)\big\| &= \Big\|\sum_{k=1}^n \alpha_k e_k^2\Big\| = \Big\|F\Big(\sum_{k=1}^n \alpha_k e_k^2\Big)\Big\|_1 = \Big\|F\Big(\sum_{k=1}^n \alpha_k k e_k\Big)\Big\|_1 \\ &= \Big\|\sum_{k=1}^n \alpha_k k\gamma(k)e_k\Big\|_1 = \sum_{k=1}^n \big|\alpha_k k\gamma(k)\big| = \sum_{k=1}^n \frac{1}{k}.\end{aligned}$$

Therefore the sequence $L_B(a_n)$ is not $\|\cdot\|$-bounded, which proves the claim. □

The next result provides a sufficient condition for continuity of L_B.

Proposition 3.5. *Let* A *be a normed evolution algebra, and let* $B = \{e_i : i \in \Lambda\}$ *be a unital natural basis. If* $\sup\{\|\sum_{i\in F} e_i\| : F \subset \Lambda, F \text{ finite}\} < \infty$, *then* L_B *is continuous.*

Proof. Let $M := \sup\{\|\sum_{i\in F} e_i\| : F \subset \Lambda, F \text{ finite}\}$. If $a = \sum_{i\in\Lambda_a} \alpha_i e_i$, then

$$\big\|L_B(a)\big\| = \Big\|\sum_{i\in\Lambda_a} \alpha_i e_i^2\Big\| = \Big\|\Big(\sum_{i\in\Lambda_a} e_i\Big)a\Big\| \leq \Big\|\sum_{i\in\Lambda_a} e_i\Big\|\|a\| \leq M\|a\|,$$

as desired. □

4. Dynamics of the evolution operator

Corollary 2.11 above shows that nondegenerate infinite-dimensional Banach evolution algebras do not exist, so we assume henceforth that A is a finite-dimensional normed evolution algebra with given algebra norm $\|\cdot\|$.

Throughout, $L(A)$ denotes the algebra (under function composition) of all linear maps on A endowed with the usual operator norm; L_a denotes the multiplication operator $L_a(b) = ab$, for $a, b \in A$, while $M_{n,m}$ is the space of all $n \times m$ matrices over $\mathbb{K}$ and $M_n := M_{n,n}$. Although A is nonassociative in general and a^m is therefore not well defined for $a \in A$, $L(A)$ is an associative algebra and we may therefore consider the iterates of L_a, namely, $L_a^1 := L_a$ and $L_a^m := L_a \circ L_a^{m-1}$, for $m \geq 2$.

Definition 4.1. Let $B = \{e_1, \ldots, e_n\}$ be a fixed natural basis of A, and let $e := e_1 + \cdots + e_n$. We call e the *evolution element* of B.

Since A is finite-dimensional, the evolution operator L_B of A (with respect to the basis B) is the multiplication operator L_e (cf. Remark 3.2).

For $b \in A$, we may postulate, to some extent, $(L_e^m(b))_m$ as a discrete-time dynamical system, whose limit points may help to describe the long-term evolutionary state of b. Our goal, therefore, is to determine when $(L_e^m(b))_m$ converges and, more crucially, to then locate its limit. In fact, the role played by e is not so central, since any nonzero multiple of e is an evolution element for another basis. We examine therefore the more general question of the convergence or otherwise of the sequence $(L_a^m(b))_m$, and the determination of the limit where it exists, for arbitrary $a, b \in A$.

Definition 4.2. We say that $a \in A$ is an *equilibrium generator* if $(L_a^m(b))_{m\in\mathbb{N}}$ converges, for all $b \in A$.

We note that since A is finite-dimensional, all norms on A are equivalent so the definition is independent of the choice of norm on A.

Let $M_A(B) = (\omega_{ij})_{ij} \in M_n$ be the evolution matrix of A with respect to B, as described in Section 2. It is straightforward to check that for $a = \sum_{i=1}^n \alpha_i e_i$, the matrix of L_a with respect to B is given by

$$W_a^B := \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{pmatrix}. \tag{4.1}$$

We call W_a^B the evolution matrix of a (with respect to B), and we write $W_a := W_a^B$ when the basis is clear from the context. We note that W_e is $M_B(A)$. As usual, we write $\sigma(W_a)$ for the set of eigenvalues of W_a and $\rho(W_a)$ for its spectral radius.

We recall a concept of spectrum for nonassociative algebras, introduced in [17] for general algebras and in [28] for evolution algebras and to which we refer for all details (see also [15], [16], [27]). We recall for a complex algebra E that $a \in E$ is said to be *m-invertible* if L_a and R_a are bijective, where R_a denotes the right multiplication map $R_a(b) = ba$, for $a, b \in A$.

Definition 4.3. Let E be a complex algebra with unit e. The *m-spectrum* of a in E is

$$\sigma_m^E(a) := \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not } m\text{-invertible}\}.$$

If E is a complex algebra without unit, then $\sigma_m^E(a) := \sigma_m^{E_1}(a)$, where E_1denotes the unitization of E, and if E is a real algebra, then $\sigma_m^E(a) := \sigma_m^{E_{\mathbb{C}}}(a)$,where $E_{\mathbb{C}}$ denotes the complexification of E.

When the context is clear, we write $\sigma_m(a)$ for $\sigma_m^E(a)$. Moreover, for a linear map $T : E \to E$, $\sigma(T)$ denotes its usual spectrum (see [7])

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bijective}\}.$$

Then $\sigma_m(a) = \sigma(L_a) \cup \sigma(R_a)$ whenever E is unital and $\sigma_m(a) = \sigma(L_a) \cup \sigma(R_a) \cup \{0\}$ otherwise. Thus, for commutative A, and evolution algebras in particular, we have that $\sigma_m(a) = \sigma(L_a)$ if A is unital and $\sigma(L_a) \cup \{0\}$ otherwise. We recall (see [28, Corollary 2.12]) that an evolution algebra A is unital if and only if A is a finite-dimensional nonzero trivial evolution algebra.

