Banach J. Math. Anal. 12 (2018), no. 4, 935–954
https://doi.org/10.1215/17358787-2018-0005
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

ON APPROXIMATION PROPERTIES OF l_1-TYPE SPACES

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Communicated by M. Mastyło

ABSTRACT. Let $(X_n\|\cdot\|_n)$ denote a sequence of real Banach spaces. Let

$$X=\bigoplus_1 X_n=\Big\{(x_n): x_n\in X_n \text{ for any } n\in\mathbb{N}, \sum_{n=1}^{\infty}\|x_n\|_n<\infty\Big\}.$$

In this article, we investigate some properties of best approximation operators associated with finite-dimensional subspaces of X. In particular, under a number of additional assumptions on (X_n), we characterize finite-dimensional Chebyshev subspaces Y of X. Likewise, we show that the set

$$\text{Nuniq}=\big\{x\in X: \text{card}\big(P_Y(x)\big)>1\big\}$$

is nowhere dense in Y, where P_Y denotes the best approximation operator onto Y. Finally, we demonstrate various (mainly negative) results on the existence of continuous selection for metric projection and we provide examples illustrating possible applications of our results.

1. Introduction

Let $(X,\|\cdot\|)$ be a Banach space, and let $Y\subset X$ be a nonempty subset. Denote by S_X (resp., B_X) the unit sphere (resp., the closed unit ball) in X. For $x\in X$ define

$$P_Y(x)=\big\{y\in Y:\|x-y\|=\text{dist}(x,Y)\big\}.$$

Any $y\in P_Y(x)$ is called a *best approximant* in Y to x, and the mapping $x\to P_Y(x)$ is called the *metric projection*. A nonempty set $Y\subset X$ is called *proximinal* if

Received Aug. 13, 2017; Accepted Mar. 1, 2018.
First published online Jul. 10, 2018.
2010 *Mathematics Subject Classification.* Primary 41A65; Secondary 41A50, 46B20.
Keywords. Banach spaces, continuous selection for the metric projection, Chebyshev subspaces.

$P_Y(x) \neq \emptyset$ for any $x \in X$. A nonempty set Y is said to be a *Chebyshev set* if it is proximinal and $P_Y(x)$ is a singleton for any $x \in X$. A continuous mapping $S : X \to Y$ is called a *continuous selection for the metric projection* if $Sx \in P_Y(x)$ for any $x \in X$.

Let $(X_n, \|\cdot\|_n)$ be a sequence of real Banach spaces. Then define

$$X = \bigoplus_1 X_n = \Big\{(x_n) : x_n \in X_n \text{ for any } n \in \mathbb{N}, \sum_{n=1}^{\infty} \|x_n\|_n < \infty\Big\}$$

equipped with the norm

$$\big\|(x_n)\big\| = \sum_{n=1}^{\infty} \|x_n\|_n.$$

Observe that if $X_n = \mathbb{R}$ for any $n \in \mathbb{N}$, then X is equal to l_1, and if $X_n = Z$ for any $n \in \mathbb{N}$, where Z is a fixed Banach space, then X is equal to $l_1(Z)$-space. In the remainder of this article, unless otherwise stated, X will denote $\bigoplus_1 X_n$.

In this article, we first characterize finite-dimensional Chebyshev subspaces of X (see Theorem 3.2) under the assumption that all spaces $(X_n, \|\cdot\|_n)$ are strictly convex. Also, under additional assumptions on the sequence (X_n), we show that the set Nuniq is nowhere dense with respect to the norm topology in X, where (see Theorem 3.5)

$$\text{Nuniq} = \big\{x \in X : \text{card}\big(P_Y(x)\big) > 1\big\}.$$

We also present some results concerning nonexistence and existence of continuous selection for the metric projection. Observe that a large number of papers exist on the investigation of Chebyshev subspaces and various concepts of selection for the metric projection (see, e.g., [1]–[14]). For a general overview concerning these topics and other problems associated with approximation theory, we refer the reader to [15]. As a product of our considerations, we present a simple example of a 4-dimensional real Banach space and its 1-dimensional subspace Y onto which there is no continuous selection for the metric projection (see Example 3.21). Our investigation of continuous metric selection is mainly inspired by results from [8] and [11].

The article is organized as follows. Following this Introduction, Section 2 contains preliminary results and technical lemmas. The main results are presented in Section 3.

2. Preliminary results

First, we recall some well-known results for the sake of completeness and the reader's convenience.

Theorem 2.1 ([16, p. 2, Theorem 1.1])**.** *Let X be a Banach space, let $x \in X$, and let $Y \subset X$ be a linear subspace. Assume that* $\text{dist}(x, Y) > 0$. *Then, $y \in P_Y(x)$ if and only if there exists $f \in S_{X^*}$ such that $f(x - y) = \text{dist}(x, Y)$ and $f|_Y = 0$. As a consequence, if $f \in S_{X^*}$, $x \in X \setminus Y$, $f|_Y = 0$, and $f(x) = \|x\|$, then $0 \in P_Y(x)$.*

We will also frequently use the following well-known fact.

Corollary 2.2. *Let X be a Banach space, let $x \in X$, and let $Y \subset X$ be a linear subspace. Assume that* $\operatorname{dist}(x, Y) > 0$*, and let $y \in P_Y(x)$. Fix $f \in S_{X^*}$ such that $f(x-y) = \|x-y\|$ and $f|_Y = 0$. Then $w \in P_Y(x)$ if and only if $f(x-w) = \|x-w\|$.*

Proof. Note that

$$\operatorname{dist}(x, Y) = \|x - y\| = f(x - y) = f(x - w) \le \|x - w\| = \operatorname{dist}(x, Y),$$

which shows our claim. □

Lemma 2.3. *Here let Y be a closed subset of a Banach space X such that* $\dim(\operatorname{Span}(Y))$ *is finite. Assume that $x \in X$ and $P_Y(x) = \{y\}$. If $x_n \in X$ and $\|x_n - x\| \to 0$, then for any $y_n \in P_Y(x_n)$, we have $\|y_n - y\| \to 0$.*

Proof. Assume, to the contrary, that there exist $\{x_n\} \subset X$, $y_n \in P_Y(x_n)$ and $x \in X$ such that $P_Y(x) = \{y\}$, $x_n \to x$ and $\{y_n\}$ does not converge to y. Passing to a subsequence if necessary, we can assume that there exists $d > 0$ such that $\|y_n - y\| > d$. Since $x_n \to x$, $\{y_n\}$ is bounded. Since $\dim(\operatorname{Span}(Y)) < \infty$ and Y is closed, passing to a convergent subsequence if necessary, we can assume that $y_n \to z \in Y$. By the continuity of the function $x \to \operatorname{dist}(x, Y)$ we get $\|x - z\| = \operatorname{dist}(x, Y)$. Since $P_Y(x) = \{y\}$, $y = z$, which leads to a contradiction. □

We will also need the following criterion.

Theorem 2.4 (see [8, Theorem 4.5])**.** *Let X be a Banach space, and let $Y \subset X$ be a 1-dimensional subspace. Then, $Y =$ span$[y]$ does not admit a continuous selection for the metric projection if and only if there exists $x \in X$ such that $0 \in P_Y(x)$, with disjoint compact intervals I_1, I_2 and two sequences $\{x_n\}$ and $\{y_n\}$ converging to x such that for any $n \in \mathbb{N}$, $P_Y(x_n) \subset I_1 y$ and $P_Y(y_n) \subset I_2 y$.*

Let $(X_n, \|\cdot\|_n)$ be a sequence of real Banach spaces. Then, define (as in the Introduction)

$$X = \bigoplus_1 X_n = \Big\{(x_n) : x_n \in X_n \text{ for any } n \in \mathbb{N}, \sum_{n=1}^{\infty} \|x_n\|_n < \infty\Big\}$$

equipped with the norm

$$\big\|(x_n)\big\| = \sum_{n=1}^{\infty} \|x_n\|_n.$$

It is well known that X is a Banach space. Moreover, it is not difficult to see that

$$X^* = \bigoplus_\infty X_n^* = \big\{(x_n^*) : x_n^* \in X_n^* \text{ for any } n \in \mathbb{N}, \sup_n \|x_n^*\|_n^* < \infty\big\}$$

equipped with the norm

$$\big\|(x_n^*)\big\|^* = \sup_n \|x_n^*\|_n^*,$$

where for any $n \in \mathbb{N}$, $\|\cdot\|_n^*$ denotes the norm in X_n^*. It is also easy to prove the following remark.

Remark 2.5. Observe that for $x = (x_n) \in X \setminus \{0\}$ and $f = (f_n) \in S_{X^*}$, we have that $f(x) = \|x\|$ if and only if $f_i(x_i) = \|x_i\|_i$ for any $i \in \mathbb{N}$.

