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PHILLIPS SYMMETRIC OPERATORS AND THEIR EXTENSIONS

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Abstract. This article is devoted to the investigation of self-adjoint (and, more generally, proper) extensions of Phillips symmetric operators (PSO). A closed densely defined symmetric operator with equal defect numbers is considered a Phillips symmetric operator if its characteristic function is a constant on $\mathbb{C}_+$. We present equivalent definitions of PSO and prove that proper extensions with real spectra of a given PSO are similar to each other. Our results imply that one-point interaction of the momentum operator $i\frac{d}{dx} + \alpha\delta(x-y)$ leads to unitarily equivalent self-adjoint operators with Lebesgue spectra. Self-adjoint operators with nontrivial spectral properties can be obtained as a result of more complicated perturbations of the momentum operator. In this way, we study special classes of perturbations which can be characterized as one-point interactions defined by the nonlocal potential $\gamma \in L_2(\mathbb{R})$.

1. Introduction

Let S be a symmetric operator with equal defect numbers, and let $\mathfrak{U}$ be a family of unitary operators in a Hilbert space $\mathfrak{H}$ such that the inclusion $U \in \mathfrak{U}$ implies $U^* \in \mathfrak{U}$. The operator S is called $\mathfrak{U}$-*invariant* if S commutes with all $U \in \mathfrak{U}$. Does there exist at least one $\mathfrak{U}$-invariant self-adjoint extension of S? The answer is definitely affirmative if S is assumed to be semibounded and the Friedrichs extension of S gives the required example.

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In the general case of nonsemibounded operators, Phillips [23, p. 382] constructed a symmetric operator S and a family $\mathfrak{U}$ of unitary operators commuting with S such that the $\mathfrak{U}$-invariant S has no $\mathfrak{U}$-invariant self-adjoint extensions. It was discovered (see [16]) that the characteristic function of the symmetric operator constructed in the Phillips work is a constant in the upper half-plane $\mathbb{C}_+$. This fact can be used for the general definition of the PSO. Namely, we say that a closed densely defined symmetric operator S with equal defect numbers is a *Phillips symmetric operator* (PSO) if its characteristic function is an operator-constant on $\mathbb{C}_+$.

The concept of characteristic function of a symmetric operator was first introduced by Shtraus [25] and later substantially developed by Kochubei [17] on the basis of the boundary triplet technique (see [12]). Section 2 contains all the necessary results about characteristic functions which are used in this article.

The present article is devoted to the investigation of PSOs as well as their self-adjoint and, more generally, proper extensions (see Section 2 for the relevant definition). Such self-adjoint extensions differ from those that are commonly studied in the literature (see [2]) and they have a lot of curious properties.

Our original definition of PSO deals with the concept of characteristic function. In many cases, an explicit calculation of a characteristic function is technically complicated. For this reason, in Section 3, we establish equivalent descriptions of PSO (see Theorems 3.1, 3.4, 3.5) which can be employed as independent definitions of PSO. These results lead to the conclusion that each simple[1] PSO coincides with the orthogonal sum of simple maximal symmetric operators having the same nonzero defect numbers in the upper $\mathbb{C}_+$ and lower $\mathbb{C}_-$ half-planes. This kind of decomposition means that every simple PSO S is unitarily equivalent to the momentum operator with one-point interaction

$$\mathrm{S} = i\frac{d}{dx}, \qquad \mathcal{D}(\mathrm{S}) = \left\{u \in W_2^1(\mathbb{R}, N) : u(0) = 0\right\}$$

acting in the Hilbert space $L_2(\mathbb{R}, N)$, where the dimension of the auxiliary Hilbert space N coincides with defect numbers of S.

Section 4 is devoted to proper extensions of PSO. The main result (Theorem 4.2) states that all proper extensions of a PSO S with real spectra are similar to each other. In fact, we can say more: each proper extension with real spectrum can be interpreted as a self-adjoint extension of S for a special choice of inner product equivalent to the initial one. Some properties of PSOs with defect numbers $\langle 1, 1\rangle$ were established in [5]. In particular, analogues of Theorems 3.4, 4.2, and Corollary 4.3 were proved.

In Section 5, PSOs are determined as the restrictions of a given self-adjoint operator A. According to Theorem 5.2, those PSO which can be obtained in this way are in one-to-one correspondence with the wandering subspaces $\mathcal{L}$ of the Cayley transform U of A. This means that the set of restrictions of A contains a PSO only in the case where A has a reducing subspace $\mathfrak{H}_0$ such that $A_0 = A\restriction_{\mathcal{D}(A)\cap\mathfrak{H}_0}$ is a self-adjoint operator in $\mathfrak{H}_0$ with Lebesgue spectrum. The existence

[1]See the definition of a simple symmetric operator in Section 2.

of a simple PSO is equivalent to the fact that A has a Lebesgue spectrum (see Corollary 5.4).

In Section 6, we consider examples of PSOs. We establish a useful (in our opinion) characterization of wavelets as functions from the defect subspace $\mathfrak{N}_{-i}$ of a simple PSO (see Proposition 6.1).

The results of Sections 4–6 show that a one-point interaction of the momentum operator $i\frac{d}{dx}+\alpha\delta(x-y)$ leads to self-adjoint operators with Lebesgue spectra which are unitarily equivalent to each other. This means that one should consider more complicated perturbations of the momentum operator for the construction of self-adjoint operators with nontrivial spectral properties. In this way, self-adjoint momentum operators acting in two intervals were studied in [15] and [22]. The momentum operators defined on oriented metric graphs were investigated in [10]. General nonlocal point interactions for first-order differential operators were introduced and studied in [4] and [21].

In Section 7, we continue investigations of [4] by focusing on special classes of perturbations which can be characterized as one-point interactions defined by the nonlocal potential $\gamma \in L_2(\mathbb{R})$.

2. Characteristic functions of symmetric operators

Let $\mathfrak{H}$ be a separable Hilbert space with inner product $(\cdot,\cdot)$ linear in the first argument, and let A be a linear operator acting in $\mathfrak{H}$. The domain of A is denoted $\mathcal{D}(A)$, while $A\restriction_{\mathcal{D}}$ stands for the restriction of A onto a set $\mathcal{D}$. An operator A is called *dissipative* if $\operatorname{Im}(Af,f) \geq 0$ for all $f \in \mathcal{D}(A)$ and *maximal dissipative* if there are no dissipative extensions of A.

Let S be a closed densely defined symmetric operator in $\mathfrak{H}$, and let S^* be the adjoint of S. We denote by

$$\mathfrak{N}_\lambda = \ker(S^* - \lambda I), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}$$

the defect subspaces of S, and we define the defect numbers $n_\pm(S)$ as

$$n_\pm(S) = \dim \mathfrak{N}_\lambda, \quad \lambda \in \mathbb{C}_\pm,$$

where $\mathbb{C}_+$ ($\mathbb{C}_-$) are the open upper (resp., lower) half-planes. In what follows, we assume that the defect numbers of S coincide, that is, $n_+(S) = n_-(S)$.

An extension A of a symmetric operator S is called *proper* if $S \subset A \subset S^*$. Self-adjoint extensions of S is a subset of proper extensions. According to the von Neumann formulas (see, e.g., [18], [24]) each proper extension A of S is uniquely determined by the choice of a subspace $M \subset \mathfrak{M}_\lambda$:

$$A = S^*\restriction_{\mathcal{D}(A)}, \quad \mathcal{D}(A) = \mathcal{D}(S) \dotplus M, \tag{2.1}$$

where

$$\mathfrak{M}_\lambda = \mathfrak{N}_\lambda \dotplus \mathfrak{N}_{\overline{\lambda}}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Let us set $M = \mathfrak{N}_\lambda$ in (2.1) and denote by

$$A_\lambda = S^*\restriction_{\mathcal{D}(A_\lambda)}, \quad \mathcal{D}(A_\lambda) = \mathcal{D}(S) \dotplus \mathfrak{N}_\lambda, \lambda \in \mathbb{C}\setminus\mathbb{R} \tag{2.2}$$

the corresponding proper extensions of S. The operators $\mathsf{sign}(\mathrm{Im}\,\lambda)A_\lambda$ are maximal dissipative, and $A_\lambda^* = A_{\overline{\lambda}}$. The resolvent set of every maximal dissipative operator contains $\mathbb{C}_-$. For this reason, the operator-function

$$\mathsf{Sh}(\lambda) = (A_\lambda - iI)(A_\lambda + iI)^{-1} \restriction_{\mathfrak{N}_i} : \mathfrak{N}_i \to \mathfrak{N}_{-i}, \quad \lambda \in \mathbb{C}_+ \tag{2.3}$$

is well defined; it is also a particular case of the characteristic function of a symmetric operator S defined by Shtraus [25] (in the Shtraus paper, (2.3) involves an arbitrary $\lambda_0 \in \mathbb{C}_+$ instead of i). The characteristic function $\mathsf{Sh}(\cdot)$ is a holomorphic operator-valued function whose values are strict contractions defined on $\mathfrak{N}_i$ and mapping into $\mathfrak{N}_{-i}$ (see [25]). Another (equivalent) definition of $\mathsf{Sh}(\cdot)$ in [25] is based on the relation

$$\mathcal{D}(A_\lambda) = \mathcal{D}(S) \dot{+} \mathfrak{N}_\lambda = \mathcal{D}(S) \dot{+} \big(I - \mathsf{Sh}(\lambda)\big)\mathfrak{N}_i, \quad \lambda \in \mathbb{C}_+, \tag{2.4}$$

which allows one to determine uniquely $\mathsf{Sh}(\cdot)$.

The explicit construction of $\mathsf{Sh}(\cdot)$ deals with the calculation of $\mathfrak{N}_\lambda$ that can at times be technically complicated. This inconvenience was overcome in [17] with the use of the boundary triplet technique. We recall (see [9], [18]) that a triplet $(\mathcal{H}, \Gamma_-, \Gamma_+)$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_\pm$ are linear mappings of $\mathcal{D}(S^*)$ into $\mathcal{H}$, is called a *boundary triplet* of S^* if

$$(S^*f, g) - (f, S^*g) = i\big[(\Gamma_+ f, \Gamma_+ g)_{\mathcal{H}} - (\Gamma_- f, \Gamma_- g)_{\mathcal{H}}\big], \quad f, g \in \mathcal{D}(S^*) \tag{2.5}$$

holds and the map $(\Gamma_-, \Gamma_+) : \mathcal{D}(S^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective. Let a boundary triplet $(\mathcal{H}, \Gamma_-, \Gamma_+)$ be given. Then the domains of definition of operators A_λ in (2.2) admit the presentation

$$\mathcal{D}(A_\lambda) = \left\{ f \in \mathcal{D}(S^*) : \begin{array}{c} \Theta(\lambda)\Gamma_+ f = \Gamma_- f, \, \lambda \in \mathbb{C}_+ \\ \Gamma_+ f = \Theta(\lambda)\Gamma_- f, \, \lambda \in \mathbb{C}_- \end{array} \right\}, \tag{2.6}$$

where $\Theta(\cdot)$ is an operator in $\mathcal{H}$.