Definition 4.4. The *m-spectral radius* of $a \in E$ is $\rho(a) := \sup\{|\lambda| : \lambda \in \sigma_m(a)\}$ if $\sigma_m(a) \neq \emptyset$ and $\rho(a) := 0$ otherwise.

An m-spectral radius formula is given in [17, Proposition 2.2].

Returning now to evolution algebras, we note that if E is a real evolution algebra, then its complexification $E_{\mathbb{C}}$ is also an evolution algebra and every natural basis of E is a natural basis of $E_{\mathbb{C}}$, so that L_B can also be regarded as an element of $L(E_{\mathbb{C}})$. In particular, we have the following, stated implicitly in [28, Propositions 5.1, 5.3].

Proposition 4.5. *Let A be a finite-dimensional evolution algebra with natural basis $B = \{e_1, \ldots, e_n\}$, and let $a \in A$. Let W_a be the evolution matrix of a with respect to B. Then $\sigma_m(a) = \sigma(W_a)$ if A is unital and $\sigma_m(a) = \sigma(W_a) \cup \{0\}$ otherwise.*

In the next definition, we introduce an operator to help us clarify the space that we are considering in our approach.

Definition 4.6. Let ϕ be the natural isomorphism from A to $\mathbb{C}^n$ given by

$$\phi\Big(\sum_i \beta_i e_i\Big) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}.$$

From (4.1) above we have $\phi(L_a(b)) = W_a \phi(b)$ for $a, b \in A$, or equivalently, as operators, $L_a = \phi^{-1} W_a \phi$ and hence by induction

$$L_a^m = \phi^{-1} W_a^m \phi \tag{4.2}$$

for $a \in A$ and $m \in \mathbb{N}$. Since the spectrum of a linear map is independent of its matrix representation, $\sigma(L_a) = \sigma(W_a)$ and hence from Proposition 4.5,

$$\rho(a) = \rho(L_a) = \rho(W_a).$$

Using the natural isomorphism ϕ, every norm on A induces a corresponding norm on $\mathbb{C}^n$ by $\|x\| := \|\phi^{-1}(x)\|$, for $x \in \mathbb{C}^n$, and for this norm on $\mathbb{C}^n$ the isomorphism ϕ then becomes an isometry. In addition, every norm on A gives a unique operator norm on $L(A)$, namely,

$$\|T\| = \sup_{\|b\| \leq 1} \big\|T(b)\big\|, \quad \text{for } T \in L(A), b \in A.$$

In fact, since A is finite-dimensional, this supremum is achieved. In exactly the same way, every norm on $\mathbb{C}^n$ (and, in particular, the norm induced from A via ϕ above) gives a unique operator norm on $L(\mathbb{C}^n)$ and we may identify $L(\mathbb{C}^n)$ with M_n in the usual way.

Of course, for any algebra norm on A, we have

$$\|L_a^m\| \leq \|L_a\|^m \leq \|a\|^m, \quad \text{for all } m \in \mathbb{N},$$

so that $\|a\| < 1$ implies $\lim_m L_a^m = 0$. Furthermore, we get the following (for the given norm on A and the above induced norms on $\mathbb{C}^n$, $L(A)$, and M_n, respectively).

Proposition 4.7. *Let A be a finite-dimensional evolution algebra. Let W_a be the evolution matrix of $a \in A$ with respect to a fixed natural basis B. The following are equivalent:*

(i) *a is an equilibrium generator, that is, $(L_a^m(b))$ converges, for all $b \in A$;*
(ii) *$\lim_m L_a^m$ exists in $L(A)$;*
(iii) *$\lim_m W_a^m$ exists in M_n.*
(iv) *$\lim_m (W_a^m)_{ij}$ exists, for all $1 \leq i, j \leq n$ (where T_{ij} denotes the ij coordinate of $T \in M_n$).*

Proof. The equivalence of (ii) and (iii) is immediate from (4.2) above. The operator norm on M_n is equivalent to the norm defined coordinatewise by $\|T\| := \max_{1 \leq i,j \leq n} |T_{ij}|$ making (iii) equivalent to (iv). Clearly (ii) implies (i). We finish by showing that (i) implies (ii). Assume therefore that $(L_a^m(b))_m$ converges, for all $b \in A$. Define $T : A \to A$ by $T(b) := \lim_m L_a^m(b)$. Clearly T is linear and hence bounded. Moreover, since for all $b \in A$, $\sup_m \|L_a^m(b)\| < \infty$, the uniform boundedness principle implies that $\sup_m \|L_a^m\| < \infty$ and, in fact, that $\|T\| \leq \sup_m \|L_a^m\|$. We finish with a standard compactness argument, given for completeness. Let $K = \sup_m \|L_a^m\|$. Fix ϵ arbitrary. By compactness of $D = \{x \in A : \|x\| \leq 1\}$ there exists $x_1, \ldots, x_k \in D$ such that

$$D \subset \bigcup_{j=1}^{k} B(x_j, \epsilon/K),$$

where $B(x, \alpha) = \{y \in A : \|x - y\| < \alpha\}$. For $1 \leq j \leq k$, there exists M_j such that $\|L_a^m(x_j) - T(x_j)\| < \epsilon$, for all $m \geq M_j$. Let $M := \max_j M_j$. Now take $x \in D$ and

$m > M$. Then $x \in B(x_j, \epsilon/K)$ for some $1 \le j \le k$, and therefore

$$\begin{aligned}\left\|L_a^m(x) - T(x)\right\| &\le \left\|L_a^m(x) - L_a^m(x_j)\right\| + \left\|L_a^m(x_j) - T(x_j)\right\| + \left\|T(x_j) - T(x)\right\| \\ &\le \|L_a^m\|\|x - x_j\| + \left\|L_a^m(x_j) - T(x_j)\right\| + \|T\|\|x - x_j\| \\ &\le K(\epsilon/K) + \epsilon + K(\epsilon/K) = 3\epsilon,\end{aligned}$$

giving the result. □

The concept of equilibrium generator is clearly independent of the natural basis chosen. As mentioned earlier, however, given two evolution elements e and $\tilde{e}$ (corresponding to different bases), one may be an equilibrium generator, while the other may not, as the following example further demonstrates.