For an element $x \in X$, we will denote

$$\mathrm{supp}(x) = \{n \in \mathbb{N} : x_n \neq 0\}.$$

Lemma 2.6. *Let $Y \subset X$ be a linear subspace. Let $y = (y_n) \in Y \setminus \{0\}$ and $x = (x_n) \in X$ be so chosen that $[-1,1]y \subset P_Y(x)$. Then, $\mathrm{supp}(y) \subset \mathrm{supp}(x)$.*

Proof. Fix $x^* = (x_n^*) \in S_{X^*}$ such that $x^*(x) = \|x\|$ and $x^*(y) = 0$. Since $[-1,1]y \subset P_Y(x)$, by Theorem 2.1 such an x^* exists. By Corollary 2.2,

$$\|x - y\| = x^*(x - y) \qquad \text{and} \qquad \|x + y\| = x^*(x + y).$$

By Remark 2.5, if $j \in \mathbb{N} \setminus \mathrm{supp}(x)$, we have that $x_j^*(-y_j) = \|y_j\|_j$ and $x_j^*(y_j) = \|y_j\|_j$, so $y_j = 0$. As a consequence, $\mathrm{supp}(y) \subset \mathrm{supp}(x)$. □

Lemma 2.7. *Let $Y \subset X$ be a linear subspace. Let $x \in X \setminus Y$, $y = (y_n) \in Y \setminus \{0\}$ be so chosen that $[-1,1]y \subset P_Y(x)$. Fix $x^* = (x_n^*) \in S_{X^*}$ such that $x^*(x) = \|x\|$ and $x^*|_Y = 0$. Let*

$$N_+ = \big\{j \in \mathbb{N} : x_j^*(y_j) > 0\big\}$$

and

$$N_- = \big\{j \in \mathbb{N} : x_j^*(y_j) < 0\big\}.$$

Then, there exists $z = (z_n) \in X$ such that $[-1,1]y \subset P_Y(z)$ and

$$\|z_j + ay_j\|_j > x_j^*(z_j + ay_j) \quad \text{for } a < -1, j \in N_+, \tag{1}$$

and

$$\|z_j + ay_j\|_j > x_j^*(z_j + ay_j) \quad \text{for } a > 1, j \in N_-. \tag{2}$$

Proof. Since $[-1,1]y \subset P_Y(x)$, by Corollary 2.2, $x_j^*(x_j + ay_j) = \|x_j + ay_j\|_j$ for any $j \in \mathbb{N}$ and $a \in [-1,1]$. Assume that $\|x_j + ay_j\|_j = x_j^*(x_j + ay_j)$ for some $a < -1$ and $j \in N_+$. Since $j \in N_+$, there exists $a_o \le a$ such that $\|x_j + a_o y_j\|_j = x_j^*(x_j + a_o y_j)$ and $\|x_j + by_j\|_j > x_j^*(x_j + by_j)$ for $b < a_o$. Put $z_j = x_j + (a_o + 1)y_j$. Observe that

$$\|z_j - y_j\|_j = \|x_j + a_o y_j\|_j = x_j^*(x_j + a_o y_j) = x_j^*(z_j - y_j).$$

But for $b < -1$,

$$\begin{aligned} \|z_j + by_j\|_j &= \big\|x_j + (a_o + 1)y_j + by_j\big\|_j \\ &= \big\|x_j + \big(a_o + (b+1)\big)y_j\big\|_j > x_j^*(z_j + by_j), \end{aligned}$$

since $a_o + b + 1 < a_o$. Since $\{z \in X_j : x_j^*(z) = \|z\|\}$ is convex, we have that

$$\|x_j + ay_j\|_j = x^*(x_j + ay_j)$$

for $a \in [a_o, 1]$. Hence, since $a_o + 2 < 1$,

$$\begin{aligned} \|z_j + y_j\|_j &= \big\|x_j + (a_o + 2)y_j\big\|_j \\ &= x_j^*\big(x_j + (a_o + 2)y_j\big) = x_j^*(z_j + y_j). \end{aligned}$$

If $j \in N_-$, reasoning in the same way we can modify (if necessary) x_j to z_j satisfying (2) such that $\|z_j - y_j\|_j = x_j^*(z_j - y_j)$. Put $z = (z_n)$, where $z_j = x_j$ for $j \notin N_+ \cup N_-$. Observe that $x^*(z + ay) = \|z + ay\|$ for any $a \in [-1,1]$. Since $x^*|_Y = 0$, by Theorem 2.1, $[-1,1]y \subset P_Y(z)$. Also, z satisfies (1) and (2). □

Lemma 2.8. *Let, $Y \subset X$ be a linear subspace. Assume that $x = (x_1, x_2, \dots) \in X$ and $y = (y_n) \in Y \setminus \{0\}$ satisfy $[-1,1]y \subset P_Y(x)$. If $\|\cdot\|_{n_o}$ is strictly convex for some $n_o \in \operatorname{supp}(y)$, then $x_{n_o} = d_{n_o} y_{n_o}$ for some $d_{n_o} \in \mathbb{R} \setminus \{0\}$. If we additionally assume that Y is finite-dimensional and $\dim(Y) = \dim(Y_{n_o})$, then*

$$P_Y(x) \subset \operatorname{span}[y].$$

Here for $n \in \mathbb{N}$,

$$Y_n = \big\{ z_n \in X_n : z_n = y_n \textit{ for some } y = (y_1, \dots, y_n, y_{n+1}, \dots) \in Y \big\}. \tag{3}$$

Proof. Fix $x \in X$, $y \in P_Y(x)$, and $n_o \in \mathbb{N}$, satisfying the assumptions of our lemma. By Theorem 2.1 and Corollary 2.2, there exists $x^* = (x_n^*) \in S_{X^*}$ such that

$$\|x\| = \|x \pm y\| = x^*(x) = x^*(x \pm y) = \operatorname{dist}(x, Y)$$

and $x^*|_Y = 0$. Since $y \in P_Y(x)$ and $n_o \in \operatorname{supp}(y)$, by Lemma 2.6, $x_{n_o} \neq 0$. By Remark 2.5,

$$x_{n_o}^*(x_{n_o}) = \|x_{n_o}\|_{n_o}$$

and

$$x_{n_o}^*(x_{n_o} \pm y_{n_o}) = \|x_{n_o} \pm y_{n_o}\|_{n_o}.$$

If $x_{n_o} = y_{n_o}$ or $x_{n_o} = -y_{n_o}$, the lemma is proved. In the other case,

$$x_{n_o}^*\Big(\frac{x_{n_o} - y_{n_o}}{\|x_{n_o} - y_{n_o}\|_{n_o}}\Big) = 1 = \|x_{n_o}^*\|_{n_o}^*$$

and

$$x_{n_o}^*\Big(\frac{x_{n_o} + y_{n_o}}{\|x_{n_o} + y_{n_o}\|_{n_o}}\Big) = 1 = \|x_{n_o}^*\|_{n_o}^*.$$

Since $\|\cdot\|_{n_o}$ is strictly convex, we get

$$\frac{x_{n_o} - y_{n_o}}{\|x_{n_o} - y_{n_o}\|_{n_o}} = \frac{x_{n_o} + y_{n_o}}{\|x_{n_o} + y_{n_o}\|_{n_o}}. \tag{4}$$

Since $y_{n_o} \neq 0$,

$$\|x_{n_o} + y_{n_o}\|_{n_o} \neq \|x_{n_o} - y_{n_o}\|_{n_o}.$$

By (4),

$$x_{n_o} - y_{n_o} = b_{n_o}(x_{n_o} + y_{n_o}),$$

where

$$b_{n_o} = \frac{\|x_{n_o} - y_{n_o}\|_{n_o}}{\|x_{n_o} + y_{n_o}\|_{n_o}}.$$

Hence,

$$x_{n_o} = \Big(\frac{1 + b_{n_o}}{1 - b_{n_o}}\Big) y_{n_o},$$

as required ($d_{n_o} = \frac{1+b_{n_o}}{1-b_{n_o}}$). Now, assume additionally that $\dim(Y) = \dim(Y_{n_o}) = k$. Let $z_1 = y_{n_o}, z_2, \dots, z_k$ be a fixed basis of Y_{n_o}. By definition of Y_{n_o}, there exists $z^1, \dots, z^k \in Y$ such that $z_{n_o}^j = z_j$ for $j = 1, \dots, k$. Observe that the $(z^j)_{j=1}^k$ form a basis of Y. Indeed, if $\sum_{j=1}^k a_j z^j = 0$, then $\sum_{j=1}^k a_j z_j = 0$ and consequently

$a_j = 0$ for $j = 1, \dots, k$, since (z_j) is a basis of Y_{n_o}. Now, fix $w \in P_Y(x)$. Then, $w = \sum_{j=1}^k a_j z^j$ and

$$w_{n_o} = \sum_{j=1}^{k} a_j z_j. \tag{5}$$

By Remark 2.5,

$$x^*_{n_o}(x_{n_o}) = \|x_{n_o}\|_{n_o}$$

and

$$x^*_{n_o}(x_{n_o} - w_{n_o}) = \|x_{n_o} - w_{n_o}\|_{n_o}.$$

If $w_{n_o} = x_{n_o}$, then by the previous part of the proof, $w_{n_o} = d_{n_o} y_{n_o}$ and by (5), $a_1 = d_{n_o}$ *and* $a_j = 0$, for $j = 2, \dots, k$, which proves that $w = d_{n_o} y$ in this case. If $w_{n_o} \neq x_{n_o}$, then reasoning as in the previous case we get that

$$\frac{x_{n_o}}{\|x_{n_o}\|_{n_o}} = \frac{x_{n_o} - w_{n_o}}{\|x_{n_o} - w_{n_o}\|_{n_o}}.$$

Hence, by the strict convexity of $\|\cdot\|_{n_o}$ we get that

$$w_{n_o} = c x_{n_o} = c d_{n_o} y_{n_o} = c d_{n_o} z_1,$$

where $c = 1 - \frac{\|x_{n_0} - w_{n_0}\|}{\|x_{n_0}\|}$. By (5), $a_1 = c d_{n_o}$ and $a_j = 0$, for $j = 2, \dots, k$, which completes the proof of the lemma. $\square$

In the sequel, the following well-known lemma is needed.

