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WEAK BOUNDEDNESS OF OPERATOR-VALUED BOCHNER–RIESZ MEANS FOR THE DUNKL TRANSFORM

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Abstract. We consider operator-valued Bochner–Riesz means with weight function h_κ^2 under a finite reflection group for the Dunkl transform. We establish the maximal inequality of the weighted Hardy–Littlewood maximal function, and we apply it to the maximal inequality of operator-valued Bochner–Riesz means $B_R^\delta(h_\kappa^2; f)(x)$ for $\delta > \lambda_\kappa := \frac{d-1}{2} + \sum_{j=1}^d \kappa_j$. Furthermore, we also obtain the corresponding pointwise convergence theorem.

1. Introduction

Given $\kappa = (\kappa_1, \ldots, \kappa_d) \in [0, \infty)^d \subset \mathbb{R}^d$, let

$$h_\kappa(x) := \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d. \tag{1.1}$$

For Borel sets $E \subset \mathbb{R}^d$, we write

$$\mathrm{meas}_\kappa(E) := \int_E h_\kappa^2(x)\, dx.$$

For $1 \leq p < \infty$, the operator-valued L_p-space is defined with respect to the measure $h_\kappa^2(x)\, dx$ on $\mathbb{R}^d$, and $\|\cdot\|_{\kappa,p}$ denotes the norm of $L_p(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M}))$ (see

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[19]). Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean norm and the Euclidean inner product on $\mathbb{R}^d$, respectively. Put

$$c_\kappa^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(y) e^{-\frac{\|y\|^2}{2}}\, dy$$

and

$$E_\kappa(-ix, y) = V_\kappa[e^{-i\langle x,\cdot\rangle}](y),$$

where $V_\kappa : C(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is the Dunkl intertwining operator associated with $h_\kappa^2(x)$ and the reflection group Z_2^d. The Dunkl transform $\widehat{f}$ of $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$ is defined by

$$\widehat{f}(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(-ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d. \tag{1.2}$$

Note that V_0 is the identity operator on $\mathbb{R}^d$. In this case, $\widehat{f}$ is the classical Fourier transform. The Dunkl transform enjoys many properties similar to those of the classical Fourier transform (see [3], [23], [24]).

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. For $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$, we define

$$f_{r,\kappa}(x) = \frac{\int_{\mathbb{R}^d} f(y)\tau_x\chi_{B_r}(y) h_\kappa^2(y)\, dy}{\int_{B_r(x)} h_\kappa^2(y)\, dy}.$$

We say that $f_{r,\kappa}$ is *weak type* $(1,1)$ if there is a positive constant C_1 such that for any $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$ and any $\lambda > 0$, there is a projection $e \in P(L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M})$ satisfying

$$\forall r > 0, \quad \|e f_{r,\kappa} e\|_\infty \le \lambda, \qquad \text{and} \qquad \tau \otimes \int e^\perp \le \frac{C_1\|f\|_{\kappa,1}}{\lambda}.$$

Here $P(A)$ denotes the set of all projections in A, $A = L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M}$. Also, we say that $f_{r,\kappa}$ is of type (p,p) (p may be equal to ∞) if there is a positive constant C_2 such that for any $f \in L_p(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M}))$,

$$\Big\|{\sup_{r>0}}^+ f_{r,\kappa}\Big\|_{\kappa,p} \le C_2\|f\|_{\kappa,p}.$$

We first consider the weak $(1,1)$-boundedness and (p,p)-boundedness of $f_{r,\kappa}$. In the scalar-valued case, that is, replacing $\mathcal{M}$ by complex numbers $\mathbb{C}$, C_1, and C_2 reduced to be the weak $(1,1)$-boundedness and (p,p)-boundedness of the weighted Hardy–Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{\int_{\mathbb{R}^d} |f(y)|\tau_x\chi_{B_r}(y) h_\kappa^2(y)\, dy}{\int_{B_r(x)} h_\kappa^2(y)\, dy}, \quad x \in \mathbb{R}^d.$$

However, this maximal function seems to be unavailable for operator-valued functions since we cannot compare any two matrices or operators, which is a source of difficulty in noncommutative analysis. Junge [13], [14] successfully overcame this obstacle by describing the interaction with operator space theory (see Section 2 for the definition of maximal function in noncommutative analysis). In fact, Junge [13] formulated a noncommutative version of Doob's maximal inequality using Pisier's theory (see [18]) of vector-valued noncommutative L_p-space. Later, Junge and Xu [14] developed a fairly complex noncommutative version of Marcinkiewicz's interpolation theorem. As a consequence, they obtained the noncommutative Dunford–Schwartz maximal ergodic inequality and the noncommutative Stein's maximal ergodic inequality (see [14]). Based on the preceding statement, Mei [17] studied the operator-valued Hardy–Littlewood maximal inequality in $\mathbb{R}^d$. He made use of the geometric property of $\mathbb{R}^d$ to reduce the Hardy–Littlewood maximal inequality to several operator-valued martingale inequalities, which can be viewed as Junge's noncommutative version of Doob's maximal inequality, or Cuculescu's weak type $(1,1)$ inequality for noncommutative martingales. Chen, Xu, and Yin [2] exploited Mei's inequality to prove maximal inequalities associated with integrable rapidly decreasing functions. (We refer the reader to [6]–[9], and [12] for more information on the development of noncommutative harmonic analysis. We also refer to [1], [10], [11], [15], and [16], which consider the boundedness of the associated operators.)

The operator-valued Bochner–Riesz means B_R^δ is defined by

$$B_R^\delta(h_\kappa^2; f)(x) = c_\kappa \int_{\|y\|\le R} \Big(1 - \frac{\|y\|^2}{R^2}\Big)^\delta \widehat{f}(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d,$$

where $R > 0$, $\delta > -1$, and $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$. The weak boundedness of the operator-valued Bochner–Riesz means B_R^δ for the classical Fourier transform was studied in [2]. In this article, we consider the weak boundedness of operator-valued Bochner–Riesz means B_R^δ for the Dunkl transform, which has applications in physics for the analysis of quantum many-body systems of Calogero–Moser–Sutherland type. From the mathematical analysis point of view, its importance lies in that it generalizes the classical Fourier transform, and plays a similar role as the Fourier transform in classical Fourier analysis.

This paper is organized as follows. In Section 2, we collect some preliminaries which are needed in this article. In Section 3, we obtain the weak type $(1,1)$-boundedness and type (p,p)-boundedness of operator-valued weighted Hardy–Littlewood maximal functions. The result can be stated as follows.

Theorem 1.1.

(i) *For any* $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$, *and* $\lambda > 0$, *there is a projection* $e \in P(L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M})$ *satisfying*

$$\|e f_{r,\kappa} e\|_\infty \le \lambda \quad \text{for } r > 0, \qquad \text{and} \qquad \tau \otimes \int e^\perp \le \frac{C\|f\|_{\kappa,1}}{\lambda}.$$

(ii) *We have that* $\|\sup_{r>0}^+ f_{r,\kappa}\|_{k,p} \le C_p \|f\|_{\kappa,p}$ *for any* $f \in L_p(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M}))$, *where* $p > 1$.

In Section 4, we show the weak type $(1,1)$-boundedness and type (p,p)-boundedness of operator-valued Bochner–Riesz means $B_R^\delta(h_\kappa^2; f)(x)$ for $\delta > \lambda_\kappa := \frac{d-1}{2} + \sum_{j=1}^d \kappa_j$, and we obtain the following result.

Theorem 1.2.