The operator-valued function $\Theta(\cdot)$ defined on $\mathbb{C} \setminus \mathbb{R}$ is called the *characteristic function* of S associated with the boundary triplet $(\mathcal{H}, \Gamma_-, \Gamma_+)$. It follows from the relation $A_\lambda^* = A_{\overline{\lambda}}$ and (2.5) that $\Theta^*(\lambda) = \Theta(\overline{\lambda})$. The explicit form of characteristic function depends on the choice of a boundary triplet. However, in every case, $\Theta(\cdot)$ is a holomorphic operator-valued function and $\|\Theta(\lambda)\| < 1$ (see [17]). We recall that a symmetric operator S is called *simple* if there does not exist a subspace of $\mathfrak{H}$ invariant under S such that the restriction of S to this subspace is self-adjoint (see [1]). The characteristic function determines a simple symmetric operator up to unitary equivalence.

Theorem 2.1 ([17, Theorem 2])**.** *Simple symmetric operators S_1 and S_2 are unitarily equivalent if and only if some of their characteristic functions coincide.*

The *Shtraus characteristic function* $\mathsf{Sh}(\cdot)$ defined in (2.3) coincides (up to the multiplication by unitary operator) with $\Theta(\cdot)$ for special choice of boundary triplet. Precisely, the simplest (inspired by the von Neumann formulas) boundary triplets $(\mathfrak{N}_\mu, \Gamma_-, \Gamma_+)$ of S^* can be constructed as

$$\Gamma_- f = \sqrt{2\mathrm{Im}\,\mu} V f_{\overline{\mu}}, \qquad \Gamma_+ f = \sqrt{2\mathrm{Im}\,\mu} f_\mu, f = u + f_\mu + f_{\overline{\mu}} \in \mathcal{D}(S^*), \tag{2.7}$$

where $\mu \in \mathbb{C}_+$ and where $V : \mathfrak{N}_{\overline{\mu}} \to \mathfrak{N}_\mu$ is a unitary mapping. Assume that $\mu = i$. Then the characteristic function $\Theta(\cdot)$ associated with the boundary triplet $(\mathfrak{N}_i, \Gamma_-, \Gamma_+)$ coincides with the function $-V\mathsf{Sh}(\cdot)$ on $\mathbb{C}_+$.

Remark 2.2. There are various approaches to the definition of boundary triplets. For instance, in [12] and [24], a triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where Γ_0, Γ_1 are linear mappings of $\mathcal{D}(S^*)$ into $\mathcal{H}$, is called a *boundary triplet* of S^* if the Green identity

$$(S^*f, g) - (f, S^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathcal{D}(S^*)$$

holds and the map $(\Gamma_0, \Gamma_1) : \mathcal{D}(S^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

The operators $\Gamma_\pm$ in (2.5) and Γ_i are related as follows: $\Gamma_\pm = \frac{1}{\sqrt{2}}(\Gamma_1 \pm i\Gamma_0)$ and, obviously, the definitions of boundary triplets $(\mathcal{H}, \Gamma_-, \Gamma_+)$ and $(\mathcal{H}, \Gamma_0, \Gamma_1)$ are equivalent.

The characteristic function $\Theta(\cdot)$ admits a natural interpretation in the Krein space setting (see [3], [6] for the basic theory of Krein spaces and terminology). To explain this point, we fix a boundary triplet $(\mathcal{H}, \Gamma_-, \Gamma_+)$ and we rewrite (2.5) as

$$(S^*f, g) - (f, S^*g) = i[\Psi f, \Psi g], \tag{2.8}$$

where

$$\Psi = \begin{bmatrix} \Gamma_+ \\ \Gamma_- \end{bmatrix} : \mathcal{D}(S^*) \to \mathsf{H} = \begin{bmatrix} \mathcal{H} \\ \mathcal{H} \end{bmatrix} \tag{2.9}$$

maps $\mathcal{D}(S^*)$ into the Krein space $(\mathsf{H}, [\cdot, \cdot])$ with the indefinite inner product

$$[\mathsf{x}, \mathsf{y}] = (x_0, y_0) - (x_1, y_1), \quad \mathsf{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \mathsf{y} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in \mathsf{H}. \tag{2.10}$$

It follows from the definition of boundary triplets that the mapping $\Psi : \mathcal{D}(S^*) \to \mathsf{H}$ is surjective and $\ker \Psi = \mathcal{D}(S)$. Because of (2.9) and (2.10), the decomposition

$$\mathsf{H} = \mathsf{H}_+ \oplus \mathsf{H}_-, \quad \mathsf{H}_+ = \Psi \ker \Gamma_- = \begin{bmatrix} \mathcal{H} \\ 0 \end{bmatrix}, \mathsf{H}_- = \Psi \ker \Gamma_+ = \begin{bmatrix} 0 \\ \mathcal{H} \end{bmatrix} \tag{2.11}$$

is a fundamental decomposition of the Krein space $(\mathsf{H}, [\cdot, \cdot])$. Here, $\mathsf{H}_\pm$ are maximal uniformly positive/negative subspaces with respect to the indefinite inner product $[\cdot, \cdot]$.

By virtue of (2.1), each proper extension A of S is completely determined by a subspace $\mathsf{L} = \Psi\mathcal{D}(A) = \Psi(\mathcal{D}(S) \dotplus M) = \Psi M$ of H. In other words, there is a one-to-one correspondence between subspaces of H and proper extensions of S. In particular, proper extensions A_λ in (2.2) are determined by the subspaces $\mathsf{L}_\lambda = \Psi\mathcal{D}(A_\lambda)$, which are maximal uniformly positive ($\lambda \in \mathbb{C}_+$) and maximal uniformly negative ($\lambda \in \mathbb{C}_-$) in $(\mathsf{H}, [\cdot, \cdot])$ (see [13]).

Taking (2.6) and (2.9) into account, we arrive at the conclusion that the maximal uniformly positive subspace L_λ is decomposed with respect to the fundamental decomposition (2.11) as

$$\mathsf{L}_\lambda = \Psi\mathcal{D}(A_\lambda) = \left\{ \begin{bmatrix} \Gamma_+ f \\ \Theta\Gamma_+ f \end{bmatrix} : f \in \mathcal{D}(A_\lambda) \right\} = \{\mathsf{h}_+ + \widetilde{\Theta}(\lambda)\mathsf{h}_+ : \mathsf{h}_+ \in \mathsf{H}_+\},$$

where $\widetilde{\Theta}(\cdot) : \mathsf{H}_+ \to \mathsf{H}_-$ acts as follows:

$$\widetilde{\Theta}(\lambda)\mathsf{h}_+ = \widetilde{\Theta}(\lambda)\begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \Theta(\lambda)h \end{bmatrix}, \quad \lambda \in \mathbb{C}_+. \tag{2.12}$$

This means that $\widetilde{\Theta}(\lambda)$ is the angular operator of the maximal uniformly positive subspace L_λ with respect to the maximal uniformly positive subspace H_+ of the fundamental decomposition (2.11) (see [6] for the concept of angular operators).

Self-adjoint extensions A of S correspond to hypermaximal neutral subspaces $\mathsf{L} = \Psi\mathcal{D}(A)$ of the Krein space $(\mathsf{H}, [\cdot,\cdot])$. Each hypermaximal neutral subspace is determined uniquely by a unitary mapping between subspaces H_+ and H_- of the fundamental decomposition (2.11). This fact leads to the conclusion that each self-adjoint extension A of S can be described as

$$A = S^* \restriction_{\mathcal{D}(A)}, \quad \mathcal{D}(A) = \{ f \in \mathcal{D}(S^*) : \mathbf{T}\Gamma_+ f = \Gamma_- f \},$$

where $\mathbf{T}$ is a unitary operator in $\mathcal{H}$.

The explicit form of characteristic function depends on the choice of boundary triplet. Let $\Theta_i(\cdot)$ $(i = 1, 2)$ be characteristic functions associated with boundary triplets $(\mathcal{H}_i, \Gamma_-^i, \Gamma_+^i)$ constructed for the same S^*. Since the dimensions of the auxiliary Hilbert spaces $\mathcal{H}_i$ coincide with the defect number of S, we may assume without loss of generality that $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$.

It is easy to see that the operator $K : \mathsf{H} \to \mathsf{H}$ defined by the formula

$$K \begin{bmatrix} \Gamma_+^1 f \\ \Gamma_-^1 f \end{bmatrix} = \begin{bmatrix} \Gamma_+^2 f \\ \Gamma_-^2 f \end{bmatrix}, \quad f \in \mathcal{D}(S^*),$$

is surjective in H and, moreover, that K is a unitary operator in the Krein space $(\mathsf{H}, [\cdot,\cdot])$, that is, $[K\mathsf{x}, K\mathsf{y}] = [\mathsf{x}, \mathsf{y}]$, $\mathsf{x}, \mathsf{y} \in \mathsf{H}$ (the latter relation follows from (2.8)–(2.10)). Each unitary operator K in $(\mathsf{H}, [\cdot,\cdot])$ determines the so-called *interspherical linear fractional transformation* (see [6, Chapter III, Section 3])

$$\Phi_K(Z) = (K_{21} + K_{22}Z)(K_{11} + K_{12}Z)^{-1}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},$$

where K_{ij} are operator components of decomposition of K with respect to (2.11), and a bounded linear operator Z maps H_+ into H_-. The interspherical transformation $\Phi_K(Z)$ is well defined for all $Z : \mathsf{H}_+ \to \mathsf{H}_-$ with $\|Z\| \le 1$ (i.e., $0 \in \rho(K_{11} + K_{12}Z)$) and $\|\Phi_K(Z)\| \le 1$. It is known (see [17], [18]) that

$$\widetilde{\Theta}_2(\lambda) = \Phi_K\big(\widetilde{\Theta}_1(\lambda)\big), \quad \lambda \in \mathbb{C}_+, \tag{2.13}$$

where $\widetilde{\Theta}_i(\cdot) : \mathsf{H}_+ \to \mathsf{H}_-$ are defined similarly to (2.12).