Lemma 2.9. *Let $y^1, \dots, y^n \in X$. Then, the set $\{y^j\}_{j=1}^n$ is linearly independent if and only if there exists $i_1 < i_2 < \dots < i_n$ such that the set $\{w^j\}_{j=1}^n$ is linearly independent, where $w^j = (y^j_{i_1}, \dots, y^j_{i_n})$ for $j = 1, \dots, n$.*

3. Main results

First, we will characterize finite-dimensional Chebyshev subspaces of $X = \bigoplus_1 X_n$.

Theorem 3.1. *Let $Y \subset X$ be a linear subspace. Assume that for any $n \in \mathbb{N}$, X_n is strictly convex. Then, there exists $x \in X$ such that* $\operatorname{card}(P_Y(x)) > 1$ *if and only if there exist $y = (y_n) \in Y \setminus \{0\}$ and $x^* = (x^*_n) \in S_{X^*}$ such that $x^*|_Y = 0$ and $x^*_n(y_n) \in \{\pm\|y_n\|_n\}$ for any $n \in \mathbb{N}$.*

Proof. Assume that there exists $x \in X$ such that $\operatorname{card}(P_Y(x)) > 1$. Let $w, z \in P_Y(x)$ and $w \neq z$. Since Y is a convex set, the segment $[w, z] \subset P_Y(x)$, where $[w, z] = \{aw + (1 - a)z : a \in [0, 1]\}$. Let $x^1 = x - \frac{w+z}{2}$. Since Y is a linear subspace of X, $P_Y(x^1) = P_Y(x) - \frac{w+z}{2}$. Hence, $w - \frac{w+z}{2} = \frac{w-z}{2} \in P_Y(x^1)$ and $z - \frac{w+z}{2} = \frac{z-w}{2} \in P_Y(x^1)$. Put $y = \frac{w-z}{2}$. Then, the segment $[-y, y] \subset P_Y(x^1)$ and $y \neq 0$. Since $0 \in P_Y(x^1)$, by Theorem 2.1 we can select $x^* \in S_{X^*}$ such that $x^*(x) = \|x\|$ and $x^*|_Y = 0$. By Lemma 2.5, $x^*_n(x_n) = \|x_n\|_n$ for any $n \in \mathbb{N}$. Since for any $n \in \mathbb{N}$, X_n is strictly convex, by Lemma 2.8, $x_n = d_n y_n$ for any $n \in \operatorname{supp}(y)$. Moreover, by Lemma 2.6, $d_n \neq 0$ for any $n \in \operatorname{supp}(y_n)$. Hence, $x^*_n(y_n) \in \{\pm\|y_n\|_n\}$ for any $n \in \mathbb{N}$, as required.

Now, assume that there exist $y = (y_n) \in Y \setminus \{0\}$ and $x^* = (x_n^*) \in S_{X^*}$ such that $x^*|_Y = 0$ and $x_n^*(y_n) \in \{\pm\|y_n\|_n\}$ for any $n \in \mathbb{N}$. Set for $n \in \mathbb{N}$, $x_n = y_n$ if $x^*(y_n) = \|y_n\|_n$ and $x_n = -y_n$ in the opposite case. Let $x = (x_n)$. Note that

$$x^*(x) = \sum_{n=1}^{\infty} x_n^*(x_n) = \sum_{n=1}^{\infty} \|x_n\|_n = \|x\|.$$

By Theorem 2.1, $0 \in P_Y(x)$. Moreover, by definition of x, for any $n \in \mathbb{N}$, $x_n^*(x_n \pm y_n) = \|x_n \pm y_n\|_n$. By Corollary 2.2 and Theorem 2.1, $[-y, y] \subset P_Y(x)$. Since $y \neq 0$, the proof is complete. □

If we additionally assume that Y is finite-dimensional, by Theorem 3.1 we immediately get the following.

Theorem 3.2. *Let $Y \subset X$ be a finite-dimensional subspace. Assume that for any $n \in \mathbb{N}$, X_n is strictly convex. Then, Y is not a Chebyshev subspace if and only if there exist $y = (y_n) \in Y \setminus \{0\}$ and $x^* = (x_n^*) \in S_{X^*}$ such that $x^*|_Y = 0$ and $x_n^*(y_n) \in \{\pm\|y_n\|_n\}$ for any $n \in \mathbb{N}$.*

Corollary 3.3. *Let $Y = \operatorname{span}[y]$ be a 1-dimensional subspace of X generated by $y = (y_n) \in X \setminus \{0\}$. Assume that for any $n \in \mathbb{N}$, X_n is strictly convex. Then, Y is not a Chebyshev subspace if and only if there exists $\sigma \in \{-1, 1\}^{\mathbb{N}}$ such that*

$$\sum_{n=1}^{\infty} \sigma_n \|y_n\|_n = 0. \tag{6}$$

Proof. Observe that if $Y = \operatorname{span}[y]$, then (6) is equivalent to the fact that there exist $x^* = (x_n^*) \in S_{X^*}$ and $y = (y_n) \in Y \setminus \{0\}$ such that $x^*|_Y = 0$ and $x_n^*(y_n) \in \{\pm\|y_n\|_n\}$ for any $n \in \mathbb{N}$. By Theorem 3.1, we get our result. □

Corollary 3.4. *Let $Y \subset X$ be a k-dimensional ($k \geq 2$) subspace spanned by $y^1, \dots, y^k$ having disjoint supports. Assume that for any $n \in \mathbb{N}$, X_n is strictly convex. Then, Y is not a Chebyshev subspace if and only if for some $j \in \{1, \dots, k\}$, $W_j = \operatorname{span}[y^j]$ is not a Chebyshev subspace.*

Proof. First, assume that for some $j \in \{1, \dots, k\}$, $W_j = \operatorname{span}[y^j]$ is not a Chebyshev subspace of X. By Corollary 3.3, there exists $\sigma \in \{-1, 1\}^{\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \sigma_n \|y_n^j\|_n = 0$. Define for $n \in \mathbb{N}$,

$$x_n = \begin{cases} -y_n^j & \text{if } n \in \operatorname{supp}(y^j), \sigma_n = -1, \\ y_n^j & \text{if } n \in \operatorname{supp}(y^j), \sigma_n = 1, \\ 0 & \text{if } n \notin \operatorname{supp}(y^j). \end{cases}$$

Put $x = (x^n)$. Observe that $[-y^j, y^j] \subset P_{W_j}(x)$. Since $\operatorname{supp}(x) = \operatorname{supp}(y^j)$ and $y^1, \dots, y^k$ have disjoint supports, $\operatorname{dist}(x, W_j) = \operatorname{dist}(x, Y) = \|x - ay^j\|$ for any $a \in [-1, 1]$. Hence, Y is not a Chebyshev subspace. Now assume that for any $j \in \{1, \dots, k\}$, W_j is a Chebyshev subspace of X and Y is a not a Chebyshev subspace of X. By Theorem 3.2, there exist $y = (y_n) \in Y \setminus \{0\}$ and $x^* = (x_n^*) \in S_{X^*}$ such that $x^*|_Y = 0$ and $x_n^*(y_n) \in \{\pm\|y_n\|_n\}$ for any $n \in \mathbb{N}$. Since $y = \sum_{j=1}^{k} a_j y^j \neq 0$, $a_j \neq 0$ for some $j \in \{1, \dots, k\}$. Since $y^1, \dots, y^k$ have disjoint supports, $x_n^*(y_n^j) \in \{\pm\|y_n^j\|_n\}$ for any $n \in \mathbb{N}$. Since $x^*(y^j) = 0$, by Theorem 3.2, W_j is not a Chebyshev subspace of X, which leads to a contradiction. □

Now, we show that under some additional assumptions on $X = \bigoplus_1 X_n$ and $Y \subset X$ being a finite-dimensional subspace of X, the set

$$\mathrm{Nuniq} = \big\{x \in X : \mathrm{card}\big(P_Y(x)\big) > 1\big\} \tag{7}$$

is nowhere dense in X, that is, $\mathrm{int}(\mathrm{cl}(\mathrm{Nuniq})) = \emptyset$, where the closure and the interior are taken with respect to the norm topology in X.