Example 4.8. Let A be the linear span of e_1 and e_2 with multiplication defined by $e_1e_2 = e_2e_1 = 0$ and $e_1^2 = e_2^2 = e_1$. Then A is an evolution algebra with natural basis $B = \{e_1, e_2\}$ and evolution element $e = e_1 + e_2$. Now let $\tilde{B} = \{\tilde{e}_1, \tilde{e}_2\}$, for $\tilde{e}_1 = e_1 + e_2$, $\tilde{e}_2 = e_1 - e_2$. Then $\tilde{B}$ is also a natural basis with evolution element $\tilde{e} = \tilde{e}_1 + \tilde{e}_2 = 2e_1$. Then

$$W_e = W_e^B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad W_{\tilde{e}} = W_{\tilde{e}}^B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

are the evolution matrices of e and $\tilde{e}$ (each taken with respect to B). Clearly $\lim_m W_e^m = W_e$, while

$$W_{\tilde{e}}^m = \begin{pmatrix} 2^m & 0 \\ 0 & 0 \end{pmatrix}$$

does not converge, so e is an equilibrium generator while $\tilde{e}$ is not.

Therefore, while the concept of equilibrium generator is independent of the basis chosen, the concept of an evolution element (of a basis) being an equilibrium generator is not. This suggests, in contrast to comments in [24, Section 3.2.1], that other operators apart from the evolution operator $L_B(= L_e)$ may be more relevant to the study of A. Nonetheless, we introduce the following (basis-dependent) definition.

Definition 4.9. Let A be a finite-dimensional evolution algebra with fixed natural basis B. Let e be the evolution element of B. We say that A reaches *B-equilibrium* if e is an equilibrium generator.

We say that $T \in L(A)$ is a projection if $T^2 = T$ and, similarly, that $C \in M_n$ is a projection if $C^2 = C$. Recall that the rank of a linear map T is well defined as the rank of any matrix representation of T.

Proposition 4.10. *Let A be a finite-dimensional evolution algebra, and let $a \in A$. If a is an equilibrium generator, then $P := \lim_m L_a^m$ commutes with L_a and is a projection onto the subspace* $\ker(L_a - I)$. *In particular,* $\operatorname{rank}(P) = \dim P(A) = \dim \ker(L_a - I)$ *and if $P \neq 0$, then $1 \in \sigma_m(a)$.*

Proof. Let a be an equilibrium generator. Then from Proposition 4.7, $P = \lim_m L_a^m$ exists in $L(A)$. The subsequence (L_a^{2m}) must then also converge to P, so that by continuity of composition in $L(A)$ we have

$$P = \lim_m L_a^{2m} = \lim_m L_a^m \circ \lim_m L_a^m = P \circ P = P^2.$$

Moreover, for $x \in A$, we have

$$L_a\big(P(x)\big) = L_a\big(\big(\lim_m L_a^m\big)(x)\big) = \big(\lim_m L_a^{m+1}\big)(x) = P(x),$$

so that $P(A) \subseteq \ker(L_a - I)$. In particular, if $P \neq 0$, then $\ker(L_a - I) \neq \emptyset$, so $1 \in \sigma(L_a)$ and hence $1 \in \sigma_m(a)$. For $y \in \ker(L_a - I)$, we have $y = L_a(y)$, so $y = L_a^m(y)$, for all $m \in \mathbb{N}$ and hence $y = P(y) \in P(A)$, so $\ker(L_a - I) \subseteq P(A)$, giving $P(A) = \ker(L_a - I)$. □

Proposition 4.10 motivates the following.

Definition 4.11. Let A be an evolution algebra, and let $a \in A$ be an equilibrium generator. We then define the *equilibrium subspace* of a as $A_a := \ker(L_a - I)$, and we define the *equilibrium rank* of a as $r(a) := \dim(\ker(L_a - I))$ if $A_a \neq \{0\}$, and $r(a) = 0$ otherwise.

We note from [19] that since $L(A)$ and M_n are finite-dimensional, the spectral radius function is continuous.

Proposition 4.12. *Let A be a finite-dimensional evolution algebra, and let $a \in A$.*

(i) *If $\rho(a) > 1$, then a is not an equilibrium generator and, in particular, $(L_a^m)_m$ has no convergent subsequences.*
(ii) *We have that $\lim_m L_a^m = 0$ if and only if $\rho(a) < 1$.*

Proof. For (i) let us first suppose that a subsequence $(L_a^{m_k})_k$ converges in $L(A)$, say, to $\tilde{P}$. Then $\rho(\tilde{P}) = \lim_k \rho(L_a^{m_k})$. As A is finite-dimensional, it is easy to see from the spectral radius formula that $\rho(L_a^{m_k}) = \rho(L_a)^{m_k}$ so $\rho(\tilde{P}) = \lim_k \rho(L_a^{m_k}) = \lim_k \rho(L_a)^{m_k} = \lim_k \rho(a)^{m_k}$. This is impossible if $\rho(a) > 1$, giving (i).