(i) *For $\delta > \lambda_\kappa$, $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$, and $\lambda > 0$, there exists a projection $e \in P(L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M})$ satisfying*

$$\left\|eB_R^\delta(h_\kappa^2; f)e\right\|_\infty \leq \lambda \quad \textit{for } R > 0, \qquad \textit{and} \qquad \tau \otimes \int e^\perp \leq \frac{C\|f\|_{\kappa,1}}{\lambda}.$$

(ii) *We have that $\|\sup^+_{R>0} B_R^\delta(h_\kappa^2; f)\|_{\kappa,p} \leq C_p \|f\|_{\kappa,p}$ for any $f \in L_p(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M}))$, where $p > 1$.*

In Section 5, we show that $B_R^\delta(h_\kappa^2; f) \xrightarrow{\text{b.a.u.}} f$ as $R \to \infty$ for $1 \leq p < \infty$ and $f \in L_p(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M}))$ for $\delta > \lambda_\kappa$. The main result is the following.

Theorem 1.3. *Let $\delta > \lambda_\kappa$. For any $f \in L_p(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M}))$, the following statements hold:*

(i) *for $1 \leq p \leq 2$, $B_R^\delta(h_\kappa^2; f) \to f$ (b.a.u.) as $R \to \infty$;*
(ii) *for $2 \leq p < \infty$, $B_R^\delta(h_\kappa^2; f) \to f$ (a.u.) as $R \to \infty$.*

Throughout the article we use the same letter C to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants C will often be specified in parentheses. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \leq CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Denote the set of all positive x in $\mathcal{M}$ by $S^+_{\mathcal{M}}$ such that $\tau(\mathrm{supp}(x)) < \infty$, where $\mathrm{supp}(x)$ denotes the support of x, that is, the least projection $e \in \mathcal{M}$ such that $ex = x$ (or $xe = x$). Let $S_{\mathcal{M}}$ be the linear span of $S^+_{\mathcal{M}}$. We define

$$\|x\|_p = \big(\tau\big(|x|^p\big)\big)^{\frac{1}{p}}, \quad \forall x \in S_{\mathcal{M}},$$

where $|x| = (x^*x)^{\frac{1}{2}}$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L_p(\mathcal{M})$, which is the usual noncommutative L_p-space associated with $(\mathcal{M}, \tau)$. For convenience, we usually set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$.

For $1 \leq p \leq \infty$, we define $L_p(\mathcal{M}, \ell_\infty)$ to be the space of all sequences $x = (x_n)_{n\geq 1}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form. There exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $y = (y_n) \subset L_\infty(\mathcal{M})$ such that

$$x_n = a y_n b, \quad n \geq 1.$$

The norm of x in $L_p(\mathcal{M}, \ell_\infty)$ is given by

$$\|x\|_{L_p(\mathcal{M},\ell_\infty)} = \inf\big\{\|a\|_{2p} \sup_n \|y_n\|_\infty \|b\|_{2p}\big\},$$

where the infimum is taken over all factorizations of x as above.

It is easy to see that $L_p(\mathcal{M},\ell_\infty)$ is a Banach space with the norm $\|\cdot\|_{L_p(\mathcal{M},\ell_\infty)}$, and a positive sequence $x=(x_n)$ belongs to $L_p(\mathcal{M},\ell_\infty)$ if and only if there is $a\in L_p^+(\mathcal{M})$ such that $x_n\le a$ for all n. Moreover, in this case,

$$\|x\|_{L_p(\mathcal{M},\ell_\infty)}=\inf\big\{\|a\|_p : a\in L_p^+(\mathcal{M}) \text{ such that } x_n\le a, \forall n\ge 1\big\}.$$

The norm of x in $L_p(\mathcal{M},\ell_\infty)$ is conventionally denoted by $\|\sup^+_{n\ge1} x_n\|_p$. Note that $\|\sup^+_{n\ge1} x_n\|_p$ is just a notation since $\sup_{n\ge1} x_n$ does not make any sense in the noncommutative setting. We use this notation only for convenience.

The definition of $L_p(\mathcal{M},\ell_\infty)$ can be extended to an arbitrary index set I. Then $L_p(\mathcal{M},\ell_\infty(I))$ can be defined similarly as before. More precisely, $L_p(\mathcal{M},\ell_\infty(I))$ consists of all families $(x_i)_{i\in I}$ in $L_p(\mathcal{M})$ which can be factorized as $x_i=ay_ib$ with $a,b\in L_{2p}(\mathcal{M})$ and a bounded family $(y_i)_{i\in I}\subset L_\infty(\mathcal{M})$. The norm of $(x_i)_{i\in I}$ in $L_p(\mathcal{M},\ell_\infty(I))$ is defined as

$$\inf\big\{\|a\|_{2p}\sup_i\|y_i\|_\infty\|b\|_{2p}\big\},$$

with the infimum running over all factorizations as above. As before, this norm is also denoted by $\|\sup^+_{i\in I} x_i\|_p$.

One can easily check that for any index set I and $1\le p\le\infty$, a family $(x_i)_{i\in I}$ in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M},\ell_\infty(I))$ if and only if

$$\sup_{J\subset I, J \text{ is a finite set}}\Big\|\sup_{i\in J}{}^+ x_i\Big\|_p<\infty.$$

If this is the case, then we have

$$\Big\|\sup_{i\in I}{}^+ x_i\Big\|_p=\sup_{J\subset I, J \text{ is a finite set}}\Big\|\sup_{i\in J}{}^+ x_i\Big\|_p.$$

We will present some necessary materials related to the Dunkl transform associated with the reflection group Z_2^d and the weight $h_\kappa^2(x)$ given in (1.1). Our main reference is [23], where materials on Dunkl analysis associated with general finite reflection groups can also be found.

Recall that $\kappa=(\kappa_1,\dots,\kappa_d)$, $\kappa_1,\dots,\kappa_d\ge 0$ and the weight $h_\kappa^2(x)$ in (1.1). For $1\le j\le d$, let σ_j denote the reflection with respect to the coordinate plane $x_j=0$, that is,

$$x\sigma_j=(x_1,\dots,x_{j-1},-x_j,x_{j+1},\dots,x_d),\quad x\in\mathbb{R}^d.$$

Let Z_2^d denote the reflection group generated by the reflections $\sigma_1,\dots,\sigma_d$. Clearly, Z_2^d is an Abelian group, and the weight $h_\kappa(x)$ is invariant under Z_2^d.

Define a family of difference operators E_j by

$$E_jf(x):=\frac{f(x)-f(x\sigma_j)}{x_j},\quad x\in\mathbb{R}^d, j=1,\dots,d.$$

Let ∂_j denote the partial derivative with respect to the jth coordinate x_j. The Dunkl operators $D_{\kappa,j}$, $j=1,\dots,d$, with respect to the weight $h_\kappa^2(x)$ and the group Z_2^d are defined by

$$D_{\kappa,j}:=\partial_j+\kappa_jE_j,\quad j=1,\dots,d.$$

A remarkable property of these operators is that they mutually commute. We denote by P_n^d the space of homogeneous polynomials of degree n in d variables, and we denote by $\Pi := \Pi(\mathbb{R}^d)$ the C-algebra of polynomial functions on $\mathbb{R}^d$. A fundamental result in Dunkl theory states that there exists a linear operator $V_k : \Pi^d \to \Pi^d$ determined uniquely by

$$V_\kappa P_n^d \subset P_n^d, \qquad V_\kappa(1) = 1, \qquad \text{and} \qquad D_{\kappa,i} V_\kappa = V_\kappa \partial_i, \quad 1 \le i \le d.$$

Such an operator is called the *Dunkl intertwining operator*.