3. Phillips symmetric operator

We say that a closed densely defined symmetric operator with equal nonzero defect numbers is a PSO if its characteristic function $\Theta(\cdot)$ is an operator-constant on $\mathbb{C}_+$. By virtue of (2.13), this definition does not depend on the choice of boundary triplet. However, in many cases, it is not easy to use (because one would have to calculate the characteristic function). For this reason a series of statements which can be used as (equivalent) definitions of PSO are presented below.

Theorem 3.1. *A symmetric operator S with equal defect numbers is a PSO if and only if*

$$\mathfrak{N}_\lambda \subset \mathcal{D}(S) \dotplus \mathfrak{N}_\mu, \quad \textit{for all } \lambda, \mu \in \mathbb{C}_+. \tag{3.1}$$

Proof. If S is a PSO, then its characteristic function $\Theta(\cdot)$ associated with the boundary triplet $(\mathfrak{N}_i, \Gamma_-, \Gamma_+)$ determined by (2.7) has to be a constant. Therefore, the Shtraus characteristic function $\mathsf{Sh}(\lambda)$ coincides with an operator $U : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ for all $\lambda \in \mathbb{C}_+$. In particular, $\mathsf{Sh}(i) = U$. By virtue of (2.4) with $\lambda = i$, $\mathcal{D}(S) \dotplus \mathfrak{N}_i = \mathcal{D}(S) \dotplus (I - U)\mathfrak{N}_i$, which is possible only for the case $U = 0$. Hence, $\mathsf{Sh}(\lambda) \equiv 0$ and (2.4) implies that

$$\mathfrak{N}_\lambda \subset \mathcal{D}(S) \dotplus \mathfrak{N}_i, \quad \forall \lambda \in \mathbb{C}_+. \tag{3.2}$$

Let us assume that there exist $f_i \in \mathfrak{N}_i$ and $\mu \in \mathbb{C}_+$ such that $f_i = v + f_\mu + f_{\overline{\mu}}$, where $v \in \mathcal{D}(S)$ and where $f_{\overline{\mu}} \in \mathfrak{N}_{\overline{\mu}}$ is nonzero. The last equality can be transformed to $\widetilde{f}_i = \widetilde{v} + f_{\overline{\mu}}$ with the use of (3.2). However, the obtained relation is impossible since $\operatorname{Im}(S^* \widetilde{f}_i, \widetilde{f}_i) = \|\widetilde{f}_i\|^2 > 0$ and, simultaneously,

$$\operatorname{Im}(S^* \widetilde{f}_i, \widetilde{f}_i) = \operatorname{Im}\big(S^*(\widetilde{v} + f_{\overline{\mu}}), \widetilde{v} + f_{\overline{\mu}}\big) = \operatorname{Im}(S^* f_{\overline{\mu}}, f_{\overline{\mu}}) = -(\operatorname{Im}\mu)\|f_{\overline{\mu}}\|^2 < 0.$$

Therefore, $f_i = v + f_\mu$ and $\mathfrak{N}_i \subset \mathcal{D}(S) \dotplus \mathfrak{N}_\mu$ for all $\mu \in \mathbb{C}$. The obtained inclusion and (3.2) justify (3.1). Conversely, if (3.1) holds, then, due to (2.4), $\mathcal{D}(S) \dotplus (I - \mathsf{Sh}(\lambda))\mathfrak{N}_i \subset \mathcal{D}(S) \dotplus \mathfrak{N}_i$, which is possible only for $\mathsf{Sh}(\lambda) \equiv 0$. □

Remark 3.2. The inclusion (3.1) and its dual counterpart in $\mathbb{C}_-$,

$$\mathfrak{N}_\nu \subset \mathcal{D}(S) \dotplus \mathfrak{N}_\xi, \quad \text{for all } \nu, \xi \in \mathbb{C}_-, \tag{3.3}$$

are equivalent. Indeed, (3.1) means that the maximal dissipative operators A_λ in (2.2) do not depend on the choice of $\lambda \in \mathbb{C}_+$, that is, $A_\lambda \equiv A_+$. Therefore, their adjoints $A^*_\lambda = A^*_\mu = A_\nu = A_\xi = A^*_+$ ($\nu = \overline{\lambda}$, $\xi = \overline{\mu}$) also do not depend on $\nu, \xi \in \mathbb{C}_-$. This fact justifies the equivalence of (3.1) and (3.3).

Corollary 3.3. *Simple PSO with the same defect numbers are unitarily equivalent.*

Proof. Let S be a PSO with the same defect numbers $n = n_+(S) = n_-(S)$. It follows from the proof of Theorem 3.1 that the Shtraus characteristic function $\mathsf{Sh}(\cdot)$ of S coincides with the zero operator. Therefore, the characteristic function of S calculated in terms of the boundary triplet $(\mathfrak{N}_i, \Gamma_-, \Gamma_+)$ (see (2.7)) is also a zero operator acting in the auxiliary space with the dimension n. By applying Theorem 2.1 for the case of simple PSO with the same defect numbers, we complete the proof. □

Theorem 3.4. *For a closed densely defined symmetric operator S with equal defect numbers, the following are equivalent:*

(i) *S is a PSO,*
(ii) $\mathfrak{N}_\lambda \perp \mathfrak{N}_\nu$ *for all* $\lambda \in \mathbb{C}_+$ *and* $\nu \in \mathbb{C}_-$,
(iii) $\mathfrak{N}_\lambda \perp \mathfrak{N}_\nu$ *for all* $\lambda \in \mathbb{C}_+$ *and some* $\nu \in \mathbb{C}_-$.

Proof. (i)⇒(ii). Let S be a PSO, and let $\lambda, \mu \in \mathbb{C}_+$, $\lambda \neq \mu$. By virtue of (3.1), $f_\lambda = u + f_\mu$, where $f_z \in \mathfrak{N}_z$ and $u \in \mathcal{D}(S)$. Therefore,

$$0 = (S^* - \lambda I)f_\lambda = (S - \lambda I)u + (\mu - \lambda)f_\mu. \tag{3.4}$$

The obtained relation means that $\mathfrak{N}_\mu \subset \mathcal{R}(S - \lambda I)$, and hence $\mathfrak{N}_\mu \perp \mathfrak{N}_\nu$, where $\nu = \overline{\lambda}$. To prove the orthogonality of $\mathfrak{N}_\mu$ and $\mathfrak{N}_{\overline{\mu}}$, we use (3.4) again in order to rewrite $f_\lambda = u + f_\mu$ as follows: $f_\lambda = (\lambda - \mu)(S - \lambda I)^{-1} f_\mu + f_\mu$. Let f_μ be fixed, and let $\lambda \to \mu$. Then $f_\lambda \to f_\mu$ due to the last formula for f_λ. Then

$$(f_\mu, f_{\overline{\mu}}) = \lim_{\lambda \to \mu} (f_\lambda, f_{\overline{\mu}}) = 0, \quad \forall f_\mu \in \mathfrak{N}_\mu, f_{\overline{\mu}} \in \mathfrak{N}_{\overline{\mu}}.$$

The implication (i)⇒(ii) is proved.

(iii)⇒(i). Let $\mathfrak{N}_\lambda \perp \mathfrak{N}_\nu$ for all $\lambda \in \mathbb{C}_+$ and some $\nu \in \mathbb{C}_-$. Each $f_\lambda \in \mathfrak{N}_\lambda$ has the decomposition $f_\lambda = u + f_\nu + f_{\overline{\nu}}$ and

$$(\lambda - \overline{\nu})f_\lambda = (S^* - \overline{\nu} I)f_\lambda = (S - \overline{\nu} I)u + (\nu - \overline{\nu})f_\nu.$$

Therefore, for every $\gamma_\nu \in \mathfrak{N}_\nu$,

$$0 = (\lambda - \overline{\nu})(f_\lambda, \gamma_\nu) = \big(u, (S^* - \nu I)\gamma_\nu\big) + (\nu - \overline{\nu})(f_\nu, \gamma_\nu) = (\nu - \overline{\nu})(f_\nu, \gamma_\nu),$$

which is possible only for $f_\nu = 0$. Therefore,

$$\mathfrak{N}_\lambda \subset \mathcal{D}(S) \dot{+} \mathfrak{N}_{\overline{\nu}}, \quad \forall \lambda \in \mathbb{C}_+.$$

This inclusion coincides with (3.2) for $\overline{\nu} = i$ and it implies (3.1) (it suffices to repeat the proof of implication (3.2) ⇒ (3.1) in Theorem 3.1 with $\overline{\nu} = i$). Then, according to Theorem 3.1, S is a PSO. By taking the trivial implication (ii)⇒(iii) into account, we complete the proof. □

Theorem 3.5. *For a closed densely defined symmetric operator S with equal defect numbers $n = n_+(S) = n_-(S)$, the following are equivalent:*

(i) *S is a PSO;*
(ii) *the Hilbert space $\mathfrak{H}$ is decomposed into the orthogonal sum $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$ of Hilbert spaces $\mathfrak{H}_j$ leaving S invariant and such that*

$$S = S_1 \oplus S_2 \oplus S_3, \quad S_j = S \restriction_{\mathfrak{H}_j}, \tag{3.5}$$

where S_1 and S_2 are simple maximal symmetric operators in $\mathfrak{H}_1$ and $\mathfrak{H}_2$ with defect numbers $n_+(S_1) = n$, $n_-(S_1) = 0$ and $n_+(S_2) = 0$, $n_-(S_2) = n$, respectively, and S_3 is a self-adjoint operator in $\mathfrak{H}_3$.

Proof. (ii)⇒(i). If S has the decomposition (3.5), then its adjoint has the form

$$S^* = S_1^* \oplus S_2^* \oplus S_3.$$

This means that

$$\mathfrak{N}_\lambda = \ker(S^* - \lambda I) = \ker(S_1^* - \lambda I) \oplus \ker(S_2^* - \lambda I) = \ker(S_1^* - \lambda I) \subset \mathfrak{H}_1$$

for every $\lambda \in \mathbb{C}_+$. Similarly, $\mathfrak{N}_\nu \subset \mathfrak{H}_2$ for each $\nu \in \mathbb{C}_-$. Therefore, $\mathfrak{N}_\lambda \perp \mathfrak{N}_\nu$ and S is a PSO due to Theorem 3.4.