Theorem 3.5. *Let $Y \subset X$ be a k-dimensional subspace of X. Fix $i_1 < i_2 < \cdots < i_k$ such that the vectors w^j from Lemma 2.9 are linearly independent. For each $j \in \mathbb{N}$, we denote by π_j the projection from X onto X_j given by $\pi_j(x) = x_j$. Set as in Lemma 2.8, for $j = 1, \dots, k$, $Y_j = \pi_{i_j}(Y)$. Assume that for any $j \in \{1, \dots, k\}$, Y_j is a proper subspace of X_{i_j} and that X_{i_j} are strictly convex for $j = 1, \dots, k$. Then, the set* Nuniq *defined by (7) is nowhere dense in X.*

Proof. Without loss of generality, we can assume that $i_j = j$ for $j = 1, \dots, k$. Define for $j = 1, \dots, k$,

$$P_j = \bigoplus_1 (Z_n)_{n=1}^{\infty},$$

where $Z_n = X_n$ for $n \neq j$ and $Z_j = Y_j$. First, we show that $\mathrm{Nuniq} \subset \bigcup_{j=1}^k P_j$. Let $x \in \mathrm{Nuniq}$. Then, there exist $w, z \in P_Y(x)$, $w \neq z$. Since the vectors w^j from Lemma 2.9 are linearly independent, $w_j \neq z_j$ for some $j \in \{1, \dots, k\}$. Let $x^1 = x - \frac{w+z}{2}$. Since Y is a linear subspace, $P_Y(x^1) = P_Y(x) - \frac{w+z}{2}$. Hence, the segment $[-y, y] \subset P_Y(x^1)$, where $y = \frac{w-z}{2}$. Since $w_j \neq z_j$ for some $j \leq k$, then $y_j \neq 0$. Applying Lemma 2.8 to x^1, we get that for some $j \leq k$, $x_j^1 = d_j y_j$ for some $d_j \neq 0$ and $y_j \in Y_j$. Hence,

$$x_j = d_j y_j + \frac{(w+z)_j}{2} \in Y_j,$$

which shows that $x_j \in Y_j$ and consequently $x \in P_j$.

To end our proof, we show that $\bigcup_{j=1}^k P_j$ is nowhere dense in X. First, we show that each set P_j is closed in X with respect to the norm topology. Since π_i is continuous for every $i \in \mathbb{N}$, and for each $j \in \mathbb{N}$ fixed $\pi_i(P_j)$ coincides with X_i or Y_i and $Y_i = \pi_i(Y)$ is a finite-dimensional subspace, $\pi_i(P_j)$ is closed in X_i. It clearly follows that P_j is closed in X for each $j \in \mathbb{N}$. Now, we show that $\mathrm{int}(\bigcup_{j=1}^k P_j) = \emptyset$. Assume that this is not true. Then, there exist $x = (x_n) \in \bigcup_{j=1}^k P_j$ and $r > 0$ such that $x + rB_X \subset \bigcup_{j=1}^k P_j$. Since $x + rB_X$ is a complete metric space (with the topology determined by the norm in X), by the Baire property, $\mathrm{int}(P_{j_o}) \neq \emptyset$ for some $j_o \in \{1, \dots, k\}$. This implies that Y_{j_0} has nonempty interior in X_{j_0}, since Y_{j_0} contains the ball centered at x_{j_0} with radius r. However, since Y_{j_0} is a proper subspace of $X_{j!_0}$, it has empty interior, which leads to a contradiction.

Finally, note that

$$\mathrm{cl}(\mathrm{Nuniq}) \subset \mathrm{cl}\Big(\bigcup_{j=1}^{k} P_j\Big) = \bigcup_{j=1}^{k} \mathrm{cl}(P_j) = \bigcup_{j=1}^{k} P_j$$

and $\mathrm{int}(\mathrm{cl}(\mathrm{Nuniq})) \subset \mathrm{int}(\bigcup_{j=1}^{k} P_j) = \emptyset$, as required. □

Corollary 3.6. *Let $Y \subset X$ be a finite-dimensional subspace. Assume that for any $n \in \mathbb{N}$, X_n is strictly convex and Y_n is a proper subspace of X_n. Then, the set* Nuniq *is nowhere dense in X.*

Proof. This follows immediately from Theorem 3.5. □

Corollary 3.7. *Let $y \in X \setminus \{0\}$, and let $Y = \mathrm{span}[y]$. Assume that there exists $n \in \mathrm{supp}(y)$ such that X_n is strictly convex and $\dim(X_n) > 1$. Then,* Nuniq *is nowhere dense in X.*

Proof. Put $i_1 = n$. Then, the result follows from Theorem 3.5. □

Observe that the assumption $\dim(X_n) > 1$ in Corollary 3.7 is essential because of the following example.

Example 3.8. Let $X = \ell_1$; that is, $X_n = \mathbb{R}$ for any $n \in \mathbb{N}$. Fix $y \in X \setminus \{0\}$ such that $\mathrm{supp}(y) = \{1, \ldots, n_o\}$, for some $n_o > 1$, $|y_{n_o}| = \min\{|y_n| : n = 1, \ldots, n_o\}$, and

$$\sum_{n=1}^{\infty} y_n = 0. \tag{8}$$

Let $Y = [y]$. Fix $c > 1$, and define $x_n = cy_n$ if $y_n \geq 0$ and $x_n = -cy_n$ if $y_n < 0$. Let $x = (x_n)$. Now, we prove that for any $z = (z_n) \in X$ such that

$$\|z - x\| < \frac{(c-1)|y_{n_o}|}{2}, \tag{9}$$

$[-y, y] \subset P_y(z)$. Fix $z \in X$ satisfying (9), and observe that for any $n \in \{1, \ldots, n_o\}$, $z_n - |y_n| > 0$. Indeed,

$$\begin{aligned} z_n - |y_n| &= z_n - x_n + x_n - |y_n| \geq (c-1)|y_n| - |z_n - x_n| \\ &\geq (c-1)|y_n| - \frac{(c-1)|y_{n_o}|}{2} \geq \frac{(c-1)|y_{n_o}|}{2} > 0. \end{aligned}$$

Define $x^* = (1, \ldots, 1_{n_o}, \mathrm{sgn}(z_{n_o+1}), \mathrm{sgn}(z_{n_o+2}), \ldots)$. Observe that $x^* \in S_{X^*}$ and $x^*(z \pm y) = \|z \pm y\|$ and by (8), $x^*(y) = 0$. By Theorem 2.1, $[-y, y] \subset P_Y(z)$, as required. Hence, the set Nuniq has nonempty interior.

From Corollary 3.6, we can easily obtain the following.

Corollary 3.9. *Let $Y \subset X$ be a finite-dimensional subspace. Assume that for any $n \in \mathbb{N}$, X_n is strictly convex and Y_n is a proper subspace of X_n. Let $S : X \to Y$ be a selection for the metric projection (i.e., $S(x) \in P_Y(x)$ for any $x \in X$). Then, the set of points in which S is discontinuous is nowhere dense in X.*

Proof. By Corollary 3.6, the set Nuniq is nowhere dense in X. By Lemma 2.3, S is continuous at any $x \in X \setminus \mathrm{Nuniq}$, which completes the proof. □

The next results show that, in general, if the set Nuniq is nonempty, the existence of a *continuous* selection for the metric projection is rather a rare situation. We start with the following theorem.

Theorem 3.10. *Let X, Y, x, y, x^*, N_-, N_+ be as in Lemma 2.7. Assume additionally that* $\dim(Y) = 1$ *and that x can be so chosen that N_- and N_+ are infinite. Then, there is no continuous selection for the metric projection onto Y.*

Proof. By Lemma 2.7 we can assume that x satisfies (1) and (2). Define for $n \in \mathbb{N}$,

$$x^n = x - 2y_n e_n \quad \text{for } n \in N_+, \tag{10}$$

$$z^n = x + 2y_n e_n \quad \text{for } n \in N_-, \tag{11}$$

and $x^n = x$ otherwise, where e_n is a sequence associated to the characteristic function of $\{n\}$ for each $n \in \mathbb{N}$. Observe that for $n \in N_+$,

$$x^n + y = (x_1 + y_1, \ldots, x_{n-1} + y_{n-1}, x_n - y_n, x_{n+1} + y_{n+1}, \ldots).$$

Since $P_Y(x) = [-1, 1]y$ and $x^*(x) = \|x\|$, by Corollary 2.2 we get that $x^*(x^n + y) = \|x^n + y\|$. Since $x^*(y) = 0$, by Theorem 2.1,

$$\|x^n + y\| = \operatorname{dist}(x^n, Y) \tag{12}$$

for any $n \in N_+$. Reasoning in the same way, we get that

$$\|z^n - y\| = \operatorname{dist}(z^n, Y)$$

for any $n \in N_-$. Observe that for $n \in N_+$ and $a > -1$, $ay \notin P_Y(x^n)$. Indeed, by (1),

$$x_n^*(x_n^n - ay_n) = x_n^*(x_n - 2y_n - ay_n) = x_n^*\big(x_n + (-2 - a)y_n\big) < \|x_n^n - ay_n\|_n,$$

since $a > -1$ if and only if $-(2 + a) < -1$. Consequently, since $x^*(y) = 0$ and N_+ is infinite, by (12), we obtain that

$$\|x^n - ay\| > \sum_{j=1}^{\infty} x_j^*(x_j^n - ay_j) = \sum_{j=1}^{\infty} x_j^*(x_j^n + y_j) = x^*(x_n + y) = \operatorname{dist}(x^n, Y).$$

Analogously, for $a < 1$ and $n \in N_-$,

$$x_n^*(z_n^n - ay_n) = x_n^*(x_n + 2y_n - ay_n) = x_n^*\big(x_n + (2 - a)y_n\big) < \|z_n^n - ay_n\|_n,$$

since $a < 1$ if and only if $2 - a > 1$. Consequently, for $n \in N_-$ and $a < 1$, $ay \notin P_Y(z^n)$. As a consequence, for $n \in N_+$, $P_Y(x^n) \subset (-\infty - 1]y$ and for $n \in N_-$, $P_Y(z^n) \subset [1, +\infty)y$. Since N_+ and N_- are infinite,

$$\lim_{n \in N_+} \|x^n - x\| = \lim_{n \in N_+} 2\|y_n\|_n = 0$$

and

$$\lim_{n \in N_-} \|z^n - x\| = \lim_{n \in N_-} 2\|y_n\|_n = 0.$$

Consequently, there is no continuous selection for the metric projection onto Y. $\square$

The following example provides 1-dimensional subspaces such that they do not admit a continuous selection for the metric projection.