For the weight function $h_\kappa^2(x)$ and the reflection group Z_2^d, the following very useful explicit formula for V_κ was obtained in [25, Theorem 4.3]:

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^d} f(x_1 t_1, \ldots, x_d t_d) \prod_{j=1}^d (1 + t_j)(1 - t_j^2)^{\kappa_j - 1}\, dt_j, \tag{2.1}$$

where $c_k = \prod_{j=1}^d \frac{\Gamma(\kappa_j + \frac{1}{2})}{\sqrt{\pi}\Gamma(\kappa_j)}$, and if any κ_j is equal to zero, then the formula holds under the limits

$$\lim_{\mu \to 0} c_\mu \int_{-1}^1 g(t)(1 - t^2)^{\mu - 1}\, dt = \frac{g(1) + g(-1)}{2}.$$

In particular, the formula (2.1) extends V_κ to a positive operator on the space of continuous functions on $\mathbb{R}^d$. It should be pointed out that such an explicit formula for V_κ is available only in the case of Z_2^d. In the case of a general reflection group, a very deep result on the operator V_κ is due to Rösler [20, Theorem 2.1], who, among other things, proved that V_κ extends to a positive operator on $C(\mathbb{R}^d)$. In fact, the Dunkl transform associated with Z_2^d and κ is defined by (1.2) with

$$E_\kappa(-ix, y) := V_\kappa[e^{-i\langle x, \cdot\rangle}](y), \quad x, y \in \mathbb{R}^d.$$

For necessity, we recall some definitions. Given $y \in \mathbb{R}^d$, the generalized translation operator $f \to \tau_y f$ is defined on $L_2(\mathbb{R}^d; h_\kappa^2; L_2(\mathcal{M}))$ by

$$\widehat{\tau_y f}(\xi) = E_\kappa(-iy, \xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^d.$$

For $f \in L_2(\mathbb{R}^d; h_\kappa^2; L_2(\mathcal{M}))$ and $g \in L^2(\mathbb{R}^d; h_\kappa^2)$, the convolution is defined by

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y) \tau_y g^\vee(y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d,$$

where $g^\vee(y) = g(-y)$. The following is cited from [21, Theorem 2.1].

Proposition 2.1. *If $f(x) = f_0(\|x\|)$ is a continuous radial function in $L^2(\mathbb{R}^d; h_\kappa^2)$, then for almost-everywhere $y \in \mathbb{R}^d$ and almost-everywhere $x \in \mathbb{R}^d$,*

$$\tau_y f(x) = V_\kappa\left[f_0\left(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, \cdot\rangle}\right)\right](y).$$

Let $A_\kappa(\mathbb{R}^d; L_1(\mathcal{M})) = \{f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M})) : \widehat{f} \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))\}$. We have the following properties whose proof is similar to the argument of [23, Section 3].

Proposition 2.2. *Assume that $f \in A_\kappa(\mathbb{R}^d; L_1(\mathcal{M}))$ and that $g \in L^1(\mathbb{R}^d; h_\kappa^2)$ is bounded. Then*

(i) *the following inverse formula holds:*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \widehat{f}(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy;$$

(ii) $\int_{\mathbb{R}^d} \tau_y f(x) g(x) h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} f(x) \tau_{-y} g(x) h_\kappa^2(x)\, dx;$

(iii) $\tau_y f(x) = \tau_{-x} f(-y)$.

Proposition 2.3.

(i) *Let* $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ *be radial and nonnegative. Then* $\tau_y f \geq 0$, $\tau_y f \in L^1(\mathbb{R}^d; h_\kappa^2)$ *and*

$$\int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x)\, dx.$$

(ii) *The generalized translation operator* τ_y, *initially defined on the intersection of* L^1 *and* L^∞, *can be extended to all radial functions in* $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leq p \leq 2$, *and* $\tau_y : L^p_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2) \to L^p(\mathbb{R}^d; h_\kappa^2)$ *is a bounded operator, where* $L^p_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$ *denotes the space of all radial functions in* $L^p(\mathbb{R}^d; h_\kappa^2)$.

(iii) *For any* $f \in L^1_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$,

$$\int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x)\, dx.$$

We see from Proposition 2.3 that the operator τ_y can be extended to a bounded operator on $L^p_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$ $(1 \leq p \leq \infty)$. Also, it follows that the definition of $f *_\kappa g$ can be extended to all $g \in L^p_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$ and $f \in L_{p'}(\mathbb{R}^d; h_\kappa^2; L_{p'}(\mathcal{M}))$ with $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{p'} = 1$. The generalized convolution satisfies the following basic property:

$$\widehat{f *_\kappa g}(\xi) = \widehat{f}(\xi) \widehat{g}(\xi). \tag{2.2}$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Before this, we recall Yeadon's weak type $(1, 1)$ maximal ergodic inequality for semigroups, which is stated as follows.

Lemma 3.1. *Let* $(T_t)_{t \geq 0}$ *be a semigroup of linear maps on* $\mathcal{M}$. *Each* (T_t) *for* $t \geq 0$ *satisfies the following properties:*

(i) (T_t) *is a contraction on* $\mathcal{M} : \|Tx\|_\infty \leq \|x\|_\infty$ *for all* $x \in \mathcal{M}$;

(ii) (T_t) *is positive:* $Tx \geq 0$ *if* $x \geq 0$;

(iii) $\tau \circ T \leq \tau : \tau(T(x)) \leq \tau(x)$ *for all* $x \in L_1 \cap \mathcal{M}^+$.

Let $x \in L_1^+(\mathcal{M})$. *Then for any* $\lambda > 0$, *there exists a projection* $e \in \mathcal{M}$ *such that*

$$eM_t(x)e \leq \lambda, \quad \text{for all } t > 0, \qquad \text{and} \qquad \tau(e^\perp) \leq \frac{\|x\|_1}{\lambda},$$

where M_t *is defined as*

$$M_t = \frac{1}{t} \int_0^t T^s\, ds, \quad \forall t \geq 0.$$

Junge and Xu [14] proved the following quite complicated noncommutative Marcinkiewicz's theorem for $L_p(\mathcal{M}; \ell_\infty)$.

Lemma 3.2. *Let $1 \leq p_0 < p_1 \leq \infty$, and let $S = (S_n)_{n\geq 0}$ be a sequence of maps from $L^+_{p_0}(\mathcal{M}) + L^+_{p_1}(\mathcal{M})$ into $L^+_0(\mathcal{M})$. Assume that S is subadditive in the sense that $S_n(x+y) \leq S_n(x) + S_n(y)$ for all $n \in \mathbb{N}$. If S is of weak type (p_0, p_0) with constant C_0 and of weak type (p_1, p_1) with constant C_1, then for any $p_0 < p < p_1$, S is of type (p, p) with constant C_p satisfying*

$$C_p \leq C C_0^{1-\theta} C_1^{\theta} \Big(\frac{1}{p_0} - \frac{1}{p}\Big)^{-2},$$

where θ is determined by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and C is a universal constant.