(i)⇒(ii). Each symmetric operator S with equal defect numbers $n = n_+(S) = n_-(S)$ is reduced by the decomposition

$$\mathfrak{H} = \mathfrak{H}_\alpha \oplus \mathfrak{H}_3, \quad \mathfrak{H}_3 = \bigcap_{\mu \in \mathbb{C}_- \cup \mathbb{C}_+} \mathcal{R}(S - \mu I), \tag{3.6}$$

where $\mathfrak{H}_3$ is the maximal invariant subspace for S on which the operator $S_3 = S \restriction_{\mathfrak{H}_3}$ is self-adjoint, while the subspace $\mathfrak{H}_\alpha$ coincides with the closed linear span of all $\ker(S^* - \mu I)$ and the restriction $S_\alpha = S \restriction_{\mathfrak{H}_\alpha}$ gives rise to a simple symmetric operator in $\mathfrak{H}_\alpha$ with the same defect numbers $n_\pm(S_\alpha) = n$ (see [11, p. 9]).

By the construction, $\mathfrak{N}_\mu = \ker(S^* - \mu I) = \ker(S_\alpha^* - \mu I) = \mathfrak{N}_\mu(S_\alpha)$ for all $\mu \in \mathbb{C}_- \cup \mathbb{C}_+$. Assume now that S is a PSO. According to Theorem 3.4, $\mathfrak{N}_\lambda(S_\alpha) \perp \mathfrak{N}_\nu(S_\alpha)$ $(\lambda \in \mathbb{C}_+, \nu \in \mathbb{C}_-)$. Therefore, S_α is a simple PSO and we can decompose $\mathfrak{H}_\alpha = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ coincide with the closed linear spans of defect subspaces $\{\mathfrak{N}_\mu\}_{\mu \in \mathbb{C}_+}$ and defect subspaces $\{\mathfrak{N}_\nu\}_{\nu \in \mathbb{C}_-}$, respectively.

To complete the proof we should verify that $S_\alpha = S_1 \oplus S_2$, where $S_j = S_\alpha \restriction_{\mathfrak{H}_j}$ $(j = 1, 2)$ are maximal symmetric operators in $\mathfrak{H}_j$ with defect numbers $n_+(S_1) = n, n_-(S_1) = 0$ and $n_+(S_2) = 0, n_-(S_2) = n$. To that end, we consider a simple symmetric operator[2]

$$\mathrm{S} = i\frac{d}{dx}, \quad \mathcal{D}(\mathrm{S}) = \{u \in W_2^1(\mathbb{R}, N) : u(0) = 0\} \tag{3.7}$$

acting in the Hilbert space $L_2(\mathbb{R}, N)$, where N is an auxiliary Hilbert space with the dimension n. The adjoint of S has the form

$$\mathrm{S}^* f = i\frac{df}{dx}, \quad \mathcal{D}(\mathrm{S}^*) = W_2^1(\mathbb{R} \setminus \{0\}, N). \tag{3.8}$$

By virtue of (3.8), the defect subspaces $\mathfrak{N}_\mu$, $\mathfrak{N}_\nu$ $(\mu \in \mathbb{C}_+,\ \nu \in \mathbb{C}_-)$ are formed, respectively, by the functions

$$\chi_{\mathbb{R}_-}(x) e^{-i\mu x} m, \qquad \chi_{\mathbb{R}_+}(x) e^{-i\nu x} m, \tag{3.9}$$

where m runs the Hilbert space N, and $\chi_I(x)$ is the characteristic function of the interval I. Therefore, S has the equal defect numbers $n = n_\pm(\mathrm{S})$ and is a PSO (since $\mathfrak{N}_\mu$ and $\mathfrak{N}_\nu$ are mutually orthogonal).

By Corollary 3.3, the symmetric operator S_α in $\mathfrak{H}_\alpha$ is unitarily equivalent to the symmetric operator S acting in $L_2(\mathbb{R}, N)$. For this reason, it sufficient to establish the decomposition $S_\alpha = S_1 \oplus S_2$ for the case where $S_\alpha = \mathrm{S}$ and $\mathfrak{H}_\alpha = L_2(\mathbb{R}, N)$. Taking (3.9) into account, we conclude that $\mathfrak{H}_1 = L_2(\mathbb{R}_-, N)$ and $\mathfrak{H}_2 = L_2(\mathbb{R}_+, N)$. Moreover, $\mathrm{S} = \mathrm{S}_1 \oplus \mathrm{S}_2$, where $\mathrm{S}_1 = i\frac{d}{dx}$, $\mathcal{D}(\mathrm{S}_1) = \{u \in W_2^1(\mathbb{R}_-, N) : u(0) = 0\}$ is a maximal symmetric operator in $L_2(\mathbb{R}_-, N)$ with defect numbers $n_+(\mathrm{S}_1) = n, n_-(\mathrm{S}_1) = 0$, while $\mathrm{S}_2 = i\frac{d}{dx}$, $\mathcal{D}(\mathrm{S}_2) = \{u \in W_2^1(\mathbb{R}_+, N) : u(0) = 0\}$ is maximal symmetric in $L_2(\mathbb{R}_+, N)$ with defect numbers $n_+(\mathrm{S}_2) = 0$, $n_-(\mathrm{S}_2) = n$. □

[2]The simplicity of S is established in [1].

Remark 3.6. It follows from the proof of Theorem 3.5 that each simple PSO S with defect numbers $n = n_{\pm}(S)$ is unitarily equivalent to the symmetric operator S defined by (3.7).

4. Proper extensions of a Phillips symmetric operator

Let S be a simple PSO. It follows from Remark 3.6 that $\ker(S^* - \lambda I) = \{0\}$ for $\lambda \in \mathbb{R}$ and that the continuous spectrum of S coincides with $\mathbb{R}$. Therefore, each proper extension of a simple PSO has a continuous spectrum on $\mathbb{R}$. Proper extensions of a nonsimple PSO have $\mathbb{R}$ contained in their spectra, but the real point spectrum may be present due to a self-adjoint part S_3 in (3.5).

Proposition 4.1. *The spectrum $\sigma(A)$ of a proper extension A of a PSO S coincides with one of the following sets:*

(i) $\sigma(A) = \mathbb{R}$,
(ii) $\sigma(A) = \mathbb{C}_- \cup \mathbb{R}$ *or* $\sigma(A) = \mathbb{R} \cup \mathbb{C}_+$,
(iii) $\sigma(A) = \mathbb{C}$.

Proof. Let us suppose that a proper extension A has a nonreal point $\lambda_0 \in \rho(A)$. Without loss of generality, we may assume that $\lambda_0 \in \mathbb{C}_-$. Then the domain of A admits the presentation

$$\mathcal{D}(A) = \big\{f = u + u_{\overline{\lambda}_0} + \Phi u_{\overline{\lambda}_0} : u \in \mathcal{D}(S), u_{\overline{\lambda}_0} \in \mathfrak{N}_{\overline{\lambda}_0}\big\},$$

where $\Phi : \mathfrak{N}_{\overline{\lambda}_0} \to \mathfrak{N}_{\lambda_0}$ is a bounded operator defined on $\mathfrak{N}_{\overline{\lambda}_0}$. The domain $\mathcal{D}(A)$ can be rewritten in terms of the boundary triplet (2.7) with $\mu = \overline{\lambda}_0$ as

$$\mathcal{D}(A) = \big\{f \in \mathcal{D}(S^*) : \mathbf{T}\Gamma_+ f = \Gamma_- f\big\}, \tag{4.1}$$

where $\mathbf{T} = V\Phi$ is a bounded operator in the auxiliary Hilbert space $\mathfrak{N}_\mu$.

By virtue of [18, Theorem 4.2] (see also [17, Theorem 3]),

$$\lambda \in \sigma(A) \iff 0 \in \sigma\big(\Theta(\lambda) - \mathbf{T}\big), \quad \lambda \in \mathbb{C}_+,$$
$$\lambda \in \sigma(A) \iff 0 \in \sigma\big(I - \Theta^*(\overline{\lambda})\mathbf{T}\big), \quad \lambda \in \mathbb{C}_-,$$

where $\Theta(\cdot)$ is the characteristic function of S associated with the boundary triplet $(\mathfrak{N}_\mu, \Gamma_-, \Gamma_+)$. Since S is a PSO, the characteristic function $\Theta(\cdot)$ is an operator-constant on $\mathbb{C}_+$, that is, $\Theta(\lambda) = \Theta$ for $\lambda \in \mathbb{C}_+$. Therefore, $\lambda \in \sigma(A) \iff 0 \in \sigma(\Theta - \mathbf{T})$. The obtained relation means that either $\mathbb{C}_+$ is contained in $\sigma(A)$ or $\mathbb{C}_+ \subset \rho(A)$. Furthermore, due to the assumption above, there is a resolvent point $\lambda_0 \in \mathbb{C}_-$ of A. Therefore, $0 \in \rho(I - \Theta^*\mathbf{T})$ and $\mathbb{C}_- \subset \rho(A)$. Summing up, the spectrum $\sigma(A)$ is described by the cases (i) or (ii) depending on if $0 \in \rho(\Theta - \mathbf{T})$ or $0 \in \sigma(\Theta - \mathbf{T})$. The case $\lambda_0 \in \mathbb{C}_+ \cap \rho(A)$ is considered in the same manner. □

Linear operators A_1, A_2 acting in $\mathfrak{H}$ are called *similar* if there exists a bounded operator Z with bounded inverse such that $Z\mathcal{D}(A_1) = \mathcal{D}(A_2)$ and $A_1 = Z^{-1}A_2Z$.

Theorem 4.2. *Let S be a PSO. Proper extensions with real spectra of S are similar to each other.*

Proof. By virtue of Corollary 3.3 and Theorem 3.5, it is sufficient to consider proper extensions of the simple PSO S determined by (3.7). In this case, the adjoint S^* is defined by (3.8) and the defect subspaces $\mathfrak{N}_\mu$, $\mathfrak{N}_{\overline{\mu}}$ ($\mu \in \mathbb{C}_+$) are formed, respectively, by the functions $\chi_{\mathbb{R}_-}(x)e^{-i\mu x}m$ and $\chi_{\mathbb{R}_+}(x)e^{-i\overline{\mu}x}m$, where m runs the auxiliary Hilbert space N.

Let us choose the unitary mapping $V : \mathfrak{N}_{\overline{\mu}} \to \mathfrak{N}_\mu$ in the definition of boundary triplet (2.7) as $V\chi_{\mathbb{R}_+}(x)e^{-i\overline{\mu}x}m = \chi_{\mathbb{R}_-}(x)e^{-i\mu x}m$, and consider the unitary mapping W between $\mathfrak{N}_\mu$ and N as

$$W\chi_{\mathbb{R}_-}(x)e^{-i\mu x}m = \frac{m}{\sqrt{2(\operatorname{Im}\mu)}}.$$

Then the modified boundary triplet $(W\mathfrak{N}_\mu, W\Gamma_-, W\Gamma_+)$ of the boundary triplet (2.7) takes the form $(N, \Gamma^1_-, \Gamma^1_+)$, where

$$\Gamma^1_- f = f(0+), \qquad \Gamma^1_+ f = f(0-), \quad f \in \mathcal{D}(\mathrm{S}^*).$$

If a proper extension A of S has real spectrum, then its domain of definition is determined by the formula (4.1), where $\mathbf{T}$ is a bounded operator in $\mathfrak{N}_\mu$ with bounded inverse. This means that

$$A_T = \mathrm{S}^* \restriction_{\mathcal{D}(A_T)}, \quad \mathcal{D}(A_T) = \left\{f \in \mathcal{D}(S^*) : Tf(0-) = f(0+)\right\}, \tag{4.2}$$

where $T = W\mathbf{T}W^{-1}$ is a bounded operator in N with bounded inverse.