Example 3.11. Fix $x_n \in S_{X_n}$ and $x_n^* \in \mathrm{ext}(S_{X_n^*})$ for $n \in \mathbb{N}$ such that $x_n^*(x_n) = 1$. Let $y_n = x_n/2^n$ for $n = 2k$, and let $y_n = -x_n/2^n$ for $n = 2k+1$, $k \geq 1$. Let $y_1 = ax_1$, where $a \in \mathbb{R}$ is so chosen that

$$\sum_{n=2}^{\infty} x_n^*(y_n) + ax_1^*(x_1) = 0.$$

Put $y = (y_n)$, $z = (|a|x_1, x_2/4, \ldots, x_n/2^n, \ldots)$, and $x^* = (x_n^*)$. Let $Y = \mathrm{span}[y]$. It is clear that $x^*(z) = \sum_{j=1}^{\infty} x_n^*(z_n) = \|z\|$. Moreover, $x^*(y) = 0$. Note that

$$\|z \pm y\| = \sum_{n=1}^{\infty} \|z_n \pm y_n\|_n = \sum_{n=1}^{\infty} x_n^*(z_n \pm y_n) = x^*(z \pm y),$$

since for any $n \in \mathbb{N}$, $z_n - y_n = 0$ or $z_n - y_n = 2z_n$. Moreover, $x^*(y) = 0$. By Theorem 2.1, $P_Y(z) = [-1,1]y$. It is clear that, in this case, $N_+ = \{2k : k \in \mathbb{N}\}$ and $\{2k+1 : k \in \mathbb{N}, k \geq 1\} \subset N_-$. By Theorem 3.10, there is no continuous selection for the metric projection onto Y.

Observe that under additional (not very restrictive) assumptions, we can prove Theorem 3.10 not only for 1-dimensional subspaces.

Theorem 3.12. *Let X, Y, x, x^*, y, N_-, N_+ be as in Lemma 2.7. Assume that x can be so chosen that N_- and N_+ are infinite. If there exists $n_o \in \mathrm{supp}(y)$ such that X_{n_o} is strictly convex and $\dim(Y_{n_o}) = \dim(Y)$, where $Y_{n_o} = \pi_{n_0}(Y)$ is defined by (3), then there is no continuous selection for the metric projection onto Y.*

Proof. The proof is similar to that of Theorem 3.10. By Lemma 2.7, we can modify x in such a way that $P_Y(x) \cap \mathrm{span}[y] = [-1,1]y$. By Lemma 2.8, $P_Y(x) = [-1,1]y$. Let x^n and z^n be defined by (10) and (11). Now, we show that $P_Y(x^n) \subset \mathrm{span}[y]$ and $P_Y(z^n) \subset \mathrm{span}[y]$ for $n > n_o$. Assume on the contrary that there exist $n_1 \in N_-$, $n_1 > n_o$ and $w = (w_n) \in Y \setminus \mathrm{span}[y]$ such that $w \in P_Y(x^{n_1})$. Since $\dim(Y) = \dim(Y_{n_o})$ and $n_o \in \mathrm{supp}(y)$, $w_{n_o} \notin \mathrm{span}[y_{n_o}]$. Hence, by Corollary 2.2, $\|x^{n_1} - w\| = x^*(x^{n_1} - w)$. By Remark 2.5 applied to $x \pm y$ and $x^n - w$, since $n_o < n_1$, we get

$$x_{n_o}^*(x_{n_o} \pm y_{n_o}) = \|x_{n_o} \pm y_{n_o}\|_{n_o}$$

and

$$x_{n_o}^*(x_{n_o} - w_{n_o}) = \|x_{n_o} - w_{n_o}\|_{n_o}.$$

Hence, reasoning as in Lemma 2.8, we get that $w_{n_o} = dy_{n_o}$ for some $d \in \mathbb{R}$, which is a contradiction. In the same way, we can show that $P_Y(z^n) \subset \mathrm{span}[y]$. By the proof of Theorem 3.10 applied to 1-dimensional subspace $\mathrm{span}[y]$, we get our result. ☐

Theorem 3.13. *Let X, Y, x, x^*, y, N_-, N_+ be as in Lemma 2.7. Assume that x can be chosen such that N_- and N_+ are infinite and that for any $n \in N_+ \cup N_-$,*

X_n is strictly convex. Assume, furthermore, that there exists a basis $y^1, \dots, y^m$ of Y such that $y^1 = y$ and for $j = 2, \dots, m$,

$$\lim_{n \in N_-} \frac{\|y_n^j\|_n}{\|y_n\|_n} = 0, \tag{13}$$

$$\lim_{n \in N_+} \frac{\|y_n^j\|_n}{\|y_n\|_n} = 0. \tag{14}$$

Then, there is no continuous selection for the metric projection onto Y.

Proof. By Lemma 2.8, for any $n \in N_- \cup N_+$, $x_n = d_n y_n$ with $d_n \neq 0$ for any $n \in N_- \cup N_+$, and consequently for any $n \in N_-$, $x_n^*(y_n) = -\|y_n\|_n$ and for any $n \in N_+$, $x_n^*(y_n) = \|y_n\|_n$. Define $z = (z_n) \in X$ by $z_j = y_j$ for $j \in N_+$ and $z_j = -y_j$ otherwise. Observe that $x^*(z) = \sum_{j=1}^{\infty} \|y_j\|_j = \|z\|$ and $x^*|_Y = 0$. Hence, by Theorem 2.1, $0 \in P_Y(z)$. Define, as in Theorem 3.10,

$$x^n = z - 2y_n e_n \quad \text{for } n \in N_+,$$
$$z^n = z + 2y_n e_n \quad \text{for } n \in N_-.$$

It is clear that $(\|x^n - z\|)_{n \in N_+} \to 0$ and $(\|z^n - z\|)_{n \in N_-} \to 0$. Observe that for $n \in N_+$, $x_n^n + y_n = 0$ and $x_j^n + y_j = z_j + y_j$ for $j \neq n$. Also for $n \in N_-$, $z_n^n - y_n = 0$ and $z_j^n - y_j = z_j - y_j$ for $j \neq n$. Since $x^*|_Y = 0$, by Theorem 2.1, $-y \in P_Y(x^n)$ for any $n \in N_+$ and $y \in P_Y(z^n)$ for any $n \in N_-$. We will argue by contradiction. Now assume that there exists a continuous selection for the metric projection $S : X \to Y$. Then, $S(z^n) \to S(z)$ and $S(x^n) \to S(z)$. By Corollary 2.2,

$$x^*\big(x^n - S(x^n)\big) = \big\|x^n - S(x^n)\big\| \quad \text{for any } n \in N_+ \tag{15}$$

and

$$x^*\big(z^n - S(z^n)\big) = \big\|z^n - S(z^n)\big\| \quad \text{for any } n \in N_-. \tag{16}$$

Let $S(z) = ay + \sum_{j=2}^m a_j y^j$, $S(x^n) = a_n y + \sum_{j=2}^m a_{n,j} y^j$ for $n \in N_+$, and $S(z^n) = b_n y + \sum_{j=2}^m b_{n,j} y^j$ for $n \in N_-$. Since $\|S(x^n) - S(z)\| \to 0$ and $\|S(z^n) - S(z)\| \to 0$,

$$a_n \to a, \qquad a_{n,j} \to a_j \quad \text{for } j = 2, \dots, m \tag{17}$$

and

$$b_n \to a, \qquad b_{n,j} \to a_j \quad \text{for } j = 2, \dots, m. \tag{18}$$

Now, we show that $a \le -1$. Since for any $n \in N_+$, $x_n^n = -y_n$, by Remark 2.5 and (15) we get

$$\big\|x_n^n - S(x^n)_n\big\|_n = x_n^*\big(x_n^n - S(x^n)_n\big) = x_n^*\Big((-1 - a_n)y_n - \sum_{j=2}^m a_{n,j} y_n^j\Big). \tag{19}$$