With this interpolation result, Junge and Xu proved that there exists a constant C_p such that

$$\big\|\sup_{t>0}{}^+ M_t(x)\big\|_p \leq C_p \|x\|_p, \quad \forall x \in L_p(\mathcal{M}). \tag{3.1}$$

Moreover, if additionally each T_t satisfies:

(iv) T_t is symmetric relative to τ : $\tau(T(y)^* x) = \tau(y^* T x)$ for all x, y in the intersection $L_2(\mathcal{M}) \cap \mathcal{M}$, then

$$\big\|\sup_{t>0}{}^+ T_t(x)\big\|_p \leq C_p \|x\|_p, \quad \forall x \in L_p(\mathcal{M}),$$

with C_p a constant depending only on p.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. (i) For any $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$. By decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with positive f_j $(j = 1, 2, 3, 4)$, we can assume that f is positive. Then we can define

$$f_{r,\kappa}(x) = \frac{\int_{\mathbb{R}^d} f(y) \tau_x \chi_{B_r}(y) h_\kappa^2(y)\, dy}{\int_{B_r(x)} h_\kappa^2(y)\, dy},$$

where $B_r(x) := \{y \in \mathbb{R}^d : \|x - y\| \leq r\}$ denotes the ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$ and $V_r(x) := \mathrm{meas}_\kappa(B_r(x)) = \int_{B_r(x)} h_\kappa^2(z)\, dz$, $r > 0$, $x \in \mathbb{R}^d$. Let $\sigma = d + 2|\kappa| + 1$, and define for $j \geq 0$, $B_{r,j} = \{x : 2^{-j-1} r \leq \|x\| \leq 2^{-j} r\}$, where $|\kappa| = \sum_{j=1}^d \kappa_j$, $\kappa = (\kappa_1, \dots, \kappa_d)$. First we recall the Poisson kernel

$$P_\varepsilon(x) = c_{d,\kappa} \frac{\varepsilon}{(\varepsilon^2 + \|x\|^2)^{\frac{\sigma}{2}}},$$

where $c_{d,\kappa} = 2^{|\kappa| + \frac{d}{2}} \frac{\Gamma(|\kappa| + \frac{d+1}{2})}{\sqrt{\pi}}$. We claim that $f *_\kappa P_t$ is weak $(1,1)$. By using the Dunkl transforms of the Poisson kernel and the heat kernel, we deduce that

$$f *_\kappa P_t(x) = \frac{t}{\sqrt{2\pi}} \int_0^\infty (f *_\kappa q_s)(x) e^{-\frac{t^2}{2s}} s^{-\frac{3}{2}}\, ds,$$

where $q_t(x) = (2t)^{-(|\kappa| + \frac{d}{2})} e^{-\frac{\|x\|^2}{4t}}$. Recall that the heat-diffusion semigroup on $\mathbb{R}^d$ is given by $T^t g = g *_\kappa q_t$, where g is a Schwartz function on $\mathbb{R}^d$. If we consider the heat-diffusion semigroup on $L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{M}$ given by $S^t = T^t \otimes \mathrm{id}_\mathcal{M}$, then it is obvious that $(S^t)_{t\geq 0}$ satisfies Lemma 3.1(i)–(iii). Hence, for any $\lambda > 0$, there exists a projection $e \in P(A)$ such that

$$eM_t(f)e \leq \lambda, \quad \forall t > 0, \qquad \text{and} \qquad \tau \otimes \int e^{\perp} \leq \frac{\|f\|_{\kappa,1}}{\lambda},$$

where

$$M_t(f)(x) = \frac{1}{t}\int_0^t S^s(f)(x)\,ds = \frac{1}{t}\int_0^t (f *_\kappa q_s)(x)\,ds.$$

On the other hand, let

$$\phi(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2s}} s^{-\frac{3}{2}}.$$

Thus by integration by parts,

$$\begin{aligned}
f *_\kappa P_t(x) &= t^{-2}\int_0^\infty \phi\Big(\frac{s}{t^2}\Big) S^s(f)(x)\,ds \\
&= -t^{-2}\int_0^\infty \Big(\int_0^s S^t(f)(x)\,dt\Big)\,d\phi\Big(\frac{s}{t^2}\Big) \\
&= -t^{-2}\int_0^\infty \Big(\frac{1}{s}\int_0^s S^t(f)(x)\,dt\Big) s\,d\phi\Big(\frac{s}{t^2}\Big) \\
&= -t^{-2}\int_0^\infty M_s f(x) s\phi'\Big(\frac{s}{t^2}\Big) t^{-2}\,ds \\
&= -\int_0^\infty M_{t^2 s} f(x) s\phi'(s)\,ds.
\end{aligned}$$

A straightforward calculation shows that $\|s\phi'(s)\|_1 < \infty$. Hence, by Lemma 3.1, for any $\lambda > 0$ there exists a projection $e \in L_\infty(\mathbb{R}^d)\,\overline{\otimes}\,\mathcal{M}$ such that

$$\sup_{t>0} \|ef *_\kappa P_t e\|_\infty \leq \big\|s\phi'(s)\big\|_1 \sup_{t>0}\big\|eM_{t^2 s}(f)e\big\|_\infty \lesssim \lambda$$

and

$$\tau \otimes \int e^{\perp} \lesssim \frac{\|f\|_{\kappa,1}}{\lambda}.$$

Now it suffices to prove that $f_{r,\kappa}$ can be controlled by the Poisson integral. Note that

$$\begin{aligned}
\chi_{B_{r,j}}(y) &= (2^{-j}r)^\sigma \cdot (2^{-j}r)^{-\sigma}\chi_{B_{r,j}}(y) \\
&\leq C(2^{-j}r)^{\sigma-1}\frac{2^{-j}r}{((2^{-j}r)^2 + \|y\|^2)^{\frac{\sigma}{2}}}\chi_{B_{r,j}}(y) \\
&\leq C(2^{-j}r)^{\sigma-1} P_{2^{-j}r}(y), \qquad (3.2)
\end{aligned}$$

where C is a constant independent of r and j. Since χ_{B_r} and P_ε are both bounded integrable radial functions, due to (3.2), it follows from Proposition 2.3 that

$$\tau_x\chi_{B_{r,j}}(y) \leq C(2^{-j}r)^{\sigma-1}\tau_x P_{2^{-j}r}(y).$$

This shows that for any positive integer m,

$$\begin{aligned}\int_{\mathbb{R}^d} f(y) \sum_{j=0}^{m} \tau_x \chi_{B_{r,j}}(y) h_\kappa^2(y)\, dy &\le C \sum_{j=0}^{\infty} (2^{-j} r)^{\sigma-1} \int_{\mathbb{R}^d} f(y) \tau_x P_{2^{-j}r}(y) h_\kappa^2(y)\, dy \\ &= C \sum_{j=0}^{\infty} (2^{-j} r)^{\sigma-1} f *_\kappa P_{2^{-j}r}(x).\end{aligned}$$

Note that $\sum_{j=0}^{m} \chi_{B_{r,j}}(y)$ converges to $\chi_{B_r}(y)$ in $L^1(\mathbb{R}^d; h_\kappa^2)$. Then the boundedness of τ_x on $L^1_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$ shows that $\sum_{j=0}^{m} \tau_x \chi_{B_{r,j}}(y)$ converges to $\tau_x \chi_{B_r}(y)$ in $L^1_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$. By passing to a subsequence, if necessary, we can assume that $\sum_{j=0}^{m} \chi_{B_{r,j}}(y)$ converges to $\chi_{B_r}(y)$ for almost every y. Thus all the functions involved are uniformly bounded by $\tau_x \chi_{B_r}(y)$. This shows that $\sum_{j=0}^{m} \tau_x \chi_{B_{r,j}}(y)$ converges to $\tau_x \chi_{B_r}(y)$ in $L^{p'}_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$. Consequently, we have

$$\lim_{m\to\infty} \int_{\mathbb{R}^d} f(y) \sum_{j=0}^{m} \tau_x \chi_{B_{r,j}}(y) h_\kappa^2(y)\, dy = \int_{\mathbb{R}^d} f(y) \tau_x \chi_{B_r}(y) h_\kappa^2(y)\, dy.$$