Let F be a bounded operator with bounded inverse in N. Then the operator

$$U_F f = \begin{cases} Ff(x), & x > 0, \\ f(x), & x < 0, \end{cases} \quad f \in L_2(\mathbb{R}, N) \tag{4.3}$$

is a bounded operator in $L_2(\mathbb{R}, N)$ with bounded inverse such that $U_F^{-1} = U_{F^{-1}}$. Furthermore, $U_F : \mathcal{D}(\mathrm{S}^*) \to \mathcal{D}(\mathrm{S}^*)$ and $U_F\mathrm{S}^* = \mathrm{S}^*U_F$. These relations and (4.2) lead to the conclusion that

$$U_F A_T f = U_F \mathrm{S}^* f = \mathrm{S}^* U_F f = A_{FT} U_F f, \quad f \in \mathcal{D}(A_T). \tag{4.4}$$

Let operators A_1, A_2 with real spectra be proper extensions of S. Then they are described in (4.2) by bounded operators T_j with $0 \in \rho(T_j)$ ($A_j \equiv A_{T_j}$). Due to (4.4), $\mathcal{D}(A_{T_2}) = U_F\mathcal{D}(A_{T_1})$, where $F = T_2T_1^{-1}$ and

$$A_{T_1} = U_F^{-1} A_{T_2} U_F. \tag{4.5}$$

Therefore, the A_j's are similar to each other. □

Self-adjoint extensions of a symmetric operator are examples of its proper extensions. For this important particular case, the statement of Theorem 4.2 can be strengthened.

Corollary 4.3. *Self-adjoint extensions of a PSO S are unitarily equivalent to each other. Precisely, there exists a collection of unitary operators $\mathfrak{U} = \{U_\xi\}_{\xi\in\mathfrak{I}}$ ($\mathfrak{I}$ is the set of indices) with the properties*

$$U_\xi \in \mathfrak{U} \iff U_\xi^* \in \mathfrak{U}, \quad U_\xi S = SU_\xi, \forall \xi \in \mathfrak{I},$$

and such that every pair of self-adjoint extensions A_1, A_2 of S satisfy the relation

$$U_\xi A_1 = A_2 U_\xi \tag{4.6}$$

for some $\xi \in \mathfrak{I}$.

Proof. Self-adjoint extensions of the symmetric operator S are uniquely distinguished in (4.2) by the set of unitary operators T acting in N (see Section 2). Therefore, the operators U_T defined by (4.3) with $F = T$ are unitary operators in $L_2(\mathbb{R}, N)$ and (4.5) can be rewritten as (4.6), where $A_i = A_{T_i}$ are self-adjoint extensions of S and $\xi = T_2 T_1^{-1}$ is a unitary operator in N.

It follows from the definition of U_T that the set $\mathfrak{U} = \{U_\xi\}$, where ξ runs the set $\mathfrak{I}$ of unitary operators in N, satisfies the conditions in the statement. Therefore, the proof is complete for the simple PSO S defined by (3.7). This result is extended to an arbitrary simple PSO S with the use of Corollary 3.3. The required set $\mathfrak{U} = \{U_\xi\}_{\xi \in \mathfrak{I}}$ for the general case of a PSO is obtained on the base of a previously constructed set (for simple PSO) by the addition of the identity operator I_3 acting in the subspace $\mathfrak{H}_3$ (see (3.5)) corresponding to the self-adjoint part of S. □

Remark 4.4. It follows from the construction of $\mathfrak{U} = \{U_\xi\}_{\xi \in \mathfrak{I}}$ in Corollary 4.3 that the symmetric operator S is $\mathfrak{U}$-invariant. However, there are no $\mathfrak{U}$-invariant self-adjoint extensions of S. First, an example of this kind was constructed by Phillips [23].

We say that a self-adjoint operator A has a *Lebesgue spectrum of multiplicity* n if A is unitarily equivalent to the operator of multiplication by independent variable in $L_2(\mathbb{R}, N)$, where $\dim N = n$.

Corollary 4.5. *Each self-adjoint extension of a simple PSO S with defect numbers $n = n_\pm(S)$ has a Lebesgue spectrum of multiplicity n.*

Proof. In view of Corollary 3.3, a simple PSO S is unitarily equivalent to the symmetric operator S in (3.7), where $\dim N = n$. The momentum operator

$$\mathrm{A} = i\frac{d}{dx}, \quad \mathcal{D}(\mathrm{A}) = W_2^1(\mathbb{R}, N) \tag{4.7}$$

is a self-adjoint extension of S in $L_2(\mathbb{R}, N)$ and it has a Lebesgue spectrum of multiplicity n (since A is unitarily equivalent to the operator of multiplication by independent variable). We now apply Corollary 4.3 to complete the proof. □

The concept of Lebesgue spectrum for a self-adjoint operator A can be defined in various (equivalent) ways (see [14], [26]) which guarantee that the spectral type of A is equivalent to the Lebesgue one and the multiplicity of $\sigma(A)$ does not change for any real point. The last condition is obviously satisfied when A is unitarily equivalent to its shifts $A - tI$ for any $t \in \mathbb{R}$. Development of this translation-invariance idea leads to the prominent Weyl commutation relation which ensures the Lebesgue spectrum property of A. Namely, due to the von Neumann theorem (see [20, p. 35]), a self-adjoint operator A has a Lebesgue spectrum if and only if there exists a strongly continuous group of unitary operators V_t such that

$$V_t A V_{-t} = A - tI, \quad \forall t \in \mathbb{R}. \tag{4.8}$$

It should be mentioned that each simple PSO is also a solution of the Weyl commutation relation (4.8). Indeed, an operator A which is the solution of (4.8) is determined up to unitary equivalence. Therefore, it is sufficient to consider the simple PSO S defined by (3.7) in $L_2(\mathbb{R}, N)$ and to verify that $A = \mathrm{S}$ is a solution of (4.8) with V_t acting as the multiplication by e^{-itx}.

5. Phillips symmetric operators as the restriction of self-adjoint ones

Lemma 5.1. *Let A be a self-adjoint operator in a Hilbert space $\mathfrak{H}$. A closed densely defined operator S is a restriction of A if and only if there is a linear subspace $\mathcal{L}$ such that $\mathcal{L} \cap \mathcal{D}(A) = \{0\}$ and $S = S_{\mathcal{L}}$, where*

$$S_{\mathcal{L}} = A \restriction_{\mathcal{D}(S_{\mathcal{L}})}, \quad \mathcal{D}(S_{\mathcal{L}}) = \big\{u \in \mathcal{D}(A) : \forall \gamma \in \mathcal{L}, \big((A - iI)u, \gamma\big) = 0\big\}. \tag{5.1}$$

The operator $S_{\mathcal{L}}$ is symmetric and its defect numbers $n_{\pm}(S_{\mathcal{L}})$ coincide with $\dim \mathcal{L}$.

Proof. Let $S_{\mathcal{L}}$ be defined by (5.1). Obviously, $S_{\mathcal{L}}$ is a closed restriction of A and, for all $u \in \mathcal{D}(S_{\mathcal{L}})$ and $p \in \mathfrak{H}$,

$$(u, p) = \big((S_{\mathcal{L}} - iI)u, (A + iI)^{-1}p\big) = \big((A - iI)u, (A + iI)^{-1}p\big).$$

This implies that $S_{\mathcal{L}}$ is nondensely defined if and only if there exists a nonzero p such that $(A + iI)^{-1}p \in \mathcal{L} \cap \mathcal{D}(A)$. Therefore, the condition $\mathcal{L} \cap \mathcal{D}(A) = \{0\}$ guarantees that $S_{\mathcal{L}}$ is densely defined. Conversely, if S is a closed densely defined restriction of A, then $S = S_{\mathcal{L}}$, where $\mathcal{L} = \mathfrak{H} \ominus \mathcal{R}(S - iI)$. The relation $\mathcal{L} \cap \mathcal{D}(A) = \{0\}$ holds since $S = S_{\mathcal{L}}$ is densely defined. By the construction, $S_{\mathcal{L}}$ is a symmetric operator with equal defect numbers. The relation $n_{\pm}(S_{\mathcal{L}}) = \dim \mathcal{L}$ follows from (5.1). □

The symmetric operator $S_{\mathcal{L}}$ in (5.1) turns out to be a PSO for a certain choice of $\mathcal{L}$. To specify the required conditions, we consider the unitary operator

$$U = (A + iI)(A - iI)^{-1}, \quad \big(A = i(U + I)(U - I)^{-1}\big), \tag{5.2}$$

which is the Cayley transform of A, and we recall that a subspace $\mathcal{L}$ is called a *wandering subspace* of U if $U^n\mathcal{L} \perp \mathcal{L}$ for all $n \in \mathbb{N}$.

Theorem 5.2. *The following statements are equivalent.*

(i) *The operator $S_{\mathcal{L}}$ defined by (5.1) is a PSO.*
(ii) *The subspace $\mathcal{L}$ is a wandering subspace of the unitary operator U.*

Proof. (ii)⇒(i). Let $\mathcal{L}$ be wandering for U. First of all we check that $\mathcal{L} \cap \mathcal{D}(A) = \{0\}$. Indeed, for all $f \in \mathcal{L}$, $(U^n f, f) = 0$ and, hence, $\int_0^{2\pi} e^{in\lambda} d(E_\lambda f, f) = 0$, where $\{E_\lambda\}$ $(0 \le \lambda \le 2\pi)$ is a spectral family of the unitary operator U. By the uniqueness theorem for the Fourier–Stieltjes series, the last equality means that $(E_\lambda f, f) = \frac{\lambda}{2\pi}\|f\|^2$ (see [27, p. 88]).