Since $x_n^*(y_n) = \|y_n\|_n$, by (14) and (17),

$$\frac{x_n^*((-1 - a_n)y_n - \sum_{j=2}^m a_{n,j} y_n^j)}{\|y_n\|_n} \to_{n \in N_+} -(1 + a).$$

By (19), $-(1+a) \geq 0$, and consequently $a \leq -1$, as required. To get a contradiction with the existence of continuous selection for the metric projection S, we show that $a \geq 1$. Since for any $n \in N_-$, $z_n^n = y_n$, by Remark 2.5 and (16),

$$\left\|z_n^n - S(z^n)_n\right\|_n = x_n^*\left(z_n^n - S(z^n)_n\right) = x_n^*\Big((1-b_n)y_n - \sum_{j=2}^m b_{n,j}y_n^j\Big). \tag{20}$$

Since $x_n^*(y_n) = -\|y_n\|_n$, by (13) and (18),

$$\frac{x_n^*((1-b_n)y_n - \sum_{j=2}^m b_{n,j}y_n^j)}{\|y_n\|_n} \to_{n\in N_-} -(1-a).$$

By (20), $-(1-a) \geq 0$, and consequently $a \geq 1$, as required. □

The following modification of Example 3.11 provides a possible application of Theorem 3.13.

Example 3.14. Let X_n be strictly convex for $n \in \mathbb{N}$. Let x^*, y be as in Example 3.11. Let $y^1 = y, y^2, \ldots, y^m \in \ker(x^*)$ be linearly independent vectors. Assume that for $j = 2, \ldots, m$, $\operatorname{supp}(y^j)$ is finite. Let $Y = \operatorname{span}[y^j, j = 1, \ldots, m]$. Then, applying Theorem 3.13 to $x = (|a|x_1, x_2/4, \ldots, x_n/2^n, \ldots)$ from Example 3.11, we can deduce that there is no continuous selection for the metric projection onto Y.

Now, we apply Theorem 3.13 to certain finite-dimensional subspaces of l_1.

Example 3.15. Let $X = l_1$. Let $y \in l_1 \setminus \{0\}$ be so chosen that $\sum_{n=1}^\infty y_n = 0$. Assume that N_+ and N_- are infinite, where $N_+ = \{n \in \mathbb{N} : y_n > 0\}$ and $N_- = \{n \in \mathbb{N} : y_n < 0\}$. Fix $y^2, \ldots, y^m$ such that $\sum_{n=1}^\infty y_n^j = 0$ for $j = 2, \ldots, m$ satisfying (13) and (14). (In our case $\|y_n\|_n = |y_n|$ for any $n \in \mathbb{N}$.) Let $Y = \operatorname{span}[y^j, j = 1, 2, \ldots, m]$. Then, by Theorem 3.13 there is no continuous, metric selection onto Y. In particular, if $\operatorname{supp}(y^j)$ is finite for $n = 2, \ldots, m$, there is no continuous selection for the metric projection onto Y.

Now, we present a class of 1-dimensional, non-Chebyshev subspaces of X onto which there exists a continuous selection for the metric projection. We start with the following.

Proposition 3.16. *Let $X = \bigoplus_1 X_n$. Let $Y = \operatorname{span}[y]$, where $y = (y_n) \in X \setminus \{0\}$. Assume that for any $n \in \operatorname{supp}(y)$, $(X_n, \|\cdot\|_n)$ is a smooth Banach space. Assume that there exist $n_o \in \mathbb{N}$ and $x^* = (x_n^*) \in S_X$ such that $\{1, \ldots, n_o\} \subset \operatorname{supp}(y)$, $x^*(y) = 0$, and*

$$x_n^*(y_n) = \begin{cases} -\|y_n\|_n & \text{for } n \leq n_o, \\ \|y_n\|_n & \text{for } n \geq n_o + 1. \end{cases}$$

Let $x = (x_n) \in X$ be such that $x_n = c_n y_n$, where $c_n \leq -1$ for $n = 1, \ldots, n_o$, $c_{m_o} = -1$ for some $m_o \in \{1, \ldots, n_o\}$, and $c_n \geq 1$ for $n \geq n_o + 1$. Assume that there is a sequence (z^j) in X converging to x such that for any $j \in \mathbb{N}$,

$$z^j = (d_{j,1}x_1, \ldots, d_{j,n_o}x_{n_o}, z_{n_o+1}^j, \ldots, z_n^j, \ldots). \tag{21}$$

Let $b_j = \inf\{b \in \mathbb{R} : by \in P_Y(z^j)\}$. Then, $b_j \to -1$.

Proof. First, assume that $\operatorname{supp}(y) = \mathbb{N}$. Observe that by our assumptions, $x^*(y) = 0$ and $x^*(x) = \|x\|$. Therefore, by Theorem 2.1, $0 \in P_Y(x)$. First, we show that

$$\inf\big\{b \in \mathbb{R} : by \in P_Y(x)\big\} = -1. \tag{22}$$

We claim that $x^*(x - by) = \|x - by\|$ for any $b \in [-1, 1]$. Indeed, for any $n \in \mathbb{N}$, $\|x_n \pm y_n\| = x_n^*(x_n \pm y_n)$. Thus, by Remark 2.5 we obtain our claim. By Theorem 2.1, $[-1, 1]y \subset P_Y(x)$. Observe that if $b < -1$, then

$$x^*_{m_o}(x_{m_o} - by_{m_o}) = x^*_{m_o}\big((-1-b)y_{m_o}\big) = -\|x_{m_o} - by_{m_o}\|_{m_o}.$$

By Corollary 2.2 and Remark 2.5, $by \notin P_Y(x)$, as required. Let $z^j \to x$, satisfy (21). Let for $j \in \mathbb{N}$,

$$b_j = \inf\big\{b \in \mathbb{R} : by \in P_Y(z^j)\big\}. \tag{23}$$

Since $P_Y(z^j)$ is closed, $b_j y \in P_Y(z^j)$. We show that $b_j \to -1$. Assume that this is not true. By (22), passing to a convergent subsequence if necessary, there exists $b \in (-1, c]$, where $c = \sup\{d \in \mathbb{R} : dy \in P_Y(x)\}$, such that $b_j \to b$. Fix $\epsilon > 0$ such that $b - \epsilon > -1$. We claim that $(b-\epsilon)y \in P_Y(z^j)$ for $j \geq j_o$. By Theorem 2.1, there exists $z^{*,j} = (z_n^{*,j}) \in X^*$ a norming functional for $z^j - b_j y$ such that $z^{*,j}(y) = 0$. Assume we have proved that for $j \geq j_o$,

$$\big\|z^j - (b-\epsilon)y\big\| = z^{*,j}\big(z^j - (b-\epsilon)y\big). \tag{24}$$

Then, by Theorem 2.1, $(b - \epsilon)y \in P_Y(z^j)$ for $j \geq j_o$. Since

$$b_j \to b > b - \epsilon,$$

we get a contradiction with (23) for $j \geq j_o$. Hence to finish our proof, we need to show (24). Observe that, by Remark 2.5 and our assumptions for any $n \in \{1, \dots, n_o\}$,

$$z_n^{*,j}(z_n^j - b_j y_n) = z_n^{*,j}(d_{j,n}x_n - b_j y_n) = \|d_{j,n}c_n y_n - b_j y_n\|_n. \tag{25}$$

Since $\|z^j - x\| \to 0$, $z_n^j = d_{j,n}x_n \to_j x_n = c_n y_n$, and consequently $d_{j,n} \to_j 1$ for $n = 1, \dots, n_o$. Since $b \in (-1, c]$, with $c \geq 1$,

$$d_{j,n}c_n - b_j \to_j c_n - b \leq -1 - b < 0$$

for $n = 1, \dots, n_o$. Hence, by (25), for $j \geq j_o$ and $n = 1, \dots, n_o$,

$$z_n^{*,j}(d_{j,n}x_n - b_j y_n) = \big\|(d_{j,n}c_n - b_j)y_n\big\|_n = x_n^*(d_{j,n}c_n y_n - b_j y_n) = \|d_{j,n}x_n - b_j y_n\|_n.$$

Since the X_n are smooth, $z_n^{*,j} = x_n^*$ for $n = 1, \dots, n_o$ and $j \geq j_o$. Consequently, for $j \geq j_o$,

$$0 = x^*(y) = \sum_{n=1}^{n_o} x_n^*(y_n) + \sum_{n=n_o+1}^{\infty} x_n^*(y_n)$$

and

$$0 = z^{*,j}(y) = \sum_{n=1}^{n_o} x_n^*(y_n) + \sum_{n=n_o+1}^{\infty} z_n^{*,j}(y_n).$$

Hence, since $x_n^*(y_n) = \|y_n\|$ for $n \geq n_o + 1$,

$$\sum_{n=n_o+1}^{\infty} \|y_n\|_n = \sum_{n=n_o+1}^{\infty} x_n^*(y_n) = \sum_{n=n_o+1}^{\infty} z_n^{*,j}(y_n).$$

Since $\mathrm{supp}(y) = \mathbb{N}$, and X_n are smooth, $z_n^{*,j} = x_n^*$ for $n > n_o$ and $j \geq j_o$ which shows that $x^* = z^{*,j}$ for $j \geq j_o$. Note that for $n = 1, \dots, n_o$ and $j \geq j_o$, $d_{j,n}c_n < b - \epsilon$, since $d_{j,n}c_n \to_j c_n \leq -1$. Hence,

$$\begin{aligned} z_n^{*,j}\big(z_n^j - (b-\epsilon)y_n\big) &= x_n^*\big(z_n^j - (b-\epsilon)y_n\big) \\ &= x_n^*\big(d_{j,n}c_n y_n - (b-\epsilon)y_n\big) \\ &= x_n^*\big((\epsilon - b + d_{j,n}c_n)y_n\big) \\ &= \big\|(\epsilon - b + d_{j,n}c_n)y_n\big\|_n \\ &= \big\|z_n^j - (b-\epsilon)y_n\big\|_n, \end{aligned}$$

since for $n = 1, \dots, n_o$, $x_n^*(-y_n) = \|-y_n\|_n$.