Thus we obtain that

$$f_{r,\kappa} \le C \sum_{j=0}^{\infty} (2^{-j})^{\sigma-1} f *_\kappa P_{2^{-j}r}(x).$$

However, we have shown that for any $\lambda > 0$, there exists a projection

$$e \in L_\infty(\mathbb{R}^d) \mathbin{\overline{\otimes}} \mathcal{M}$$

such that, for any $r > 0$,

$$\|e f *_\kappa P_r e\|_\infty \lesssim \lambda \qquad \text{and} \qquad \tau \otimes \int e^\perp \lesssim \frac{\|f\|_{\kappa,1}}{\lambda}.$$

Therefore, we infer that

$$\sup_{r>0} \|e f_{r,\kappa} e\|_\infty \le C \sum_{j=0}^{\infty} (2^{-j})^{\sigma-1} \sup_{r>0} \|e f *_\kappa P_{2^{-j}r} e\|_\infty \lesssim \lambda$$

and

$$\tau \otimes \int e^\perp \lesssim \frac{\|f\|_{\kappa,1}}{\lambda}.$$

(ii) If instead of using Yeadon's inequality we use Junge and Xu's inequality (3.1), then, in the same spirit, we can deduce that for $1 < p \le \infty$, there exists an absolute constant $C_p > 0$ such that

$$\Big\| {\sup_{r>0}}^+ f_{r,\kappa} \Big\|_{\kappa,p} \le C_p \|f\|_{\kappa,p},$$

which finishes the proof of Theorem 1.1. □

Remark 3.3. Indeed, Theorem 1.1 also holds for general reflection groups on $\mathbb{R}^d$ in the noncommutative setting. But in this article, we only consider the operator-valued Bochner–Riesz means under the group Z_2^d. Therefore, we do not include the proof for the case of general reflection groups here.

The Hardy–Littlewood maximal function can be used to study the maximal estimate of $f *_\kappa \phi_\varepsilon$ under certain conditions on ϕ, where

$$A_\kappa(\mathbb{R}^d) = \left\{\widehat{\phi} \in L^1\big(\mathbb{R}^d; h_\kappa^2(x)\big) : \phi \in L^1\big(\mathbb{R}^d; h_k^2(x)\big)\right\}.$$

Theorem 3.4. *Let $\phi \in A_\kappa(\mathbb{R}^d)$ be a real-valued radial function which satisfies*

$$\big|\phi(x)\big| \leq c\big(1 + \|x\|\big)^{-2\lambda_\kappa - 1}.$$

Let $\phi_\varepsilon(s) = \frac{1}{\varepsilon^{d+2|\kappa|}}\phi(\frac{s}{\varepsilon})$ for $s \in \mathbb{R}^d$ and $\varepsilon > 0$. Then we have the following.

(i) *If $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$, then for any $\alpha > 0$, there exists a projection $e \in P(A)$ such that*

$$\sup_{\varepsilon>0}\big\|e(f *_\kappa \phi_\varepsilon)e\big\|_\infty \leq \alpha \qquad \textit{and} \qquad \tau \otimes \int e^\perp \leq C\frac{\|f\|_{\kappa,1}}{\alpha}.$$

(ii) *If $1 < p \leq \infty$, then*

$$\big\|{\sup_{\varepsilon>0}}^+ f *_\kappa \phi_\varepsilon\big\|_{\kappa,p} \leq C_p\|f\|_{\kappa,p}, \quad f \in L_p\big(\mathbb{R}^d, h_\kappa^2; L_p(\mathcal{M})\big),$$

where $A_\kappa(\mathbb{R}^d) = \{\widehat{\phi} \in L^1(\mathbb{R}^d; h_\kappa^2(x)) : \phi \in L^1(\mathbb{R}^d; h_\kappa^2(x))\}$.

Proof. First we prove part (i). Let $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$. We assume without loss of generality that f is positive. On the other hand, it is easy to reduce the problem to the case when ϕ is positive too. Indeed, by decomposing ϕ into its real and imaginary parts, we need only consider each part separately. Since $f \geq 0$, we have

$$f *_\kappa \operatorname{Re}(\phi_\varepsilon) \leq f *_\kappa \big|\operatorname{Re}(\phi_\varepsilon)\big| \leq f *_\kappa |\phi_\varepsilon|.$$

This gives the announced reduction. Thus in the sequel, we assume that $\phi \geq 0$. Given a function $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$ and a cube $Q \in \mathbb{R}^d$ centered at x and with sides parallel to the axes, we put

$$f_{Q,\kappa}(x) = \frac{1}{|Q|}\int_Q f(y)h_\kappa^2(y)\,dy.$$

Let $I_0 = [-1, 1]^d$, and let $I_j = \{t \in \mathbb{R}^d : 2^{j-1} \leq |t| \leq 2^j\}$ for $j = 1, 2, \ldots$. Also, let $\widetilde{I_j} = [-2^j, 2^j]$ and

$$\phi(y) = \sum_{j=0}^{\infty} \phi(y)\chi_{2^{j-1}\leq\|y\|\leq 2^j}(y).$$

Then we obtain

$$\sum_{j=0}^{m} \tau_x\big(\phi(y)\tau_x\chi_{2^{j-1}\leq\|y\|\leq 2^j}(y)\big) \leq c\sum_{j=0}^{m}(1 + 2^j)^{-2\lambda_\kappa - 1}\tau_x\chi_{2^{j-1}\leq\|y\|\leq 2^j}(y),$$

which implies that

$$\begin{aligned}
&\int_{\mathbb{R}^d} f(\varepsilon y) \sum_{j=0}^{m} \tau_x\big(\phi(y)\chi_{2^{j-1}\le\|y\|\le 2^j}(y)\big) h_\kappa^2(y)\, dy \\
&\le c\sum_{j=0}^{m}(1+2^j)^{-2\lambda_\kappa-1}\int_{\mathbb{R}^d} f(\varepsilon y)\tau_x\big(\chi_{2^{j-1}\le\|y\|\le 2^j}(y)\big)h_\kappa^2(y)\,dy \\
&\le c\sum_{j=0}^{\infty}(1+2^j)^{-2\lambda_\kappa-1}\int_{\mathbb{R}^d} f(\varepsilon y)\tau_x\big(\chi_{2^{j-1}\le\|y\|\le 2^j}(y)\big)h_\kappa^2(y)\,dy \\
&\le c\sum_{j=0}^{\infty}(1+2^j)^{-2\lambda_\kappa-1}(2^j)^{2\lambda_k}\frac{\int_{\mathbb{R}^d} f(\varepsilon y)\tau_x\chi_{\tilde{I}_j}(y)h_\kappa^2(y)\,dy}{\int_{\tilde{I}_j(x)} h_\kappa^2(y)\,dy} \\
&\le C\sum_{j=0}^{\infty}(2^{-j})f_{\varepsilon\tilde{I}_j,\kappa}(x).
\end{aligned}$$

Since $\phi(y) \le c(1+\|y\|)^{-2\lambda_k-1} \le cP_1(y)$, we infer that $\tau_x\phi(y) \le \tau_x P_1(y)$ is bounded. Arguing as in Theorem 3.4, we can show that the left-hand side of the above inequality converges to $f *_\kappa \phi_\varepsilon$. Note that Theorem 1.1 remains true with balls replaced by cubes. Thus we obtain that for any $\alpha > 0$, there exists a projection $e \in P(A)$ such that

$$\sup_{\varepsilon>0}\big\|e(f *_\kappa \phi_\varepsilon)e\big\|_\infty \le C\sum_{j=0}^{\infty}(2^{-j})\sup_{\varepsilon>0}\|ef_{\varepsilon\tilde{I}_j,\kappa}e\|_\infty \lesssim \alpha$$

and

$$\tau\otimes\int e^\perp \le C\frac{\|f\|_{\kappa,1}}{\alpha}.$$

Now we prove part (ii). It is clear that the map $f \to f *_\kappa \phi_\varepsilon$ is of type (∞,∞) with constant $\|\phi\|_{k,1}$. On the other hand, since we have assumed that $\phi \ge 0$, we obtain that $f *_\kappa \phi_\varepsilon \ge 0$, for $f \ge 0$. Thus by the interpolation theorem (see Lemma 3.2), we deduce the desired (p,p)-type maximal inequality, that is, part (ii). □