It follows from (5.2) that

$$A = i\int_0^{2\pi} \frac{e^{i\lambda} + 1}{e^{i\lambda} - 1}\, dE_\lambda = \int_0^{2\pi} \cot(\lambda/2)\, dE_\lambda$$

with the domain $\mathcal{D}(A) = \{f \in \mathfrak{H} : \int_0^{2\pi} \cot^2(\lambda/2)\, d(E_\lambda f, f) < \infty\}$. In the case of $f \in \mathcal{L}$ ($f \neq 0$),

$$\int_0^{2\pi} \cot^2(\lambda/2)\, d(E_\lambda f, f) = \frac{\|f\|^2}{2\pi} \int_0^{2\pi} \cot^2(\lambda/2)\, d\lambda = \infty.$$

Therefore, $\mathcal{L} \cap \mathcal{D}(A) = \{0\}$ and the operator $S_{\mathcal{L}}$ is densely defined and closed by Lemma 5.1.

It follows from (5.1) that the defect subspace $\mathfrak{N}_{-i}$ of $S_{\mathcal{L}}$ coincides with $\mathcal{L}$. In order to describe other defect subspaces $\mathfrak{N}_\alpha$ of $S_{\mathcal{L}}$, we consider the operator

$$T_\alpha = (A + iI)(A - \alpha I)^{-1}, \quad \alpha \in \mathbb{C}_- \cup \mathbb{C}_+.$$

The formula $\mathfrak{N}_\alpha = T_\alpha \mathcal{L}$ is verified directly. Again using (5.2), we get $T_\alpha = 2U[(1+i\alpha)U + (1 - i\alpha)I]^{-1}$. In particular, if $\alpha = \lambda \in \mathbb{C}_+$, then the obtained expression for T_α can be rewritten as

$$T_\lambda = \frac{2U}{1 - i\lambda}[I - tU]^{-1} = \frac{2U}{1 - i\lambda} \sum_{n=0}^{\infty} t^n U^n, \quad t = \frac{i\lambda + 1}{i\lambda - 1} \tag{5.3}$$

since $\|tU\| = |t| < 1$. Since $U^n \mathcal{L} \perp \mathcal{L}$ for all $n \in \mathbb{N}$, the relation (5.3) yields that $T_\lambda \mathcal{L} \perp \mathcal{L}$ for $\lambda \in \mathbb{C}_+$. Therefore, $\mathfrak{N}_\lambda \perp \mathfrak{N}_{-i}$. Due to Theorem 3.4, the operator $S_{\mathcal{L}}$ is PSO. The implication (ii)⇒(i) is proved.

(i)⇒(ii). The operator $S_{\mathcal{L}}$ in (5.1) is completely determined by the pair $\{A, \mathcal{L}\}$ of the given self-adjoint operator A and the subspace $\mathcal{L} \subset \mathfrak{H}$. If $S_{\mathcal{L}}$ is a PSO, then the decomposition (3.5) implies that $\mathcal{L}$ is a subspace of $\mathfrak{H}_1 \oplus \mathfrak{H}_2$. Therefore, it is sufficient to assume that $S_{\mathcal{L}}$ is a simple PSO. Another important fact is that the operator $S_{\mathcal{L}}$ can be determined by a pair $\{A_1, \mathcal{L}_1\}$, where A_1 is an *arbitrary* self-adjoint extension of $S_{\mathcal{L}}$. Indeed, due to Corollary 4.3, there exists a unitary operator U_ξ such that $U_\xi S_{\mathcal{L}} = S_{\mathcal{L}} U_\xi$ and $U_\xi A = A_1 U_\xi$. Hence, the operator $S_{\mathcal{L}}$ can be described as

$$S_{\mathcal{L}} = A_1 \restriction_{\mathcal{D}(S_{\mathcal{L}})}, \quad \mathcal{D}(S_{\mathcal{L}}) = \big\{v \in \mathcal{D}(A_1) : \forall g \in \mathcal{L}_1, \big((A_1 - iI)v, g\big) = 0\big\}, \tag{5.4}$$

where $\mathcal{L}_1 = U_\xi \mathcal{L}$.

Since the Cayley transformations U and U_1 of the operators A and A_1 are related as $U_\xi U = U_1 U_\xi$, the existence of a wandering subspace $\mathcal{L}$ for U implies that $\mathcal{L}_1 = U_\xi \mathcal{L}$ is a wandering subspace for U_1. Therefore, it suffices to prove the implication (i)⇒(ii) assuming that $S_{\mathcal{L}}$ is determined by a pair $\{A_1, \mathcal{L}_1\}$, where A_1 is an arbitrary self-adjoint extension of $S_{\mathcal{L}}$. Such a flexibility of the choice of A_1 allows one to simplify the argumentation below.

Simple Phillips symmetric operators with the same defect numbers are unitarily equivalent (see Corollary 3.3). For this reason, we can consider a concrete PSO in (5.1). It is useful to work with an operator $S_{\mathcal{L}}$ defined in $\mathfrak{H} = l_2(\mathbb{Z}, N)$ (N is an auxiliary Hilbert space) as

$$S_{\mathcal{L}} u = i(\ldots, x_{-3} + x_{-2}, x_{-2} + x_{-1}, \underline{x_{-1}}, x_1, x_1 + x_2, \ldots), \quad x_j \in N, \tag{5.5}$$

where the element at the zero position is underlined and

$$u \in \mathcal{D}(S_{\mathcal{L}}) \iff u = (\ldots, x_{-3} - x_{-2}, x_{-2} - x_{-1}, \underline{x_{-1}}, -x_1, x_1 - x_2, \ldots),$$

where $\sum_{i\in\mathbb{Z}}\|x_i\|_N^2 < \infty$. The operator $S_{\mathcal{L}}$ defined by (5.5) is a simple PSO in $l_2(\mathbb{Z}, N)$, and the operator

$$A_1 u = i(\ldots, x_{-3}+x_{-2}, x_{-2}+x_{-1}, \underline{x_{-1}+x_0}, x_0+x_1, x_1+x_2, \ldots) \tag{5.6}$$

with the domain of definition $u \in \mathcal{D}(A_1) \iff$

$$u = (\ldots, x_{-3}-x_{-2}, x_{-2}-x_{-1}, \underline{x_{-1}-x_0}, x_0-x_1, x_1-x_2, \ldots), \quad \sum_{i\in\mathbb{Z}}\|x_i\|_N^2 < \infty$$

is a self-adjoint extension of $S_{\mathcal{L}}$ (see [18], [19]).

It follows from (5.5) and (5.6) that $\mathcal{D}(S_{\mathcal{L}})$ consists of those $u \in \mathcal{D}(A_1)$ for which $x_0 = 0$. Direct calculation with use of (5.6) shows that $(A_1 - iI)u = 2i(\ldots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \ldots)$. Therefore, (5.4) gives $S_{\mathcal{L}}$ with the choice

$$\mathcal{L}_1 = \left\{(\ldots, 0, 0, \underline{x_0}, 0, 0, \ldots) : x_0 \in N\right\} \subset l_2(\mathbb{Z}, N).$$

It is easy to see that the Cayley transform U_1 of A_1 coincides with the unitary operator

$$U_1(\ldots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \ldots) = (\ldots, x_{-3}, x_{-2}, \underline{x_{-1}}, x_0, x_1, \ldots)$$

in $l_2(\mathbb{Z}, N)$. The subspace $\mathcal{L}_1$ is a wandering subspace for U_1. The proof is complete. □

A unitary operator U in a Hilbert space $\mathfrak{H}$ is called a *bilateral shift* if there exists a wandering subspace $\mathcal{L}$ of U such that $\mathfrak{H} = \bigoplus_{n\in\mathbb{Z}} U^n\mathcal{L}$. Every such subspace $\mathcal{L}$ is called a *generating* subspace, and the dimension of $\mathcal{L}$ is called the *multiplicity* of the bilateral shift U. A bilateral shift is determined by its multiplicity up to unitary equivalence (see [27, p. 5]).

The next auxiliary result is folklore in operator theory.

Lemma 5.3. *A self-adjoint operator A has a Lebesgue spectrum of multiplicity n if and only if its Cayley transform U is a bilateral shift of multiplicity n.*

Proof. A self-adjoint operator with Lebesgue spectrum of multiplicity n is, by definition, unitarily equivalent to the operator of multiplication by independent variable: $Af(\delta) = \delta f(\delta)$ in $L_2(\mathbb{R}, N)$, where $\dim N = n$. The Cayley transform of A coincides with the multiplication operator $Uf(\delta) = \frac{\delta+iI}{\delta-iI}f(\delta)$. It follows from [20, p. 49] that U is a bilateral shift in $L_2(\mathbb{R}, N)$ with the generating wandering subspace $\mathcal{L} = \frac{1}{\delta+i}N$. The multiplicity of U is $\dim\mathcal{L} = \dim N = n$.

The inverse statement is obvious because each bilateral shift of multiplicity n is unitarily equivalent to the bilateral shift $Uf(\delta) = \frac{\delta+iI}{\delta-iI}f(\delta)$ in $L_2(\mathbb{R}, N)$ (it follows from the fact that bilateral shifts of the same multiplicity are unitarily equivalent; see [27, p. 5]). □

Corollary 5.4. *If the set of closed densely defined restrictions of a given self-adjoint operator A contains a PSO, then there exists a reducing subspace $\mathfrak{H}_0$ of A such that the operator $A_0 = A\restriction_{\mathcal{D}(A)\cap\mathfrak{H}_0}$ is self-adjoint in $\mathfrak{H}_0$ and A_0 has a Lebesgue spectrum. The existence of a simple PSO S with defect numbers $n = n_\pm(S)$ among restrictions of A is equivalent to the fact that A has a Lebesgue spectrum of multiplicity n.*

Proof. Due to Theorem 5.2, a PSO can appear only in the case where there exists a wandering subspace $\mathcal{L}$ of U. Denote $\mathfrak{H}_0 = \bigoplus_{n\in\mathbb{Z}} U^n\mathcal{L}$. The operator $U_0 = U \upharpoonright_{\mathfrak{H}_0}$ is a bilateral shift in $\mathfrak{H}_0$, its Cayley transform A_0 coincides with $A \upharpoonright_{\mathcal{D}(A)\cap\mathfrak{H}_0}$, and it is a self-adjoint operator with Lebesgue spectrum in the Hilbert space $\mathfrak{H}_0$ (Lemma 5.3).