For (ii) let us assume that $\rho(a) < 1$. Then $\rho(L_a) < 1$ and hence $\|L_a^m\| < 1$, for all m sufficiently large (otherwise $\|L_a^{m_k}\| \geq 1$, for some subsequence $(m_k)_k$, and then $\rho(L_a) = \lim_k \|L_a^{m_k}\|^{1/m_k} \geq 1$). Then L_a^m, for m large, lies in the closed unit ball of $L(A)$ which is compact, and thus every subsequence of (L_a^m) has itself a convergent subsequence. Consider the limit of any such convergent subsequence, say, $\tilde{P} := \lim_k L_a^{m_k}$. As in (i) above, $\rho(\tilde{P}) = \lim_k \rho(a)^{m_k}$ and hence $\tilde{P} = 0$. Since the limit of all such convergent subsequences of (L_a^m) is thus zero, it follows by compactness that the sequence (L_a^m) itself must also converge to zero. In other words, $\rho(a) < 1$ implies $\lim_m L_a^m = 0$. In the opposite direction, if $\lim_m L_a^m = 0$, then continuity of the spectral radius gives

$$0 = \rho(0) = \lim_m \rho(L_a^m) = \lim_m \rho(L_a)^m = \lim_m \rho(a)^m$$

and hence $\rho(a) < 1$, and we are done. □

It remains to examine the case $\rho(a) = 1$. To this end, we use the Jordan normal form of a matrix, considered folklore in the literature (see [23]), but recalled here for convenience.

Proposition 4.13. *For $W \in M_n$, there exists an invertible matrix Q and Jordan block matrix J such that $W = Q^{-1}JQ$, where*

$$J = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_t \end{pmatrix}.$$

Each J_i is a Jordan matrix corresponding to eigenvalue λ_i, that is, a square matrix with λ_i on the diagonal, 1 on the superdiagonal, and zeros elsewhere. Moreover, the eigenvalues of the blocks $J_1, \dots, J_t$, counting multiplicities, are precisely the eigenvalues of the matrix J and hence of W. In particular, for eigenvalue λ_i, we recall the following.

(i) *The geometric multiplicity $m_g(\lambda_i, W) = \dim(\ker(W - \lambda_i I))$ gives the number of Jordan blocks corresponding to λ_i.*
(ii) *The algebraic multiplicity $m_a(\lambda_i, W)$ gives the sum of the sizes of all Jordan blocks corresponding to λ_i.*
(iii) *The index, denoted $\nu(\lambda_i, W)$, gives the size of the largest Jordan block corresponding to λ_i.*
(iv) *In particular, we have $\nu(\lambda_i, W) = 1$ if and only if*

$$\dim\big(\ker(W - \lambda_i I)\big) = m_g(\lambda_i, W) = m_a(\lambda_i, W).$$

In this case, putting together all the Jordan matrices corresponding to λ_i gives $\lambda_i I_{r_i}$, a diagonal matrix of size $r_i := \dim(\ker(W - \lambda_i I))$.

Since the eigenvalues of W_a determine the multiplicative spectrum $\sigma_m(a)$ of a (see Proposition 4.5 above), the following definitions are natural.

Definition 4.14. For $a \in A$ and λ an eigenvalue of W_a, we define the *multiplicative a-index* of λ as $\nu(\lambda, a) := \nu(\lambda, W_a)$. If $\lambda \in C$ is not an eigenvalue of W, then we define the multiplicative a-index of λ as $\nu(\lambda, a) := 0$.

Since W_a is unique up to similarity and the Jordan form is unique up to order of its blocks, the index $\nu(\lambda, a)$ is well defined and independent of the basis.

Proposition 4.15. *Let A be a finite-dimensional evolution algebra, and let $a \in A$ with $\rho(a) = 1$. Then, a is an equilibrium generator if and only if $\sigma_m(a) \cap \partial\Delta = \{1\}$ and $\nu(1, a) = 1$.*

Proof. Let $a \in A$ and $\rho(a) = 1$. Let W_a be the evolution matrix of a as above. Let $\lambda \in \sigma_m(a) \cap \partial\Delta$. From Proposition 4.5, $\lambda \in \sigma(W_a)$. Consider the Jordan normal form of W_a as above, and let J be any Jordan matrix corresponding to λ. The $(1, 1)$ entry of J^m is λ^m, for $m \in \mathbb{N}$. Since $\lim_m \lambda^m$ only exists if and only if $\lambda = 1$, we have that if $\lambda \neq 1$, then $(J^m)_{1,1}$ cannot converge and hence W_a^m cannot converge; then from Proposition 4.7, a is not an equilibrium generator. If on the other hand $\lambda = 1$ and $\nu(1, a) = \nu(1, W_a) > 1$, then it means that there is a Jordan matrix J corresponding to eigenvalue 1 of size $s > 1$. Then J^m has

m on its first superdiagonal, so $(J^m)_m$ and hence also $(W_a^m)_m$ cannot converge and again a is not an equilibrium generator. In other words, if a is an equilibrium generator, then $\sigma_m(a) \cap \partial\Delta = \{1\}$ and $\nu(1, a) = 1$.

In the opposite direction, if $\rho(a) = 1$, $\sigma_m(a) \cap \partial\Delta = \{1\}$, and $\nu(1, a) = 1$, then, from Proposition 4.13(iv), putting all Jordan blocks corresponding to eigenvalue 1 together gives the $r \times r$ identity matrix $I_r \in M_r$, where

$$r = \dim\big(\ker(W_a - I)\big) = \dim\big(\ker(L_a - I)\big) = r(a).$$

Write $R \oplus T$ for the matrix

$$\begin{pmatrix} R & 0 \\ 0 & T \end{pmatrix},$$

for $R \in M_r$ and $T \in M_{n-r}$. Then we have $W_a = Q^{-1}(I \oplus T)Q$, for $I \in M_r$, $T \in M_{n-r}$ with $\rho(T) < 1$, and some invertible $Q \in M_n$. Then

$$W_a^m = Q^{-1}(I \oplus T^m)Q.$$

Since $\rho(T) < 1$ gives $\lim_m T^m = 0$ (see Proposition 4.12(ii)), we then have $\lim_m W_a^m = Q^{-1}(I \oplus 0)Q$. From (4.2),

$$\lim_m L_a^m = \lim_m (\phi^{-1} \circ W_a^m \circ \phi) = \phi^{-1} \circ Q^{-1}(I \oplus 0)Q \circ \phi,$$

and we are done. □

We note that Propositions 4.12 and 4.15 can also be derived from Proposition 4.7 and known results in different formats for matrices (see, e.g., [20]). Propositions 4.10, 4.12, and 4.15 together now give the following.