4. Proof of Theorem 1.2

Recall that the Bochner–Riesz means of order δ of $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$ are defined by

$$B_R^\delta(h_\kappa^2; f)(x) = c\int_{\mathbb{R}^d}\widehat{f}(\xi)E_\kappa(ix,\xi)\Phi^\delta(R^{-1}\xi)h_\kappa^2(\xi)\,d\xi, \quad R>0,$$

where $\Phi^\delta(x) := (1-\|x\|^2)_+^\delta$. It is known that $\widehat{\Phi^\delta}(x) = \phi^\delta(x)$, where

$$\phi^\delta(x) = 2^{\lambda_\kappa}\|x\|^{-\lambda_\kappa-\delta-\frac{1}{2}}J_{\lambda_\kappa+\delta+\frac{1}{2}}\big(\|x\|\big) =: \phi^{\delta,0}\big(\|x\|\big).$$

Here J_α denotes the Bessel function of the first kind. Since $\phi^\delta(x)$ is radial, we have $\widehat{\phi^\delta}(x) = \Phi^\delta(x)$.

Lemma 4.1. *Let* $\phi_R^\delta(x) := R^{2\lambda_k+1}\phi^\delta(Rx)$ *for* $R>0$. *Then* $\widehat{\phi_R^\delta}(\xi) = \widehat{\phi^\delta}(R^{-1}\xi) = \Phi^\delta(\frac{\xi}{R})$.

Proof. By the definition of $\phi_R^\delta(x)$, $t=Ry$, and the properties of $E_\kappa(x,y)$, we have

$$\begin{aligned}\widehat{\phi_R^\delta}(\xi) &= c_\kappa\int_{\mathbb{R}^d} E_\kappa(-i\xi,y)\phi_R^\delta(y)h_\kappa^2(y)\,dy\\ &= c_\kappa\int_{\mathbb{R}^d} E_\kappa(-i\xi,y)R^{2\lambda_\kappa+1}\phi^\delta(Ry)h_\kappa^2(y)\,dy\\ &= c_\kappa\int_{\mathbb{R}^d} E_\kappa\Big(-i\xi,\frac{t}{R}\Big)\phi^\delta(t)h_\kappa^2(t)\,dt\\ &= c_\kappa\int_{\mathbb{R}^d} E_\kappa\Big(-i\frac{\xi}{R},t\Big)\phi^\delta(t)h_\kappa^2(t)\,dt\\ &= \Phi^\delta\Big(\frac{\xi}{R}\Big).\end{aligned}$$

□

Remark 4.2. Using Lemma 4.1, the inverse formula, and (2.2), we obtain that

$$\begin{aligned}B_R^\delta(h_\kappa^2;f)(x) &= c_\kappa\int_{\mathbb{R}^d}\widehat{f}(\xi)E_\kappa(ix,\xi)\Phi^\delta(R^{-1}\xi)h_\kappa^2(\xi)\,d\xi\\ &= c_\kappa\int_{\mathbb{R}^d}\widehat{f}(\xi)E_\kappa(ix,\xi)\widehat{\phi_R^\delta}(\xi)h_\kappa^2(\xi)\,d\xi\\ &= c_\kappa\int_{\mathbb{R}^d}(\widehat{f*_\kappa\phi_R^\delta})(\xi)E_\kappa(ix,\xi)h_\kappa^2(\xi)\,d\xi\\ &= f*_\kappa\phi_R^\delta(x).\end{aligned}$$

The following lemma is cited from [24, Lemma 4.3].

Lemma 4.3.

(i) *For each* $\alpha\in R$, $z^{-\alpha}J_\alpha(z)$ *is an even entire function of* $z\in\mathbb{C}$ *and*

$$\frac{d}{dz}\big[z^{-\alpha}J_\alpha(z)\big] = -z^{-\alpha}J_{\alpha+1}(z).$$

(ii) *For each* $\alpha\in R$,

$$\big|x^{-\alpha}J_\alpha(x)\big| \le C\big(1+|x|\big)^{-\alpha-\frac{1}{2}},$$

where J_α *denotes the Bessel function of the first kind.*

If $f\in\mathscr{S}(\mathbb{R}^d;L_1(\mathcal{M}))$, then we can define the spherical mean operator on $\mathscr{S}(\mathbb{R}^d;L_1(\mathcal{M}))$ by

$$S_rf(x) = \frac{1}{\omega_{d-1}}\int_{S^{d-1}}\tau_{ry}f(x)h_\kappa^2(y)\,dw(y),$$

where S denotes the Schwartz class, $d\omega$ is the usual measure on S^{d-1}, and ω_{d-1} is its total mass.

The generalized convolution of f with a radial function can be expressed in terms of the spherical means S_rf. In fact, if $f\in\mathscr{S}(\mathbb{R}^d;L_1(\mathcal{M}))$ and $g(x)=$

$g_0(\|x\|)$ is an integrable radial function, then, using the spherical-polar coordinates and Proposition 2.2, we have

$$\begin{aligned}(f *_\kappa g)(x) &= \int_{\mathbb{R}^d} f(y)\tau_x g^\vee(y) h_\kappa^2(y)\,dy \\ &= \int_{\mathbb{R}^d} \tau_y f(x) g(y) h_\kappa^2(y)\,dy \\ &= \int_0^\infty r^{2\lambda_\kappa} g_0(r) \int_{S^{d-1}} \tau_{ry'} f(x) h_\kappa^2(y')\,dy'\,dr \\ &= \omega_{d-1} \int_0^\infty S_r(f) g_0(r) r^{2\lambda_\kappa}\,dr.\end{aligned}$$

We also need the following lemma.

Lemma 4.4. *Let $G = Z_2^d$, and let $\phi(x) = \phi_0(\|x\|) \in L^1(\mathbb{R}^d, h_\kappa^2)$ be a radial function. Assume that ϕ_0 is differentiable, $\lim_{r\to\infty}\phi_0(r) = 0$, $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$, and $\int_0^\infty r^{2\lambda_\kappa+1}|\phi_0'(r)|\,dr < \infty$. Let $\phi_\varepsilon(s) = \frac{1}{\varepsilon^{d+2|\kappa|}}\phi(\frac{s}{\varepsilon})$ for $s \in \mathbb{R}^d$ and $\varepsilon > 0$. Then we have the following.*

(i) *Let $f \in L_1(\mathbb{R}^d; h_\kappa^2; L_1(\mathcal{M}))$. Then for any $\alpha > 0$, there exists a projection $e \in P(A)$ such that*

$$\sup_{\varepsilon>0}\big\|e(f *_\kappa \phi_\varepsilon)e\big\|_\infty \le \alpha \qquad \textit{and} \qquad \tau \otimes \int e^\perp \le C\frac{\|f\|_{\kappa,1}}{\alpha}.$$

(ii) *Let $1 < p \le \infty$. Then*

$$\big\|{\sup_{\varepsilon>0}}^+ f *_\kappa \phi_\varepsilon\big\|_{\kappa,p} \le C_p\|f\|_{\kappa,p}, \quad f \in L_p\big(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M})\big).$$