If A is a self-adjoint extension of a simple PSO S with defect numbers $n = n_\pm(S)$, then A has a Lebesgue spectrum of multiplicity n (see Corollary 4.5). Conversely, the Lebesgue spectrum of multiplicity n of A means that A is unitarily equivalent to the momentum operator A in (4.7). The simple PSO S defined in (3.7) has defect numbers $n = n_\pm(\mathrm{S})$ and it is the restriction of A. □

Corollary 5.5. *For a self-adjoint operator A the following are equivalent.*

(i) *The Cayley transform U of A is a bilateral shift and a subspace $\mathcal{L}$ of $\mathfrak{H}$ is a generating wandering subspace for U.*

(ii) *The operator $S_\mathcal{L}$ defined by (5.1) is a simple PSO.*

Proof. (i)⇒(ii). Since $\mathcal{L}$ is a wandering subspace for U, the operator $S_\mathcal{L}$ defined by (5.1) is a PSO (see Theorem 5.2). The PSO $S_\mathcal{L}$ is also defined by (3.5) with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \mathfrak{H}_3$ (see Theorem 3.5). The subspaces $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ and $\mathfrak{H}_3$ are invariant for the Cayley transform U of A and $\mathcal{L}$ is a subspace of $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ (the latter follows from (5.1)). Taking into account that $\mathcal{L}$ is a generating subspace for U, we obtain $\mathfrak{H} = \bigoplus_{n\in\mathbb{Z}} U^n\mathcal{L} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Therefore, $\mathfrak{H}_3 = \{0\}$ in the decomposition (3.5) and $S_\mathcal{L}$ is a simple PSO.

(ii)⇒(i). Corollary 5.4 implies that A has a Lebesgue spectrum. Hence, U is a bilateral shift in $\mathfrak{H}$ (see Lemma 5.3). By Theorem 5.2, $\mathcal{L}$ is a wandering subspace for U. Denote $\mathfrak{H}_0 = \bigoplus_{n\in\mathbb{Z}} U^n\mathcal{L}$ and $\mathfrak{H}' = \mathfrak{H} \ominus \mathfrak{H}_0$. The orthogonal sum $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}'$ reduces U. Therefore, the operator A is decomposed as an orthogonal sum of self-adjoint operators $A = A \upharpoonright_{\mathfrak{H}_0} \oplus A \upharpoonright_{\mathfrak{H}'}$ acting in the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}'$, respectively. By the construction, $\mathcal{L}$ is a subspace of $\mathfrak{H}_0$. Therefore, the operator $S_\mathcal{L}$ defined by (5.1) contains the self-adjoint part $A \upharpoonright_{\mathfrak{H}'}$. The simplicity of $S_\mathcal{L}$ yields that $\mathfrak{H}' = \{0\}$. Therefore, $\mathcal{L}$ is a generating subspace for U. □

Corollary 5.6. *Assume that a self-adjoint operator A has a Lebesgue spectrum of multiplicity $n = \dim \mathcal{L} < \infty$, where $\mathcal{L}$ is a subspace of $\mathfrak{H}$ such that the operator $S_\mathcal{L}$ defined by (5.1) is a PSO. Then $S_\mathcal{L}$ is a simple PSO.*

Proof. The Cayley transform U of A is a bilateral shift of multiplicity n (see Lemma 5.3). The subspace $\mathcal{L}$ is wandering for U (see Theorem 5.2). Moreover, $\mathfrak{H} = \bigoplus_{n\in\mathbb{Z}} U^n\mathcal{L}$ since the dimension of $\mathcal{L}$ is finite and it coincides with multiplicity n of U. Therefore, $\mathcal{L}$ is a generating wandering subspace for U. By Corollary 5.5, $S_\mathcal{L}$ is a simple PSO. □

6. Examples of PSOs

(I) Let $S_\mathcal{L}$ be a PSO that is determined by (5.1) as the restriction of a self-adjoint operator A. Consider a unitary operator W that commutes with A. It is easy to see that $S' = WS_\mathcal{L}W^{-1}$ is also a PSO and that S' is determined by (5.1) with the new wandering subspace $\mathcal{L}' = W\mathcal{L}$, that is, $S' = S_{W\mathcal{L}}$. This simple observation gives rise to infinitely many unitarily equivalent PSOs which are

restrictions of a given self-adjoint operator A. If the operator A has a Lebesgue spectrum and $\mathcal{L}$ is a generating wandering subspace for the Cayley transform U of A, then the constructed PSOs $S_{W\mathcal{L}}$ are simple (see Corollary 5.5).

Let us consider the simplest example of self-adjoint operators with Lebesgue spectra assuming that A is the operator of multiplication by independent variable: $Af(\delta) = \delta f(\delta)$ in $L_2(\mathbb{R}, N)$. The Cayley transform of A is a bilateral shift $Uf(\delta) = \frac{\delta+iI}{\delta-iI}f(\delta)$ in $L_2(\mathbb{R}, N)$ with generating wandering subspace $\mathcal{L} = \frac{1}{\delta+i}N$ (see the proof of Lemma 5.3).

If W is a unitary operator in $L_2(\mathbb{R}, N)$ that commutes with A, then W can be realized as a multiplicative operator-valued function $w(\delta) : N \to N$ which is unitary for almost all δ (see, e.g., [20, Corollary 4.2, p. 53]) :

$$Wf = w(\delta)f(\delta), \quad f \in L_2(\mathbb{R}, N).$$

This means that the subspaces

$$\mathcal{L}_w = W\mathcal{L} = W\frac{1}{\delta + i}N = \frac{w(\delta)}{\delta + i}N$$

are generating wandering subspaces for the bilateral shift U in $L_2(\mathbb{R}, N)$ and that they determine infinitely many unitarily equivalent simple PSOs $S_w \equiv S_{W\mathcal{L}}$, which, due to (5.1), are restrictions of A onto linear manifolds

$$\mathcal{D}(S_w) = \Big\{u \in \mathcal{D}(A) : \forall v \in N, \int_{\mathbb{R}} \big(u(\delta), w(\delta)v\big)_N \, d\delta = 0\Big\}.$$

This result can be reformulated for the restrictions of self-adjoint momentum operator A (see (4.7)) with the use of Fourier transformation

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\delta x} f(\delta) \, d\delta$$

that relates $Af(\delta) = \delta f(\delta)$ and $\mathrm{A}f = i\frac{df}{dx}$. Taking into account that $\mathrm{A}F = FA$, we conclude that $F\mathcal{L}_w$ are generating wandering subspaces for the Cayley transform of A in $L_2(\mathbb{R}, N)$. Simple PSOs $\mathrm{S}_w = FS_wF^{-1}$ are the restrictions of A onto those functions $f \in W_2^1(\mathbb{R}, N)$ that satisfy the relation

$$\mathrm{S}_w = \mathrm{A} \restriction_{\mathcal{D}(\mathrm{S}_w)}, \mathcal{D}(\mathrm{S}_w) = \big\{f \in W_2^1(\mathbb{R}, N) : \forall \gamma \in F\mathcal{L}_w, \big((\mathrm{A} - iI)f, \gamma\big) = 0\big\}. \quad (6.1)$$

Let us set $w(\delta) \equiv 1$. Then $F\mathcal{L}_w$ coincides with the subspace $\chi_{\mathbb{R}_+}(x)e^{-x}N$ and (6.1) gives the simple PSO S defined by (3.7). The operator S deals with one-point interaction at $x = 0$ of the momentum operator A (see [2]). The simple PSO

$$\mathrm{S}_w = i\frac{d}{dx}, \quad \mathcal{D}(\mathrm{S}_w) = \big\{u \in W_2^1(\mathbb{R}, N) : u(y) = 0\big\} \quad (6.2)$$

corresponding to one-point interaction at real point $x = y$ is obtained from (6.1) with $w(\delta) = e^{i\delta y}$.

Assume now that $w(\delta) = \frac{\delta+iI}{\delta-iI}$. Then $F\mathcal{L}_w = F(\frac{1}{\delta-iI}N) = \chi_{\mathbb{R}_-}(x)e^{x}N$ and the formula (6.1) gives rise to the simple PSO

$$\mathrm{S}_w = i\frac{d}{dx}, \quad \mathcal{D}(\mathrm{S}_w) = \Big\{u \in W_2^1(\mathbb{R}, N) : u(0) = 2\int_{-\infty}^{0} u(x)e^x\, dx\Big\}, \tag{6.3}$$

which is an example of nonlocal point interaction of the momentum operator A.

(II) Let $Df = \sqrt{2}f(2x)$ be the dilation operator in $L_2(\mathbb{R})$, and let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis for $L_2(\mathbb{R})$ (see [8, Definition 8.2.1]). The operator D is a bilateral shift in $L_2(\mathbb{R})$ with generating wandering subspace $W_0 = V_1 \ominus V_0$ (this follows from [8, Lemma 9.2.3]). Denote $A_D = i(D+I)(D-I)^{-1}$. The operator A_D is self-adjoint in $L_2(\mathbb{R})$.

Proposition 6.1. *Let S be a simple PSO that is a restriction of the self-adjoint operator A_D, and let $\psi \in \mathfrak{N}_{-i} = \ker(S^* + iI)$ be a function in $L_2(\mathbb{R})$ such that $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathfrak{N}_{-i}$. Then ψ is a wavelet.*

Proof. If S is a restriction of A, then S is determined by (5.1), that is, $S = S_{\mathcal{L}}$, with $\mathcal{L} = \mathfrak{N}_{-i}$. According to Corollary 5.5, $\mathcal{L}$ is a generating wandering subspace for the bilateral shift D (since S is a simple PSO). Therefore, $L_2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} D^j\mathcal{L}$ and $\{D^j\psi(x-k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L_2(\mathbb{R})$. The latter means that ψ is a wavelet (see [8, Definition 8.1.1]). $\square$

7. Nonlocal point interactions

The preceding results show that one-point interaction of the momentum operator $\mathrm{A} + \alpha\delta(x-y)$ leads to self-adjoint operators which are unitarily equivalent to each other and have Lebesgue spectra. This means that nontrivial spectral properties of self-adjoint operators associated with the momentum operator should be obtained with the help of more complicated perturbations. In the present section, we consider special classes of general nonlocal one-point interactions (see [4]) which can be characterized as one-point interactions defined by the nonlocal potential $\gamma(x) \in L_2(\mathbb{R})$.