Theorem 4.16. *Let A be a finite-dimensional evolution algebra, and let $a \in A$. Then a is an equilibrium generator if and only if*

$$\sigma_m(a) \subseteq \Delta \cup \{1\} \qquad \textit{and} \qquad \nu(1, a) \leq 1.$$

Moreover, if a is an equilibrium generator, then $P = \lim_m L_a^m$ is a projection onto the a-equilibrium subspace $A_a = \ker(L_a - I)$, and if $\nu(1, a) = 0$, then $P = 0$.

Note that if $\rho(a) < 1$, then trivially we have $\nu(1, a) = 0$.

Corollary 4.17. *Let A be a finite-dimensional evolution algebra with evolution element e with respect to a natural basis B. Then A reaches B-equilibrium if and only if*

$$\sigma_m(e) \subseteq \Delta \cup \{1\} \qquad \textit{and} \qquad \nu(1, e) \leq 1.$$

The following two corollaries are reformulations of the above two results using the Jordan normal form and, in particular, Proposition 4.13(iv). Recall also Definitions 4.9 and 4.11.

We write $I_r \oplus T$ to denote the matrix

$$\begin{pmatrix} I_r & 0 \\ 0 & T \end{pmatrix}$$

if $r \neq 0$ and $I_r \oplus T = T$ if $r = 0$.

Corollary 4.18. *Let A be a finite-dimensional evolution algebra, and let $a \in A$. Then $a \in A$ is an equilibrium generator if and only if its evolution matrix W_a (with respect to any basis B) is similar to a matrix of the form*

$$I_r \oplus T, \quad \text{where } \rho(T) < 1, r = r(a), I_r \in M_r, T \in M_{n-r}, n = \dim(A).$$

Corollary 4.19. *Let A be a finite-dimensional evolution algebra, and let $a \in A$. Then a is an equilibrium generator if and only if*

$$L_a = P + S, \quad \text{for linear maps } P, S \text{ in } L(A)$$

satisfying $P^2 = P$, $PS = SP = 0$, and $\rho(S) < 1$.

Proof. If a is an equilibrium generator, then from Corollary 4.18 and the proof of Theorem 4.15 there is an invertible matrix $Q \in M_n$ such that

$$W_a = Q^{-1}(I_r \oplus T)Q,$$

where $\rho(T) < 1$, $r = r(a)$, $I_r \in M_r$, $T \in M_{n-r}$. Then, from (4.2),

$$L_a = \phi^{-1}Q^{-1}(I \oplus T)Q\phi.$$

Let

$$P := \phi^{-1}Q^{-1}(I_r \oplus 0)Q\phi \qquad \text{and} \qquad S := \phi^{-1}Q^{-1}(0 \oplus T)Q\phi$$

(recall that $P = 0$ if $r = 0$). Then $L_a = P + S$ and it is easy to see that P and S have the required properties. In the opposite direction, if $L_a = P + S$ with properties as stated, then $L_a^m = P + S^m$, and since $\rho(S) = \rho(T) < 1$, then $\lim_m S^m = 0$ giving $\lim_m L_a^m = P$, and a is an equilibrium generator. □

We now examine the situation where a type of recurrent behavior can arise, namely, when W_a has eigenvalues that are pth roots of unity. Let

$$\Omega = \{e^{\frac{2\pi i}{p}} : p \in \mathbb{N}\}.$$

Lemma 4.20. *Let $W \in M_n$ with $\sigma(W) \subset \Delta \cup \Omega$ and*

$$\sigma(W) \cap \Omega = \{e^{\frac{2\pi i}{p_1}}, \dots, e^{\frac{2\pi i}{p_s}}\} \neq \emptyset.$$

Let $\lambda_k := e^{\frac{2\pi i}{p_k}}$, $1 \le k \le s$. If

$$\nu(\lambda_k, W) = 1, \quad \text{for } 1 \le k \le s,$$

then for any choice of i and k_i, $1 \le i \le s$, $1 \le k_i \le p_i$, there exist a subsequence $(m_l)_l$ of $\mathbb{N}$ and coefficients $\alpha_j \in \{1, \lambda_j, \dots, \lambda_j^{p_j-1}\}$, for $j \neq i$, $1 \le j \le s$, such that

$$\lim_l W^{m_l} = \lambda_i^{k_i}\tilde{P}_i + \sum_{j \neq i} \alpha_j \tilde{P}_j,$$

where $\tilde{P}_1, \dots, \tilde{P}_s$ are mutually orthogonal projections onto the eigenspaces of W for $\lambda_1, \dots, \lambda_s$.

Proof. Let $W \in M_n$ satisfy the conditions in the statement of the lemma. We note that the case $s = 1$, $p_1 = 1$ is covered by Theorem 4.16. Writing $R_1 \oplus \cdots \oplus R_t$ for the block diagonal matrix with blocks $R_1, \dots, R_t$, Proposition 4.13 gives an invertible $Q \in M_n$ such that

$$W = Q^{-1} J Q \qquad \text{and} \qquad J = J_1 \oplus \cdots \oplus J_t,$$

where each $J_1, \dots, J_t$ is a Jordan matrix corresponding to some eigenvalue of W, $1 \leq t \leq n$. Let $\lambda_k := e^{\frac{2\pi i}{p_k}}$, for $1 \leq k \leq s$, as in the statement. Since $\nu(\lambda_k, W) = 1$, Proposition 4.13(iv) implies that putting together all Jordan matrices corresponding to eigenvalue λ_k gives a diagonal matrix $\lambda_k I_{r_k}$ of size $r_k := \dim(\ker(W - \lambda_k I))$.