Proof. (i) Let $f \in \mathscr{S}(\mathbb{R}^d; L_1(\mathcal{M}))$. By an argument similar to that in Theorem 3.4, we can assume that f and ϕ are positive. By definition of the spherical means $S_t f$, we can write

$$f_{r,\kappa} = \frac{\int_0^r t^{2\lambda_\kappa} S_t f(x)\,dt}{\int_0^r t^{2\lambda_\kappa}\,dt}.$$

The assumption on ϕ_0 implies that

$$\begin{aligned}\lim_{r\to\infty}\phi_0(r)\int_0^r S_t f(x) t^{2\lambda_\kappa}\,dt &= \lim_{r\to\infty}\phi_0(r)\int_{\mathbb{R}^d}\tau_y f(x) h_\kappa^2(y)\,dy \\ &= \lim_{r\to\infty}\phi_0(r)\int_{\mathbb{R}^d} f(x) h_\kappa^2(y)\,dy \\ &= 0.\end{aligned}$$

Therefore, using the spherical-polar coordinates and integrating by parts, we get

$$\begin{aligned}(f *_\kappa \phi_\varepsilon)(x) &= \frac{1}{\varepsilon^{d+2|k|}}\int_{\mathbb{R}^d}\tau_y f(x)\phi\Big(\frac{y}{\varepsilon}\Big)h_\kappa^2(y)\,dy \\ &= \omega_{d-1}\frac{1}{\varepsilon^{d+2|\kappa|}}\int_0^\infty S_r(f)(x)\phi_0\Big(\frac{r}{\varepsilon}\Big)r^{2\lambda_\kappa}\,dr\end{aligned}$$

$$= \omega_{d-1} \frac{1}{\varepsilon^{d+2|\kappa|}} \int_0^\infty \phi_0\Big(\frac{r}{\varepsilon}\Big) d \int_0^r t^{2\lambda_\kappa} S_t(f)(x)\, dt$$

$$= -\omega_{d-1} \frac{1}{\varepsilon^{d+2|\kappa|}} \int_0^\infty \int_0^r t^{2\lambda_k} S_t(f)(x)\, dt \frac{1}{\varepsilon} \phi_0'\Big(\frac{r}{\varepsilon}\Big)\, dr$$

$$= -\frac{\omega_{d-1}}{2\lambda_\kappa + 1} \int_0^\infty f_{\varepsilon R,\kappa} R^{2\lambda_\kappa+1} \phi_0'(R)\, dR.$$

Hence, using Theorem 1.1, for any $\alpha > 0$, there exists a projection $e \in P(A)$ such that

$$\sup_{\varepsilon>0} \|e f_{\varepsilon,\kappa} e\|_\infty \le \alpha \qquad \text{and} \qquad \tau \otimes \int e^\perp \le C \frac{\|f\|_{\kappa,1}}{\alpha},$$

which implies that

$$\sup_{\varepsilon>0} \big\| e(f *_\kappa \phi_\varepsilon) e \big\|_\infty \le \sup_{\varepsilon>0} \big\| e(f_{\varepsilon R,\kappa}) e \big\|_\infty \frac{\omega_{d-1}}{2\lambda_\kappa + 1} \int_0^\infty R^{2\lambda_\kappa+1} \phi_0'(R)\, dR \lesssim \alpha$$

and

$$\tau \otimes \int e^\perp \le C \frac{\|f\|_{\kappa,1}}{\alpha}.$$

(ii) The proof of part (ii) is by the interpolation theorem (see Lemma 3.2), where we deduce the desired (p,p)-type maximal inequality. We thus complete the proof of the lemma. □

Proof of Theorem 1.2. Note that $\phi(x)$ satisfies the conditions of Lemma 4.4. Then we obtain the weak type $(1,1)$ and (p,p) type of the Bochner–Riesz means. Indeed, since

$$\phi^\delta(x) = 2^{\lambda_\kappa} \|x\|^{-\lambda_\kappa - \delta - \frac{1}{2}} J_{\lambda_\kappa + \delta + \frac{1}{2}}\big(\|x\|\big) =: \phi^{\delta,0}\big(\|x\|\big), \quad \text{for } \delta > \lambda_\kappa,$$

it is easy to see that $\phi^\delta(x)$ is radial and $\lim_{r\to\infty} \phi^{\delta,0}(r) = 0$. Due to Lemma 4.3, we infer that

$$\phi^{(\delta,0)'}(r) = -2^{\lambda_\kappa} r^{-\lambda_\kappa - \delta - \frac{1}{2}} J_{\lambda_\kappa + \delta + \frac{3}{2}}(r).$$

Note that $J_\alpha(r) = O(r^{-\frac{1}{2}})$. Then we obtain that

$$r^{2\lambda_\kappa+1} r^{-\lambda_\kappa - \delta - \frac{1}{2}} J_{\lambda_\kappa + \delta + \frac{3}{2}}(r) \le C \frac{1}{r^{\delta + 1 - \lambda_\kappa}}.$$

Therefore, the proof of Theorem 1.2 is complete. □

5. Proof of Theorem 1.3

Recall the noncommutative version of the almost-everywhere convergence. Let $\mathcal{M}$ be a von Neumann algebra equipped with a semifinite normal faithful trace τ. Let $x_n, x \in L_0(\mathcal{M})$. Then

(i) (x_n) is said to converge *bilaterally almost uniformly* (b.a.u. for short) to x if for every $\varepsilon > 0$, there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) < \varepsilon \qquad \text{and} \qquad \lim_{n\to\infty} \big\| e(x_n - x) e \big\|_\infty = 0;$$

(ii) (x_n) is said to converge *almost uniformly* (a.u. for short) to x if for every $\varepsilon > 0$, there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^{\perp}) < \varepsilon \qquad \text{and} \qquad \lim_{n\to\infty} \big\|(x_n - x)e\big\|_{\infty} = 0.$$

Obviously, $x_n \xrightarrow{\text{a.u.}} x$ implies $x_n \xrightarrow{\text{b.a.u.}} x$. In the commutative case, both convergences in the definition above are equivalent to the usual almost-everywhere convergence by virtue of Egorov's theorem. However they are different in the non-commutative case. Similarly, we can introduce these notions of convergence for functions with values in $L_0(\mathcal{M})$ and for nets in $L_0(\mathcal{M})$.

Recall that the map $x \to x^p$ $(1 \leq p \leq 2)$ is convex on the positive cone $\mathcal{M}_+$ of $\mathcal{M}$. Thus, for $f \in L_p(L_\infty(\mathbb{R}^d \otimes \mathcal{M}))$ $(1 \leq p \leq 2)$, we get

$$\int_A |f|\, dt \leq \Big(\int_A |f|^p\, dt\Big)^{\frac{1}{p}}, \quad \forall A \subseteq \mathbb{R}^d, |A| = 1.$$

By iterating this inequality, we obtain the following lemma.

Lemma 5.1 ([17, Lemma 3.5])**.** *If $f \in L_p(L_\infty(\mathbb{R}^d \otimes \mathcal{M}))$, $1 \leq p < \infty$, then*

$$\int_A |f|\, dt \leq \Big(\int_A |f|^p\, dt\Big)^{\frac{1}{p}}, \quad \forall A \subseteq \mathbb{R}^d, |A| = 1. \tag{5.1}$$

Recall also that for any bounded linear operators a, b on a Hilbert space H, where a is positive and $\|b\| \leq 1$, if T is an operator monotone function defined for positive operators (e.g., $T(a) = a^{\frac{1}{p}}, p \geq 1$), then

$$b^* T(a) b \leq T(b^* a b).$$

This is the so-called *Hansen's inequality* (see [5, Theorem 2.1]). In particular, we have

$$b^* a b \leq (b^* a^p b)^{\frac{1}{p}}. \tag{5.2}$$

We also need the following lemma whose proof can be done by imitating the classical proof in [22].