(I) Let us consider the maximal operator $S_{\max}$ which is determined on $W_2^1(\mathbb{R}\setminus\{0\})$ by the differential expression

$$S_{\max}f = i\frac{df}{dx} + \gamma(x)f_r \quad (x \neq 0),\, f_r = \frac{1}{2}\big(f(0+) + f(0-)\big),$$

where the nonlocal potential $\gamma(x)$ belongs to $L_2(\mathbb{R})$. Direct calculation shows that, for all $f, g \in \mathcal{D}(S_{\max}) = W_2^1(\mathbb{R}\setminus\{0\})$,

$$(S_{\max}f, g) - (f, S_{\max}g) = i[\Gamma_+ f\overline{\Gamma_+ g} - \Gamma_- f\overline{\Gamma_- g}],$$

where $\Gamma_\pm$ are determined by

$$\Gamma_+ f = f(0-) + \frac{i}{2}(f,\gamma), \qquad \Gamma_- f = f(0+) - \frac{i}{2}(f,\gamma). \tag{7.1}$$

Lemma 7.1. *The operator*

$$S_{\min} = S_{\max}\restriction_{\mathcal{D}(S_{\min})}, \quad \mathcal{D}(S_{\min}) = \ker\Gamma_- \cap \ker\Gamma_+$$

is a closed densely defined symmetric operator in $L_2(\mathbb{R})$ and such that $S_{\min}^* = S_{\max}$. A triplet $(\mathbb{C}, \Gamma_-, \Gamma_+)$, where the linear mappings $\Gamma_\pm : W_2^1(\mathbb{R}\setminus\{0\}) \to \mathbb{C}$ are determined by (7.1), is a boundary triplet of $S_{\max}$.

Proof. To complete the proof, by virtue of [7, Corollary 2.5] and Remark 2.2, it is sufficient to verify that (i) there is a unimodular c such that the operator $A = S_{\max} \upharpoonright_{\ker(c\Gamma_+ - \Gamma_-)}$ is self-adjoint in $L_2(\mathbb{R})$, and (ii) the map $(\Gamma_-, \Gamma_+) : \mathcal{D}(S_{\max}) \to \mathbb{C}^2$ is surjective.

The condition (i) is satisfied if we choose $c = -1$. In this case,

$$A = i\frac{d}{dx}, \quad \mathcal{D}(A) = \{f \in W_2^1(\mathbb{R} \setminus \{0\}) : f(0-) = -f(0+)\} \tag{7.2}$$

is a self-adjoint operator.

The surjectivity of the map $(\Gamma_-, \Gamma_+) : \mathcal{D}(S_{\max}) \to \mathbb{C}^2$ is obvious for $\gamma = 0$. Assume that $\gamma \neq 0$, and consider an arbitrary element $< h_1, h_2 >$ of $\mathbb{C}^2$. There exists $f \in W_2^1(\mathbb{R} \setminus \{0\})$ such that $f(0+) = h_1$ and $f(0-) = h_2$. Let us fix $u \in W_2^1(\mathbb{R} \setminus \{0\})$ such that $u(0-) = u(0+) = 0$ and $(u, \gamma) \neq 0$. Using now (7.1) we conclude that the vector $\widetilde{f} = f - \frac{(f,\gamma)}{(u,\gamma)} u$ solves the equation $(\Gamma_-, \Gamma_+)\widetilde{f} = \langle h_1, h_2 \rangle$, which justifies (ii). □

The boundary triplet $(\mathbb{C}, \Gamma_-, \Gamma_+)$ constructed in Lemma 7.1 allows us to determine self-adjoint operators

$$A_\theta f = i\frac{df}{dx} + \gamma(x) f_r, \quad f \in \mathcal{D}(A_\theta) \subset W_2^1(\mathbb{R} \setminus \{0\}), \theta \in [0, 2\pi) \tag{7.3}$$

whose domains $\mathcal{D}(A_\theta)$ consist of all functions $f \in W_2^1(\mathbb{R} \setminus \{0\})$ that satisfy the nonlocal boundary-value condition

$$e^{i\theta}\Big[f(0-) + \frac{i}{2}(f, \gamma)\Big] = f(0+) - \frac{i}{2}(f, \gamma).$$

These operators are mathematical models of one-point interaction defined by the nonlocal potential $\gamma(x)$. Each operator A_θ is a self-adjoint extension of the symmetric operator $S_{\min} = S_{\max}^* = i\frac{d}{dx}$ with domain of definition

$$\mathcal{D}(S_{\min}) = \left\{ f \in W_2^1(\mathbb{R} \setminus \{0\}) : \begin{matrix} f(0-) + \frac{i}{2}(f, \gamma) = 0 \\ f(0+) - \frac{i}{2}(f, \gamma) = 0 \end{matrix} \right\}. \tag{7.4}$$

The symmetric operator $S_{\min}$ has equal defect numbers $n_\pm(S_{\min}) = 1$, and its defect subspaces $\mathfrak{N}_\lambda$, $\mathfrak{N}_\nu$ ($\lambda \in \mathbb{C}_+$, $\nu \in \mathbb{C}_-$) coincide with the linear span of the functions

$$f_\lambda(x) = g^\lambda(x) - 2\big[1 + g^\lambda(0)\big] G_\lambda^+(x) \qquad \text{and} \qquad f_\nu(x) = g^\nu(x) - 2\big[1 + g^\nu(0)\big] G_\nu^-(x),$$

respectively. Here

$$g^z = (\mathrm{A} - zI)^{-1}\gamma = \begin{cases} ie^{-izx}\int_x^\infty e^{iz\tau}\gamma(\tau)\,d\tau & z \in \mathbb{C}_+, \\ -ie^{-izx}\int_{-\infty}^x e^{iz\tau}\gamma(\tau)\,d\tau & z \in \mathbb{C}_-, \end{cases} \tag{7.5}$$

and $G_\lambda^+(x) = \chi_{\mathbb{R}_-}(x)e^{-i\lambda x}$, $G_\nu^-(x) = \chi_{\mathbb{R}_+}(x)e^{-i\nu x}$.

By virtue of (2.6) and (7.1), the characteristic function $\Theta(\cdot)$ has the form

$$\Theta(\lambda)=\frac{\Gamma_-f_\lambda}{\Gamma_+f_\lambda}=\frac{f_\lambda(0+)-\frac{i}{2}(f_\lambda,\gamma)}{f_\lambda(0-)+\frac{i}{2}(f_\lambda,\gamma)}$$
$$=-I+\frac{2}{2+g^\lambda(0)-\frac{i}{2}(f_\lambda,\gamma)},\quad \lambda\in\mathbb{C}_+. \tag{7.6}$$

Let us consider a particular case assuming that $\gamma=\alpha\chi_{\mathbb{R}_+}(x)e^{-x}$, $\alpha\in\mathbb{C}$. Then

$$g^\lambda(x)=\frac{i\alpha}{1-i\lambda}\begin{cases}e^{-x} & x>0,\\ e^{-i\lambda x} & x<0\end{cases}$$

and

$$g^\lambda(0)-\frac{i}{2}(f_\lambda,\gamma)=\frac{i\alpha}{1-i\lambda}(1-i\overline{\alpha}/4).$$

Therefore, the characteristic function (7.6) turns out to be a constant on $\mathbb{C}_+$ when $\alpha=4i$. In this case, the symmetric operator $S_{\min}$ in (7.4) is a PSO. Its simplicity can be established with the use of Corollary 5.5. Indeed, the operator A in (7.2) is a self-adjoint extension of the simple PSO S in (3.7) (with $N=\mathbb{C}$). By Corollary 4.5, A has a Lebesgue spectrum of multiplicity 1. Our PSO $S_{\min}$ has defect numbers $n_\pm(S_{\min})=1$, and $S_{\min}$ is a restriction of A. By Corollary 5.6, $S_{\min}$ is a simple PSO and its self-adjoint extensions (7.3) have Lebesgue spectra of multiplicity 1 (see Corollary 4.5).

(II) Let the maximal operator $S_{\max}$ be determined by the differential expression

$$S_{\max}f=i\frac{df}{dx}+\gamma(x)f_s\quad (x\neq 0), f_s=f(0+)-f(0-),$$

where the nonlocal potential $\gamma(x)$ belongs to $L_2(\mathbb{R})$. Similarly to the previous case, the Green's formula can be established as

$$(S_{\max}f,g)-(f,S_{\max}g)=i[\Gamma_+f\overline{\Gamma_+g}-\Gamma_-f\overline{\Gamma_-g}],\quad f,g\in\mathcal{D}(S_{\max})=W_2^1\big(\mathbb{R}\setminus\{0\}\big),$$

where $\Gamma_+f=f(0-)-i(f,\gamma)$ and $\Gamma_-f=f(0+)-i(f,\gamma)$. The same arguments as in the proof of Lemma 7.1 lead to the conclusion that $(\mathbb{C},\Gamma_-,\Gamma_+)$ is a boundary triplet of $S_{\max}$ and the corresponding symmetric operator $S_{\min}=S_{\max}\restriction_{\mathcal{D}(S_{\min})}$, $\mathcal{D}(S_{\min})=\ker\Gamma_-\cap\ker\Gamma_+$ has the form

$$S_{\min}=i\frac{d}{dx},\quad \mathcal{D}(S_{\min})=\big\{f\in W_2^1(\mathbb{R}):f(0)=i(f,\gamma)\big\}. \tag{7.7}$$

Each self-adjoint extension A_θ of $S_{\min}$ is determined by the formula

$$A_\theta f=i\frac{df}{dx}+\gamma(x)f_s,$$

where $\mathcal{D}(A_\theta)=\{f\in\mathcal{D}(S_{\max}):e^{i\theta}[f(0-)-i(f,\gamma)]=f(0+)-i(f,\gamma)\}$. The defect subspaces $\mathfrak{N}_\lambda$, $\mathfrak{N}_\nu$ ($\lambda\in\mathbb{C}_+$, $\nu\in\mathbb{C}_-$) of $S_{\min}$ coincide with the linear span of vectors

$$f_\lambda(x)=g^\lambda(x)+G_\lambda^+(x)\qquad\text{and}\qquad f_\nu(x)=g^\nu(x)-G_\nu^-(x),$$

respectively.

Let us fix $\gamma = \alpha\chi_{\mathbb{R}_-}(x)e^x$ and find $\alpha \in \mathbb{C}$ for which the symmetric operator $S_{\min}$ in (7.7) is a PSO. It follows from (7.5) that

$$g^\lambda(x) = \frac{i\alpha\chi_{\mathbb{R}_-}(x)}{1+i\lambda}(e^{-i\lambda x} - e^x), \qquad g^\nu(x) = -\frac{i\alpha}{1+i\nu}\begin{cases} e^{-i\nu x} & x>0, \\ e^x & x<0. \end{cases}$$

The obtained expressions allow one to calculate

$$(f_\lambda, f_\nu) = \frac{\overline{\alpha}}{2(1-i\lambda)(1-i\overline{\nu})}(2i-\alpha), \quad \lambda \in \mathbb{C}_+, \quad \nu \in \mathbb{C}_-.$$

The obtained expression and Theorem 3.4 mean that the symmetric operator $S_{\min}$ defined in (7.7) is a PSO if and only if $\alpha = 2i$. In this case, the PSO $S_{\min}$ coincides with the operator S_w determined by (6.3).

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