If $\sum_{k=1}^{s} r_k < n$, then $\sigma(W) \cap \Delta \neq \emptyset$. In this case, for $q = n - \sum_{k=1}^{s} r_k$, putting together all Jordan matrices corresponding to eigenvalues in Δ gives a matrix $T \in M_q$ (also block diagonal) with $\rho(T) < 1$. We may therefore assume, without loss of generality, that

$$W = Q^{-1}(\lambda_1 I_{r_1} \oplus \cdots \oplus \lambda_s I_{r_s} \oplus T) Q, \quad \text{if } q \neq 0$$

and

$$W = Q^{-1}(\lambda_1 I_{r_1} \oplus \cdots \oplus \lambda_s I_{r_s}) Q, \quad \text{if } q = 0.$$

Now fix i and k_i, $1 \leq i \leq s$, $1 \leq k_i \leq p_i$. Then

$$(\lambda_i I_{r_i})^{k_i + m p_i} = \lambda_i^{k_i} I_{r_i}, \quad \text{for all } m \in \mathbb{N}.$$

Moreover, for $1 \leq k \leq s$, each of the following sets is finite:

$$\left\{ (\lambda_k I_{r_k})^m : m \in \mathbb{N} \right\} = \{ I_{r_k}, \lambda_k I_{r_k}, \dots, \lambda_k^{p_k - 1} I_{r_k} \}.$$

Therefore there is a subsequence $(m_l)_l$ of $(k_i + m p_i)_m$ such that for all $j \neq i$,with $1 \leq j \leq s$, there is $\alpha_j \in \{1, \lambda_j, \dots, \lambda_j^{p_j - 1}\}$ with

$$(\lambda_j I_{r_j})^{m_l} = \alpha_j I_{r_j}.$$

Of course, α_j may depend on the fixed i and k_i chosen. For convenience (reordering if necessary), we will assume that $i = 1$. Then

$$(\lambda_1 I_{r_1} \oplus \cdots \oplus \lambda_s I_{r_s} \oplus T)^{m_l} = \lambda_1^{k_1} I_{r_1} \oplus \alpha_2 I_{r_2} \oplus \cdots \oplus \alpha_s I_{r_s} \oplus T^{m_l}, \quad \text{if } q \neq 0,$$

and equals

$$\lambda_1^{k_1} I_{r_1} \oplus \alpha_2 I_{r_2} \oplus \cdots \oplus \alpha_s I_{r_s}, \quad \text{if } q = 0.$$

Now let, for $1 \leq k \leq s$,

$$\tilde{P}_k := Q^{-1}(0_{r_1} \oplus \cdots 0_{r_{k-1}} \oplus I_{r_k} \oplus 0_{r_{k+1}} \cdots \oplus 0_{r_s} \oplus 0_q) Q \in M_n, \quad \text{if } q \neq 0,$$

and

$$\tilde{P}_k := Q^{-1}(0_{r_1} \oplus \cdots 0_{r_{k-1}} \oplus I_{r_k} \oplus 0_{r_{k+1}} \cdots \oplus 0_{r_s}) Q, \quad \text{if } q = 0.$$

Clearly then $\tilde{P}_1, \dots, \tilde{P}_s$ are mutually orthogonal projections in M_n (and $\tilde{P}_k(\mathbb{C}^n)$ is exactly the λ_k-eigenspace of W). Then

$$W^{m_l} = \lambda_1^{k_1} \tilde{P}_1 + \alpha_2 \tilde{P}_2 + \cdots + \alpha_s \tilde{P}_s + Q^{-1}(0_{n-q} \oplus T^{m_l}) Q, \quad \text{for } l \in \mathbb{N},\ \text{if } q \neq 0,$$

and

$$W^{m_l} = \lambda_1^{k_1} \tilde{P}_1 + \alpha_2 \tilde{P}_2 + \cdots + \alpha_s \tilde{P}_s, \quad \text{for all } l \in \mathbb{N},\ \text{if } q = 0. \tag{4.3}$$

Of course, if $q \neq 0$, then $T \in M_q$ has $\rho(T) < 1$ and hence $\lim_l T^{m_l} = 0$, giving the required result. $\square$

Lemma 4.20 also covers the case where the spectrum contains only pth roots of unity and since then $q = 0$, the next result follows immediately from (4.3).

Corollary 4.21. *Let $W \in M_n$ with $\sigma(W) \subset \Omega$ and*

$$\sigma(W) \cap \Omega = \{e^{\frac{2\pi i}{p_1}}, \dots, e^{\frac{2\pi i}{p_s}}\}.$$

Let $\lambda_k := e^{\frac{2\pi i}{p_k}}$, $1 \leq k \leq s$. If

$$\nu(\lambda_k, W) = 1, \quad \text{for } 1 \leq k \leq s,$$

then for any choice of i and k_i, $1 \leq i \leq s, 1 \leq k_i \leq p_i$, there exist a subsequence $(m_l)_l$ of $\mathbb{N}$ and coefficients $\alpha_j \in \{1, \lambda_j, \dots, \lambda_j^{p_j-1}\}$, for $j \neq i$, $1 \leq j \leq s$, such that

$$W^{m_l} = \lambda_i^{k_i} \tilde{P}_i + \sum_{j \neq i} \alpha_j \tilde{P}_j, \quad \text{for all } l \in \mathbb{N}, \tag{4.4}$$

where $\tilde{P}_1, \dots, \tilde{P}_s$ are mutually orthogonal projections onto the eigenspaces of W for $\lambda_1, \dots, \lambda_s$.

Lemma 4.20 and Proposition 4.7 now give the following.