Lemma 5.2 ([22, Theorem 1.2.19])**.** *If $f \in L_p(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M}))$, $1 \leq p < \infty$, moreover, for $f \in C_0(\mathbb{R}^d) \overline{\otimes} \mathcal{M}$, $p = \infty$, and $\delta > \lambda_\kappa$, then*

$$\big\|B_R^\delta(h_\kappa^2; f) - f\big\|_{\kappa,p} \to 0, \quad \text{as } R \to \infty.$$

Proof of Theorem 1.3. (i) Without loss of generality, we can assume that f is self-adjoint. For any given $f \in L_p(\mathbb{R}^d; h_\kappa^2; L_p(\mathcal{M}))$ and $\varepsilon > 0$, we choose $g_n = \sum_{k=1}^n \varphi_k x_k$, where $x_k \in S_{\mathcal{M}}^+$ and $\varphi_k : \mathbb{R}^d \to C$ are continuous functions with compact support, such that

$$\big\||f - g_n|^p\big\|_{\kappa,1} = \|f - g_n\|_{\kappa,p}^p < \Big(\frac{1}{2^n}\Big)^p \frac{\varepsilon}{2^n}. \tag{5.3}$$

Choose $e_{1,n}^\varepsilon \in P(L_\infty(\mathbb{R}^d) \otimes \mathcal{M})$ such that

$$\big\|e_{1,n}^\varepsilon |f - g_n|^p e_{1,n}^\varepsilon\big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} < \Big(\frac{1}{2^n}\Big)^p \qquad \text{and} \qquad \tau \otimes \int (e_{1,n}^\varepsilon)^\perp < \frac{\varepsilon}{2^n}.$$

Taking $e_1^\varepsilon = \bigwedge_n e_{1,n}^\varepsilon$, we have

$$\tau \otimes \int (e_1^\varepsilon)^\perp < \varepsilon.$$

And by using (5.2),

$$\begin{aligned} \left\| e_1^\varepsilon (g_n - f) e_1^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} &\le \left\| e_1^\varepsilon |g_n - f| e_1^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\ &\le \left\| e_1^\varepsilon |g_n - f|^p e_1^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}}^{\frac{1}{p}} \\ &< \frac{1}{2^n}, \quad \forall n \ge 1. \end{aligned}$$

On the other hand, using (5.3) and Theorem 1.2, we can find a sequence $(e_{2,n}^\varepsilon)_n \in P(L_\infty(\mathbb{R}^d) \otimes \mathcal{M})$ such that

$$\left\| e_{2,n}^\varepsilon B_R^\delta\big(h_k^2; |g_n - f|^p\big) e_{2,n}^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} < \Big(\frac{1}{2^n}\Big)^p. \tag{5.4}$$

Let $e_2^\varepsilon = \bigwedge_n e_{2,n}^\varepsilon$. Then we have

$$\tau \otimes \int (e_2^\varepsilon)^\perp < \varepsilon.$$

By (5.1), (5.2), and (5.4), we obtain that

$$\begin{aligned} &\left\| e_2^\varepsilon B_R^\delta\big(h_\kappa^2; (g_n - f)\big) e_2^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\ &\quad \le \left\| e_{2,n}^\varepsilon B_R^\delta\big(h_\kappa^2; |g_n - f|\big) e_{2,n}^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\ &\quad \le \left\| e_{2,n}^\varepsilon \Big(B_R^\delta\big(h_\kappa^2; |g_n - f|^p\big)^{\frac{1}{p}}\Big) e_{2,n}^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\ &\quad \le \left\| e_{2,n}^\varepsilon B_R^\delta\big(h_\kappa^2; |g_n - f|^p\big) e_{2,n}^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}}^{\frac{1}{p}} \\ &\quad < \frac{1}{2^n}, \quad \forall n \ge 0. \end{aligned}$$

Using Lemma 5.2, we have

$$\lim_{R\to\infty} \left\| B_R^\delta(h_\kappa^2; g_n) - g_n \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} = 0, \quad \forall n \ge 1.$$

Set $e^\varepsilon = e_1^\varepsilon \wedge e_2^\varepsilon$. Then we obtain that

$$\tau \otimes \int (e^\varepsilon)^\perp < 2\varepsilon.$$

Hence, we deduce that

$$\begin{aligned} &\left\| e^\varepsilon \big(B_R^\delta(h_\kappa^2; f) - f\big) e^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\ &\quad \le \left\| e^\varepsilon (g_n - f) e^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} + \left\| e^\varepsilon \big(B_R^\delta(h_\kappa^2; g - g_n)\big) e^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\ &\qquad + \left\| e^\varepsilon \big(B_R^\delta(h_\kappa^2; g_n) - B_R^\delta(h_\kappa^2; f)\big) e^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\ &\quad \le \left\| e_1^\varepsilon (g_n - f) e_1^\varepsilon \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} + \left\| B_R^\delta(h_\kappa^2; g_n) - g_n \right\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \end{aligned}$$

$$
\begin{aligned}
&+ \big\| e_2^\varepsilon \big(B_R^\delta(h_\kappa^2; g_n) - B_R^\delta(h_\kappa^2; f) \big) e_2^\varepsilon \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\
&\leq \frac{3}{2^n}.
\end{aligned}
$$

Thus we have

$$
\lim_{R\to\infty} \big\| e^\varepsilon \big(B_R^\delta(h_\kappa^2; f) - f \big) e^\varepsilon \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} = 0.
$$

This completes the proof of (i).

(ii) The proof of (i) works well for part (ii) of the theorem with some minor changes. Let g_n and $e_1^\varepsilon, e_2^\varepsilon, e^\varepsilon$ be as above. Since $p > 2$, by (5.1) and (5.2), we obtain that

$$
\begin{aligned}
\big\| e_1^\varepsilon (g_n - f) \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} &= \big\| e_1^\varepsilon |g_n - f|^2 e_1^\varepsilon \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}}^{\frac{1}{2}} \\
&\leq \big\| e_1^\varepsilon |g_n - f|^p e_1^\varepsilon \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}}^{\frac{1}{p}} \\
&< \frac{1}{2^n}, \quad \forall n \geq 1,
\end{aligned}
$$

and also

$$
\begin{aligned}
&\big\| e_2^\varepsilon \big(B_R^\delta(h_\kappa^2, g_n) - B_R^\delta(h_\kappa^2, f) \big) \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \\
&\quad = \big\| e_2^\varepsilon \big| B_R^\delta(h_\kappa^2, g_n) - B_R^\delta(h_\kappa^2, f) \big|^2 e_2^\varepsilon \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}}^{\frac{1}{2}} \\
&\quad \leq \big(\big\| e_2^\varepsilon B_R^\delta \big(h_\kappa^2, |g_n - f|^2\big) e_2^\varepsilon \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \big)^{\frac{1}{2}} \\
&\quad \leq \big(\big\| e_2^\varepsilon B_R^\delta \big(h_\kappa^2, |g_n - f|^p\big) e_2^\varepsilon \big\|_{L_\infty(\mathbb{R}^d)\otimes\mathcal{M}} \big)^{\frac{1}{p}} \\
&\quad < \frac{1}{2^n}, \quad \forall n \geq 1.
\end{aligned}
$$

Hence, as in the proof of (i), we can complete the proof. □

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