Note that for $n > n_o$ and $j \geq j_o$,

$$\begin{aligned} z_n^{*,j}\big(z_n^j - (b-\epsilon)y_n\big) = x_n^*\big(z_n^j - (b-\epsilon)y_n\big) &= x_n^*(z_n^j - b_j y_n) + x_n^*\big((b_j - b + \epsilon)y_n\big) \\ &= \|z_n^j - b_j y_n\|_n + \big\|(b_j - b + \epsilon)y_n\big\|_n \\ &= \big\|z_n^j - (b-\epsilon)y_n\big\|_n, \end{aligned}$$

since for $n > n_o$, $x_n^*(y_n) = \|y_n\|$ and $b_j - b + \epsilon > 0$ for $j \geq j_o$. Consequently, $\|z^j - (b-\epsilon)y\| = z^{*,j}(z^j - (b-\epsilon)y)$ for $j \geq j_o$, which proves (24), as required.

Now assume that $\mathrm{supp}(y) \neq \mathbb{N}$. Reasoning as in the previous part of the proof, we can show that for $j \geq j_o$ and $n \in \mathrm{supp}(y)$,

$$z_n^{*,j}\big(z_n^j - (b-\epsilon)y_n\big) = \big\|z_n^j - (b-\epsilon)y_n\big\|_n.$$

Since $z^{*,j}$ is a norming functional for $z^j - b_j y$, for any $n \notin \mathrm{supp}(y)$,

$$z_n^{*,j}\big(z_n^j - (b-\epsilon)y_n\big) = z_n^{*,j}(z_n^j - b_j y_n) = \|z_n^j - b_j y_n\|_n = \big\|z_n^j - (b-\epsilon)y_n\big\|_n,$$

which proves (24) in this case. The proof is complete. $\square$

Applying Proposition 3.16, we will show the existence of continuous selection for the metric projection onto some 1-dimensional non-Chebyshev subspaces of X.

Theorem 3.17. *Let $y = (y_n) \in X \setminus \{0\}$. Assume that* $\mathrm{supp}(y) = \mathbb{N}$ *and that there exists $n_o \in \mathbb{N}$ such that*

$$-\sum_{j=1}^{n_o} \|y_j\|_j + \sum_{j=n_o+1}^{\infty} \|y_j\|_j = 0 \tag{26}$$

and such that, for any nonempty set $N_1 \subseteq \mathbb{N}$, $N_1 \neq \{1, \dots, n_o\}$, and $N_1 \neq \mathbb{N} \setminus \{1, \dots, n_o\}$,

$$-\sum_{n \in N_1} \|y_j\|_j + \sum_{n \in \mathbb{N} \setminus N_1} \|y_j\|_j \neq 0. \tag{27}$$

If the X_n's are strictly convex and smooth for any $n \in \mathbb{N}$ and $\dim(X_n) = 1$ *for* $n = 1, \ldots, n_o$*, then there exists a continuous selection for the metric projection onto* Y*.*

Proof. We apply Theorem 2.4. To do that, fix $x = (x_n) \in X$ such that $0 \in P_Y(x)$, and fix two sequences $(z^n) \subset X$ and $(w^n) \subset X$ converging to x and two compact intervals I_1, I_2 such that $P_Y(z^n) \subset I_1 y$ for any $n \in \mathbb{N}$ and $P_Y(w^n) \subset I_2 y$ for any $n \in \mathbb{N}$. We need to show that $I_1 \cap I_2 \neq \emptyset$. If $P_Y(x) = \{0\}$, then by Lemma 2.3, $0 \in I_1 \cap I_2$. If $\{0\} \neq P_Y(x)$, we can assume without loss of generality that $P_Y(x) = [-y, y]$. By Lemma 2.8, $x_n = c_n y_n$ for any $n \in \mathbb{N}$, where $c_n \in \mathbb{R} \setminus \{0\}$. By Theorem 2.1, there exists $x^* = (x_n^*) \in S_{X^*}$ such that for any $n \in \mathbb{N}$,

$$x_n^*(x_n) = \|x_n\|_n = |c_n| \|y_n\|_n = c_n x_n^*(y_n), \qquad x_n^*(x_n \pm y_n) = \|x_n \pm y_n\|_n,$$

and $x^*(y) = 0$. Since $\operatorname{supp}(y) = \mathbb{N}$, by (26) and (27), $\operatorname{sign}(c_n) = -a$ for $n = 1, \ldots, n_o$ and $\operatorname{sign}(c_n) = a$ for $n > n_o$, where $a \in \{-1, 1\}$. Assume that $a = 1$. Since $P_Y(x) = [-1, 1]y$, we claim that $c_n \leq -1$ for $n = 1, \ldots, n_o$, $c_{m_o} = -1$ for some $m_o \in \{1, \ldots, n_o\}$, and $c_n \geq 1$ for $n > n_o$. If $0 > c_n > -1$ for some $n \in \{1, \ldots, n_0\}$, then

$$\|x_n + y_n\|_n = (c_n + 1) x_n^*(y_n) < 0,$$

which is a contradiction. If $c_n < -1$ for all $n \in \{1, \ldots, n_0\}$, then $m = \min\{c_n : n = 1, \ldots, n_0\} < -1$. Hence, for any $n \in \mathbb{N}$,

$$x_n^*(x_n - my) = \|x_n - my\|_n.$$

This shows that

$$[my, y] \subset P_Y(x_0) = [-y, y],$$

so we get a contradiction. Analogously, if $c_n < 1$ for some $n > n_0$, then

$$\|x_n - y_n\|_n = (c_n - 1) x_n^*(y_n) < 0,$$

which gives a contradiction. Since $\dim(X_n) = 1$ for $n = 1, \ldots, n_o$, we can apply Proposition 3.16 to (z^n) and (w^n). Hence, $-1 \in I_1 \cap I_2$. If $a = -1$, by applying Proposition 3.16 to $-x$ we get that $1 \in I_1 \cap I_2$. By Theorem 2.4, we get our claim. ☐

Remark 3.18. If $y \in X \setminus \{0\}$ satisfies (26) and Y is the space generated by y, then Y is never a Chebyshev subspace. Indeed, if we define the element $x = (x_n) \in X$ by $x_n = -y_n$ for $n = 1, \ldots, n_o$ and $x_n = y_n$ for $n > n_o$, then $\operatorname{dist}(x, Y) = 2\sum_{k=1}^{n_o} \|y_k\|_k$ and also $P_Y(x) = [-y, y]$.

Now, we present a possible application of Theorem 3.17.

Example 3.19. Let X be so chosen that $\dim(X_1) = 1$ and X_n are smooth and strictly convex for $n \in \mathbb{N}$. Fix $y_n \in S_{X_n}$ for $n \in \mathbb{N}$. Let $y = (-y_1, \frac{y_2}{2}, \ldots, \frac{y_n}{2^n}, \ldots)$. It is easy to see that y satisfies (26) and (27) for $n_o = 1$. Applying Theorem 3.17, we get that there exists a continuous selection for the metric projection onto $Y = \operatorname{span}[y]$.

Observe that the assumption (21) from Proposition 3.16 is essential because of the following.