Theorem 4.22. *Let A be a finite-dimensional evolution algebra, and let $a \in A$ with $\sigma_m(a) \subset \Delta \cup \Omega$ and*

$$\sigma_m(a) \cap \Omega = \{e^{\frac{2\pi i}{p_1}}, \dots, e^{\frac{2\pi i}{p_s}}\} \neq \emptyset.$$

Let $\lambda_k := e^{\frac{2\pi i}{p_k}}$, $1 \leq k \leq s$. If

$$\nu(\lambda_k, a) = 1, \quad \text{for } 1 \leq k \leq s,$$

then for any choice of i and k_i, $1 \leq i \leq s$, $1 \leq k_i \leq p_i$, there exist a subsequence $(m_l)_l$ of $\mathbb{N}$ and coefficients $\alpha_j \in \{1, \lambda_j, \dots, \lambda_j^{p_j-1}\}$, for $j \neq i$, $1 \leq j \leq s$, such that

$$\lim_l L_a^{m_l} = \lambda_i^{k_i} P_i + \sum_{j \neq i} \alpha_j P_j,$$

where $P_1, \dots, P_s$ are mutually orthogonal projections onto the L_a-eigenspaces for $\lambda_1, \dots, \lambda_s$, respectively.

Proof. Let $a \in A$ satisfy the conditions in the statement of the theorem, and let W_a be its evolution matrix with respect to a fixed natural basis. Then W_a satisfies the conditions of Lemma 4.20. Fixing i and k_i, $1 \leq i \leq s$, $1 \leq k_i \leq p_i$, Lemma 4.20 then yields a subsequence $(m_l)_l$ of $\mathbb{N}$, mutually orthogonal projection matrices $\tilde{P}_1, \dots, \tilde{P}_s$, and scalars $\alpha_j \in \{1, \lambda_j, \dots, \lambda_j^{p_j-1}\}$, for $j \neq i$, $1 \leq j \leq s$, such that (4.4) holds, namely,

$$\lim_l W_a^{m_l} = \lambda_i^{k_i} \tilde{P}_i + \sum_{j \neq i} \alpha_j \tilde{P}_j.$$

For $1 \le k \le s$, let

$$P_k := \phi^{-1} \circ \tilde{P}_k \circ \phi, \tag{4.5}$$

where ϕ is the isometry in (4.2) above. It follows that $P_1, \dots, P_s$ are mutually orthogonal projections onto the L_a-eigenspaces for $\lambda_1, \dots, \lambda_s$, respectively. Proposition 4.7 and (4.2) then give

$$\lim_l L_a^{m_l} = \lambda_i^{k_i} P_i + \sum_{j \ne i} \alpha_j P_j,$$

as required. □

If $\sigma_m(a)$ contains only pth roots of unity, then the next result follows from (4.5) in Theorem 4.22 and (4.3) in Lemma 4.20 above.

Corollary 4.23. *Let A be a finite-dimensional evolution algebra, and let $a \in A$ with $\sigma_m(a) \subset \Omega$ and*

$$\sigma_m(a) \cap \Omega = \{e^{\frac{2\pi i}{p_1}}, \dots, e^{\frac{2\pi i}{p_s}}\}.$$

Let $\lambda_k := e^{\frac{2\pi i}{p_k}}$, $1 \le k \le s$. If

$$\nu(\lambda_k, a) = 1, \quad \text{for } 1 \le k \le s,$$

then for any choice of i and k_i, $1 \le i \le s, 1 \le k_i \le p_i$, there exist a subsequence $(m_l)_l$ of $\mathbb{N}$ and coefficients $\alpha_j \in \{1, \lambda_j, \dots, \lambda_j^{p_j-1}\}$, for $j \ne i$, $1 \le j \le s$, such that

$$L_a^{m_l} = \lambda_i^{k_i} P_i + \sum_{j \ne i} \alpha_j P_j, \quad \text{for all } l \in \mathbb{N}, \tag{4.6}$$

where $P_1, \dots, P_s$ are mutually orthogonal projections onto the L_a-eigenspaces for $\lambda_1, \dots, \lambda_s$.

In Corollary 4.23 above, for fixed i and k_i, $1 \le i \le s$, $1 \le k_i \le p_i$ then the subsequence in $L(A)$ obtained in (4.6) above, namely,

$$L_{a,i,k_i} := (L_a^{m_l})_l,$$

is constant. In particular, this means that for all $b \in A$, the sequence $(L_a^n(b))_n$ will return to the value $L_{a,i,k_i}(b)$ infinitely often. Borrowing from the language of Markov processes, we would say that $L_{a,i,k_i}(b)$ is a *recurrent state* of the system.

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References

1. Y. Cabrera-Casado, M. Siles-Molina, and M. V. Velasco, *Evolution algebras of arbitrary dimension and their decompositions*, Linear Algebra Appl. **495** (2016), 122–162. Zbl 06545267. MR3462991. DOI 10.1016/j.laa.2016.01.007. 114, 115, 118
2. L. M. Camacho, J. R. Gómez, B. A. Omirov, and R. M. Turdibaev, *The derivations of some evolution algebras*, Linear Multilinear Algebra **61** (2013), no. 3, 309–322. Zbl 1311.17001. MR3003426. DOI 10.1080/03081087.2012.678342. 114
3. L. M. Camacho, J. R. Gómez, B. A. Omirov, and R. M. Turdibaev, *Some properties of evolution algebras*, Bull. Korean Math. Soc. **50** (2013), no. 5, 1481–1494. Zbl 1278.05120. MR3116885. DOI 10.4134/BKMS.2013.50.5.1481. 114
4. J. M. Casas, M. Ladra, B. A. Omirov, and U. A. Rozikov, *On nilpotent index and dibaricity of evolution algebras*, Linear Algebra Appl. **439** (2013), no. 1, 90–105. Zbl 1343.17021. MR3045225. DOI 10.1016/j.laa.2013.03.006. 114
5. J. M. Casas, M. Ladra, B. A.Omirov, and U. A. Rozikov, *On evolution algebras*, Algebra Colloq. **21** (2014), no. 2, 331–342. Zbl 1367.17026. MR3192352. DOI 10.1142/S1005386714000285. 114
6. J. M. Casas, M. Ladra, and U. A. Rozikov, *A chain of evolution algebras*, Linear Algebra Appl. **435** (2011), no. 4, 852–870. Zbl 1220.17026. MR2807239. DOI 10.1016/j.laa.2011.02.012. 114
7. H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. (N.S.) **24**, Oxford Univ. Press, New York, 2000. Zbl 0981.46043. MR1816726. 121
8. A. Elduque and A. Labra, *Evolution algebras and graphs*, J. Algebra Appl. **14** (2015), no. 7, art. ID 1550103. Zbl 1356.17028. MR3339402. DOI 10.1142/S0219498815501030. 114
9. I. M. H. Etherington, *Genetic algebras*, Proc. Roy. Soc. Edinburgh Sect. A **59** (1939), 242–258. Zbl 0027.29402. MR0000597. 113
10. I. M. H. Etherington, *Non-associative algebra and the symbolism of genetics*, Proc. Roy. Soc. Edinburgh. Sect. B. **61** (1941), 24–42. Zbl 0063.01290. MR0003557. 113
11. V. Glivenko, *Algèbre mendélienne*, C. R. Acad. Sci. URSS **4** (1936), no. 4, 385–386. Zbl 0016.00503. 113
12. H. S. Jennings, *The numerical results of diverse systems of breeding, with respect to two pairs of characters, linked or independent, with special relation to the effects of linkage*, Genetics **2** (1917), no. 2, 97–154. 113
13. A. Labra, M. Ladra, and U. A. Rozikov, *An evolution algebra in population genetics*, Linear Algebra Appl. **45** (2014), 348–362. Zbl 1292.17030. MR3230450. DOI 10.1016/j.laa.2014.05.036. 114
14. M. Ladra, B. A. Omirov, and U. A. Rozikov, *Dibaric and evolution algebras in biology*, Lobachevskii J. Math. **35** (2014), no. 3, 198–210. Zbl 1370.17034. MR3259516. DOI 10.1134/S199508021403007X. 114
15. J. C. Marcos and M. V. Velasco, *The Jacobson radical of an non-associative algebra and the uniqueness of the complete norm topology*, Bull. Lond. Math. Soc. **42** (2010), no. 6, 1010–1020. Zbl 1217.46030. MR2740021. DOI 10.1112/blms/bdq060. 121
16. J. C. Marcos and M. V. Velasco, *Continuity of homomorphisms into power-associative complete normed algebras*, Forum Math. **25** (2013), no. 6, 1109–1125. Zbl 1291.46047. MR3199725. DOI 10.1515/form.2011.169. 121
17. J. C. Marcos and M. V. Velasco, *The multiplicative spectrum and the uniqueness of the complete norm topology*, Filomat **28** (2014), no. 3, 473–485. Zbl 06704774. MR3360022. DOI 10.2298/FIL1403473M. 114, 121
18. G. Mendel, "Experiments in plant-hybridization" in *Classic Papers in Genetics*, Prentice-Hall, Englewood Cliffs, N.J., 1959, 1–20. 113
19. G. Murphy, *Continuity of the spectrum and spectral radius*, Proc. Amer. Math. Soc. **82** (1981), no. 4, 619–621. Zbl 0474.46038. MR0614889. DOI 10.2307/2043782. 124

20. R. Oldenburger, *Infinite powers of matrices and characteristic roots*, Duke Math. J. **6** (1940), 357–361. Zbl 0023.19702. MR0001949. 126
21. U. A. Rozikov and S. N. Murodov, *Dynamics of two-dimensional evolution algebras*, Lobachevskii J. Math. **34** (2013), no. 4, 344–358. Zbl 1297.37011. MR3146637. DOI 10.1134/S199508021304015X. 114
22. A. Serebrowsky, *On the properties of the Mendelian equations* (in Russian), Doklady A. N. SSSR. **2** (1934), 33–36. 113
23. G. E. Shilov, *Linear Algebra*, revised English ed., Dover, New York, 1977. Zbl 0218.15003. MR0466162. 125
24. J. P. Tian, *Evolution Algebras and Their Applications*, Lecture Notes in Math. **1921**, Springer, Berlin, 2008. Zbl 1136.17001. MR2361578. DOI 10.1007/978-3-540-74284-5. 114, 115, 116, 118, 123
25. J. P. Tian and P. Vojtěchovský, *Mathematical concepts of evolution algebras in non-Mendelian genetics*, Quasigroups Related Systems **14** (2006), no. 1, 111–122. Zbl 1112.17001. MR2268830. 114, 115, 117
26. J. P. Tian and Y. M. Zou, *Finitely generated nil but not nilpotent evolution algebras*, J. Algebra Appl. **13** (2014), no. 1, art. ID 1350070. Zbl 1296.17011. MR3096851. DOI 10.1142/S0219498813500709. 114
27. M. V. Velasco, *Spectral theory for non-associative complete normed algebras and automatic continuity*, J. Math. Anal. Appl. **351** (2009), no. 1, 97–106. Zbl 1170.46043. MR2472923. DOI 10.1016/j.jmaa.2008.09.036. 121
28. M. V. Velasco, *The Jacobson radical of an evolution algebra*, to appear in J. Spectr. Theory, preprint, arXiv:1805.08812v1 [math.FA]. 121

[1]School of Mathematics and Statistics, University College Dublin, Dublin 4, Ireland.
E-mail address: pmellon@maths.ucd.ie

[2]Departmento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071-Granada (Spain).
E-mail address: vvelasco@ugr.es