Example 3.20. Let $X = \bigoplus_1 X_n$, where for any $n \in \mathbb{N}$, $X_n = \mathbb{R}^2$, endowed with the Euclidean norm $\{e_1, e_2\}$ the usual basis of $\mathbb{R}^2$ and $e_1^* \in X_n^*$ the unique element in $S_{X_n^*}$ such that $e_1^*(e_1) = 1$. Let

$$y = \Big(\frac{-1}{2}e_1, \frac{1}{4}e_1, \dots, \frac{1}{2^n}e_1, \dots\Big),$$

$Y = \text{span}[y]$, and

$$x = \Big(\frac{1}{2}e_1, \frac{1}{4}e_1, \dots, \frac{1}{2^n}e_1, \dots\Big).$$

Note that

$$\|x\| = \sum_{n=1}^{\infty} \|x_n\|_2 = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

and that $x^* = (e_1^*)_{n\geq 1}$ is a norming functional for x. Moreover,

$$x^*(y) = \sum_{n=1}^{\infty} x_i^* y_i = 0.$$

By Theorem 2.1, $0 \in P_Y(x)$. Now, we show that $P_Y(x) = [-1,1]y$. Indeed, for any $b \in \mathbb{R}$,

$$x - by = \Big(\frac{1}{2}(1+b)e_1, \frac{1}{4}(1-b)e_1, \dots, \frac{1}{2^n}(1-b)e_1, \dots\Big).$$

Hence, for any $b \in [-1,1]$,

$$\begin{aligned}\|x - by\| &= \frac{1}{2}(1+b) + \Big(\sum_{n=2}^{\infty} \frac{1}{2^n}\Big)(1-b) \\ &= \frac{1}{2}(1+b) + \frac{1}{2}(1-b) = 1 = \text{dist}(x, Y).\end{aligned}$$

If $|b| > 1$, then it is easy to see that $\|x - by\| > x^*(x - by)$ and by Corollary 2.2, we get that $by \notin P_Y(x)$. Now, we give two sequences (x^j) and (z^j) such that $\|x^j - x\| \to 0$, $\|z^j - x\| \to 0$, $P_Y(x^j) = \{-y\}$, and $P_Y(z^j) = \{0\}$. For each $j \in \mathbb{N}$ define the element $x^j \in X$ given by

$$x_k^j = x_k \quad \text{for } k \neq j, \qquad x_j^j = x_j - \frac{1}{2^{(j-1)}}e_1.$$

Note that $\|x^j - x\| = \frac{1}{2^{(j-1)}}$ for every $j \in \mathbb{N}$, so (x^j) converges to x. Now, we show that $P_Y(x^j) = \{-y\}$. Observe that for any $b \in \mathbb{R}$,

$$x^j - by = \Big(\frac{1}{2}(1+b)e_1, \frac{1}{4}(1-b)e_1, \dots, \frac{-1}{2^j}(1+b)e_1, \frac{1}{2^{(j+1)}}(1-b)e_1, \dots\Big).$$

Hence,

$$x^j - (-y) = \Big(0, \frac{1}{2}e_1, \dots, \frac{1}{2^{(j-1)}}e_1, 0, \frac{1}{2^j}e_1, \dots\Big).$$

This shows that

$$\big\|x^j - (-y)\big\| = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 = x^*(x^j).$$

Since $x^*(y) = 0$, $-y \in P_Y(x^j)$. Observe that for $b > -1$, $(x^j - by)_j < 0$. Hence, by Corollary 2.2,

$$\|x - by\| > x^*(x - by) = x^*(x) = \|x\| = \text{dist}(x, Y).$$

Analogously, if $b < -1$, then $(x^j - by)_1 < 0$ and again by Corollary 2.2,

$$\|x - by\| > x^*(x - by) = x^*(x) = \|x\| = \text{dist}(x, Y).$$

This shows that $P_Y(x^j) = \{-y\}$ for any $j \in \mathbb{N}$. Now, fix a sequence of positive numbers $\{a_j\}$ tending to zero. Define $z^j = x + (\frac{a_j e_2}{2^n})_{n\geq 1}$. Note that

$$\|z^j - x\| = a_j \Big\| \Big(\frac{e_2}{2^n} \Big) \Big\| = a_j \sum_{n=1}^{\infty} \frac{1}{2^n} = a_j.$$

Hence, $\|z^j - x\| \to 0$ as $j \to \infty$. Observe that for any $b \in \mathbb{R}$ and $j \in \mathbb{N}$,

$$\begin{aligned} \|z^j - by\| &= \frac{1}{2}\sqrt{(1+b)^2 + a_j^2} + \Big(\sum_{n=2}^{\infty} \frac{1}{2^n} \Big) \sqrt{(1-b)^2 + a_j^2} \\ &= \frac{1}{2}\Big(\sqrt{(1+b)^2 + a_j^2} + \sqrt{(1-b)^2 + a_j^2} \Big). \end{aligned}$$

Define for $b \in \mathbb{R}$ and $j \in \mathbb{N}$,

$$g_j(b) = \sqrt{(1+b)^2 + a_j^2} + \sqrt{(1-b)^2 + a_j^2} = 2\|z^j - by\|.$$

To show that $P_Y(z^j) = \{0\}$, it is enough to show that g_j attains its global strict minimum at $b = 0$. Since $a_j > 0$, g_j is a convex differentiable function. Note that

$$g_j'(b) = \frac{b+1}{\sqrt{(1+b)^2 + a_j^2}} + \frac{b-1}{\sqrt{(1-b)^2 + a_j^2}}.$$

Therefore, $g_j'(b) = 0$ if and only if

$$(b+1)\sqrt{(1-b)^2 + a_j^2} = (1-b)\sqrt{(1+b)^2 + a_j^2}.$$

Hence,

$$(b+1)^2\big((1-b)^2 + a_j^2\big) = (1-b)^2\big((1+b)^2 + a_j^2\big)$$

and consequently,

$$(1-b)^2 a_j^2 = (1+b)^2 a_j^2.$$

Since $a_j > 0$, the only solution of the last equation is $b = 0$, which shows that g_j attains its global strict minimum at $b = 0$. Thus, $P_Y(z^j) = \{0\}$ for any $j \in \mathbb{N}$. Since $P_Y(x^j) = \{-y\}$ for any $j \in \mathbb{N}$ and the sequences $\{z^j\}$ and $\{x^j\}$ converge to x, by Theorem 2.4, there is no continuous selection for the metric projection from X onto Y.

We conclude this article by modifying a bit of reasoning from Example 3.20 to show an example of 4-dimensional real Banach space X and its 1-dimensional subspace Y such that there is no continuous selection for the metric projection from X onto Y.

Example 3.21. Let $X = l_2^{(2)} \bigoplus_1 l_2^{(2)}$, and let $Y = \text{span}[y]$, where $y = (-1, 0, 1, 0)$. Let $x = (1, 0, 1, 0)$. Observe that for any $b \in \mathbb{R}$,

$$\|x - by\| = |1 + b| + |1 - b|.$$

Hence, it is easy to see that $P_Y(x) = [-1, 1]y$ and $\text{dist}(x, Y) = \|x\| = 2$. Now, fix $k \in \mathbb{R}$ and a sequence (a_n) of positive real numbers tending to zero. Let $x^{k,n} = (1, a_n, 1, ka_n)$. It is clear that $\|x^{k,n} - x\| \to 0$ for $n \to +\infty$. Now, we show that for any $n \in \mathbb{N}$, $P_Y(x^{k,n}) = \{\frac{1-|k|}{|k|+1}y\}$. To do that, for fixed k and n we will minimize the function $f_{k,n}(b) = \|x^{k,n} - by\|$. Observe that

$$f_{k,n}(b) = \sqrt{(1 + b)^2 + a_n^2} + \sqrt{(1 - b)^2 + k^2 a_n^2}.$$

To show that $P_Y(x^{k,n}) = \{\frac{|k|-1}{|k|+1}y\}$, it is enough to show that $f_{k,n}$ attains its global strict minimum at $b = \frac{1-|k|}{|k|+1}$. Since $a_n > 0$, $f_{k,n}$ is a convex differentiable function for any $k \neq 0$. Note that

$$f'_{k,n}(b) = \frac{b + 1}{\sqrt{(1 + b)^2 + a_n^2}} + \frac{b - 1}{\sqrt{(1 - b)^2 + k^2 a_n^2}}.$$

Therefore, $f'_{k,n}(b) = 0$ if and only if

$$(b + 1)\sqrt{(1 - b)^2 + k^2 a_n^2} = (1 - b)\sqrt{(1 + b)^2 + a_n^2}. \tag{28}$$

Hence,

$$(b + 1)^2\big((1 - b)^2 + k^2 a_n^2\big) = (1 - b)^2\big((1 + b)^2 + a_n^2\big)$$

and consequently,

$$(1 + b)^2 k^2 a_n^2 = (1 - b)^2 a_n^2.$$

Since $a_n > 0$, the above equation is equivalent to

$$(1 + b)^2 k^2 = (1 - b)^2. \tag{29}$$

If $k = 0$, then $f_{o,n}$ is differentiable at any $b \neq 1$. By the above reasoning, $P_Y(x^{o,n}) = \{y\}$. If $k = \pm 1$, the only solution of (29) is $b = 0$. For $k \notin \{-1, 0, 1\}$, after elementary calculations, we get that $b_1 = \frac{|k|+1}{1-|k|}$ and $b_2 = \frac{1-|k|}{1+|k|}$ are two solutions of (29). By (28), b_2 is the only solution of (28). Since $f_{k,n}$ is a convex and differentiable function, $P_Y(x^{k,n}) = \{\frac{1-|k|}{|k|+1}y\}$, as required. In particular, $\lim_n P_Y(x^{o,n}) = y$ and $\lim_n P_Y(x^{1,n}) = 0$. This shows that there is no continuous selection for the metric projection onto Y.

Acknowledgment. Ciesielski's work was supported by grant 2017/01/X/ST1/01036 from the National Science Centre, Poland.

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